Recommended by the University of Calcutta, Dacca, Patna, Utkal etc., as a text-book for B. A. & B. Sc. Examinations

DIFFERENTIAL CALCULUS

BY

B. C. DAS, M. Sc.

PROFESSOR OF MATHEMATICS,
PRESIDENCY COLLEGE, CALCUTTA (RETD.) ;
EX-LECTURER IN APPLIED MATHEMATICS, CALCUTTA UNIVERSITY

AND

B. N. MUKHERJEE, M. A.

Premchand Roychand Scholar

PROFESSOR OF MATHEMATICS,
SCOTTISH CHURCH COLLEGE, CALCUTTA (RETD.)

SIXTEENTH EDITION

U. N. DHUR & SONS, PRIVATE LTD.
15, BANKIM CHATTERJEE STREET,
CALCUTTA 12

Published by
DWIJENDRANATH DHUR, LL. B.
For U. N. DHUR & SONS, PRIVATE LTD.,
15, Bankim Chatterjee St., Calcutta 12.

1ST EDITION—1949
2ND EDITION—1950
3RD EDITION—1952
4TH EDITION—1954
5TH EDITION—1955
6TH EDITION—1957
7TH EDITION—1958
8TH EDITION—1960
9TH EDITION—1961
10TH EDITION—1962
11TH EDITION—1963
12TH EDITION—1964
13TH EDITION—1965
14TH EDITION—1967
15TH EDITION—1968
16TH EDITION—1970

Printed by
TRIDIBESH BASU,
THE K. P. BASU PRINTING WORKS,
11, Mohendra Gossain Lane, Calcutta 6.

PREFACE TO THE FIRST EDITION

THIS book is prepared with a view to be used as a text-book for the B. A. and B. Sc. students of the Indian Universities. We have tried to make the expositions of the fundamental principles clear as well as concise without going into unnecessary details ; and at the same time an attempt has been made to make the treatment as much rigorous and up-to-date as possible within the scope of the elementary work.

Proofs of certain important fundamental results and theorems which are assumed in the earlier chapters of the book for the convenience of the beginners and average students, have been given in the Appendix. A brief account of the theory of infinite series, especially the power series is given in the Appendix in order to emphasise their peculiarity in the application of the principles of Calculus. Important formulæ of this book are given in the beginning for ready reference. A good number of typical examples have been worked out by way of illustrations.

Many varied types of examples have been given for exercise, in order that the students might acquire a good grasp of the applications of the principles of Calculus. University questions of recent years have been added at the end to give the students an idea of the standard of the examination. Our thanks are due to the authorities and the staff of the K. P. Basu Printing Works, Calcutta, who in spite of their various preoccupations, had the kindness to complete the printing so efficiently in a short period of time. Any criticism, correction and suggestion towards the improvement of the book from teachers and students will be thankfully received.

CALCUTTA, *June, 1949* — B. C. D. B. N. M.

PREFACE TO THE EIGHTH EDITION

In this edition we have thoroughly revised the book in accordance with the syllabus of the Three-Year Degree Course for Differential Calculus (Pass Course). In some places chapters have been re-arranged and a few well-chosen examples have been added here and there to make the book more useful. Differentiation of Determinants and a proof of the converse of the Euler's Theorem on Homogeneous Functions have been added. We take this opportunity of thanking Prof. Joyti Choudhuri, M. Sc. and Prof. Tapen Moulik, M.Sc. for their help in the revision of the text.

CALCUTTA, *July, 1960* — B. C. D., B. N. M.

PREFACE TO THE SIXTEENTH EDITION

This edition is practically a reprint of the fifteenth edition with some minor additions and alterations.

Our thanks are due to Sri Balen Mukherjee B. A. and Sm. Kalpana Sircar M. Sc. for helping us considerably in bringing out this edition promptly.

Our thanks are also due to the authorities and staff of Messrs K. P. Basu Printing Works for efficient and prompt discharge of their duties in spite of their various pre-occupations.

CALCUTTA, *March, 1970* — B. C. D., B. N. M.

GREEK ALPHABETS USED IN THE BOOK

α (alpha)	λ (lambda)	ξ (xi)
β (bētā)	μ (mu)	η (ētā)
γ (gamma)	ν (nu)	ζ (zetā)
δ (deltā)	π (pi)	Δ (cap. deltā)
ϕ (phai)	ρ (rho)	Σ (cap. sigma)
ψ (psi)	σ (sigma)	Γ (cap. gamma)
κ (kappa)	τ (tau)	ϵ (epsilon)
		θ (thetā)

CONTENTS

SYLLABUS FOR THREE-YEAR DEGREE COURSE

DIFFERENTIAL CALCULUS (B.A. & B.Sc. Pass)

Rational, irrational and real numbers Linear continuum (Rigorous treatment is not required).

Functions of a single variable. Limits of functions. Continuity of functions. Existence and attainment of bounds and of all intermediate values in a closed interval (Examples to be given, but no proof required). Inverse functions.

Derivative. Its geometrical interpretation and the meaning of its sign. Rules of differentiation. Rolle's theorem. Mean Value Theorems of Lagrange and Cauchy. Successive differentiation. Leibnitz's Theorem. Differential.

Taylor's and Maclaurin's theorems with Cauchy's and Lagrange's form of the remainder. Taylor's series for such functions as

e^x, $\sin x$, $\cos x$ and for $(1+x)^n$, $\log_e (1+x)$ when $|x| < 1$. Maxima and Minima. Indeterminate forms.

Functions of two or more variables. Geometric notion of their continuity. Successive partial derivatives. Statement of a set of sufficient conditions (without proof) for the commutative property of partial derivatives. Euler's Theorem on homogeneous functions. Acquaintance with the rule for differentiation of implicit functions.

Applications : Tangent, normal, rectilinear asymptote and curvature of plane curves. Envelopes.

List of Important Formulæ

I. Important Limits.

(i) $\underset{x\to 0}{Lt} \dfrac{\sin x}{x} = 1$, where x is in radian measure.

(ii) $\underset{n\to \pm\infty}{Lt} \left(1+\dfrac{1}{n}\right)^n = e$ or, $\underset{x\to 0}{Lt} (1+x)^{\frac{1}{x}} = e$.

(iii) $\underset{x\to 0}{Lt} \dfrac{1}{x} \log(1+x) = 1$. (iv) $\underset{x\to 0}{Lt} \dfrac{e^x - 1}{x} = 1$.

(v) $\underset{x\to 0}{Lt} \dfrac{(1+x)^n - 1}{x} = n$. (vi) $\underset{n\to \infty}{Lt} \dfrac{x^n}{n!} = 0$.

(vii) $\underset{n\to \infty}{Lt} x^n = 0 \ (-1 < x < 1)$.

II. Standard Derivatives.

$\dfrac{d}{dx}(x^n) = nx^{n-1}$; $\dfrac{d}{dx}\left(\dfrac{1}{x^n}\right) = -\dfrac{n}{x^{n+1}}$.

$\dfrac{d}{dx}(x) = 1$; $\dfrac{d}{dx}(\sqrt{x}) = \dfrac{1}{2\sqrt{x}}$.

$\dfrac{d}{dx}(e^x) = e^x$; $\dfrac{d}{dx}(a^x) = a^x \log_e a$.

$\dfrac{d}{dx}(\log x) = \dfrac{1}{x}$; $\dfrac{d}{dx}(\log_a x) = \dfrac{1}{x} \log_a e$.

$\dfrac{d}{dx}(\sin x) = \cos x$; $\dfrac{d}{dx}(\cos x) = -\sin x$.

$\dfrac{d}{dx}(\tan x) = \sec^2 x$; $\dfrac{d}{dx}(\cot x) = -\operatorname{cosec}^2 x$.

$\dfrac{d}{dx}(\sec x) = \sec x \tan x$; $\dfrac{d}{dx}(\operatorname{cosec} x) = -\operatorname{cosec} x \cot x$.

$\dfrac{d}{dx}(\sin^{-1} x) = \dfrac{1}{\sqrt{1-x^2}}$; $\dfrac{d}{dx}(\cos^{-1} x) = -\dfrac{1}{\sqrt{1-x^2}}$.

$\dfrac{d}{dx}(\tan^{-1} x) = \dfrac{1}{1+x^2}$; $\dfrac{d}{dx}(\cot^{-1} x) = -\dfrac{1}{1+x^2}$.

$$\frac{d}{dx}(\sec^{-1}x)=\frac{1}{x\sqrt{x^2-1}}; \qquad \frac{d}{dx}(\operatorname{cosec}^{-1}x)=-\frac{1}{x\sqrt{x^2-1}}.$$

$$\frac{d}{dx}(\sinh x)=\cosh x; \qquad \frac{d}{dx}(\cosh x)=\sinh x.$$

$$\frac{d}{dx}(\tanh x)=\operatorname{sech}^2 x; \qquad \frac{d}{dx}(\coth x)=-\operatorname{cosech}^2 x.$$

$$\frac{d}{dx}(\operatorname{sech} x)=-\operatorname{sech} x\tanh x; \quad \frac{d}{dx}(\operatorname{cosech} x)=-\operatorname{cosech} x\coth x.$$

$$\frac{d}{dx}(\sinh^{-1}x)=\frac{1}{\sqrt{1+x^2}}; \qquad \frac{d}{dx}(\cosh^{-1}x)=\frac{1}{\sqrt{x^2-1}}\ (x>1).$$

$$\frac{d}{dx}(\tanh^{-1}x)=\frac{1}{1-x^2}(x<1); \quad \frac{d}{dx}(\coth^{-1}x)=-\frac{1}{x^2-1}\ (x>1).$$

$$\frac{d}{dx}(\operatorname{sech}^{-1}x)=-\frac{1}{x\sqrt{1-x^2}}(x<1); \quad \frac{d}{dx}(\operatorname{cosech}^{-1}x)=-\frac{1}{x\sqrt{x^2+1}}.$$

III. Fundamental theorems on Differentiation.

(i) $\frac{d}{dx}(c)=0.$ (ii) $\frac{d}{dx}\{c\phi(x)\}=c\phi'(x).$

(iii) $\frac{d}{dx}\{\phi(x)\pm\psi(x)\}=\phi'(x)\pm\psi'(x).$

(iv) $\frac{d}{dx}\{\phi(x)\times\psi(x)\}=\phi(x)\psi'(x)+\psi(x)\phi'(x).$

Derivative of the product of two functions
= first function × derivative of the second
+ second function × derivative of the first.

(v) $$\frac{d}{dx}\{\phi_1(x)\phi_2(x)\ldots\phi_n(x)\}$$
$$=\phi_1'(x)\{\phi_2(x)\phi_3(x)\cdots\}+\phi_2'(x)\{\phi_1(x)\phi_3(x)\cdots\}+\cdots+\phi_n'(x)\{\phi_1(x)\phi_2(x)\cdots\}.$$

(vi) $$\frac{d}{dx}\left\{\frac{\phi(x)}{\psi(x)}\right\}=\frac{\phi'(x)\psi(x)-\psi'(x)\phi(x)}{\{\psi(x)\}^2},\ \psi(x)\neq 0.$$

Derivative of the quotient of two functions

$$=\frac{(\text{Deriv. of Num.})\times\text{Denom.}-(\text{Deriv. of Denom.})\times\text{Num}}{(\text{Denom.})^2}.$$

(vii) If $y=f(v)$, $v=\phi(x)$,

$$\frac{dy}{dx}=\frac{dy}{dv}\cdot\frac{dv}{dx}.$$

(viii) $\frac{dy}{dx}\times\frac{dx}{dy}=1$, *i.e.*, $\frac{dy}{dx}=1\Big/\frac{dx}{dy}\left(\frac{dx}{dy}\text{ and }\frac{dy}{dx}\neq 0\right)$.

(ix) If $x=\phi(t)$, $y=\psi(t)$,

$$\frac{dy}{dx}=\frac{dy}{dt}\Big/\frac{dx}{dt}\left(\frac{dx}{dt}\neq 0\right).$$

IV. Meaning of the derivatives and differential.

$\frac{dy}{dx}=\tan\psi$, where ψ is the angle which the tangent at any point to the curve $y=f(x)$ makes with the x-axis.

$\frac{dy}{dx}=$ rate of change of y with respect to x.

$dy=f'(x)\,dx$, if $y=f(x)$

V. The nth derivatives of some special functions.

$D^n(x^n)=n$

$D^n(x^m)=\frac{m\,!}{(m-n)\,!}\,x^{m-n}$ (m being a positive integer $> n$)

$D^n(e^{ax})=a^n e^{ax}$; $D^n(e^x)=e^x$.

$D^n\cdot\frac{1}{x+a}=\frac{(-1)^n\,n\,!}{(x+a)^{n+1}}$; $D^n\log(x+a)=\frac{(-1)^{n-1}(n-1)\,!}{(x+a)^n}$.

$D^n\sin(ax+b)=a^n\sin\left(\frac{n\pi}{2}+ax+b\right)$.

$D^n\cos(ax+b)=a^n\cos\left(\frac{n\pi}{2}+ax+b\right)$.

$$D^n \sin ax = a^n \sin\left(\frac{n\pi}{2} + ax\right); \; D^n \cos ax = a^n \cos\left(\frac{n\pi}{2} + ax\right).$$

$$\sin bx) = (a^2 + b^2)^{\frac{n}{2}} e^{ax} \sin(bx + n \tan^{-1} b/a).$$

$$D^n (e^{ax} \cos bx) = (a^2 + b^2)^{\frac{n}{2}} e^{ax} \cos(bx + n \tan^{-1} b/a).$$

$$D^n \left(\frac{1}{x^2 + a^2}\right) = \frac{(-1)^n \, n\,!}{a^{n+2}} \sin^{n+1}\theta \sin(n+1)\theta,$$

where $\theta = \tan^{-1}(a/x)$.

$$D^n (\tan^{-1} x) = (-1)^{n-1} (n-1)\,! \sin^n\theta \sin n\theta,$$

where $\theta = \cot^{-1} x$.

Leibnitz's Theorem.

$$(uv)_n = u_n v + {}^n c_1 u_{n-1} v_1 + {}^n c_2 u_{n-2} v_2 + \cdots\cdots + u v_n.$$

VI. (i) Mean Value Theorem.

$$f(x+h) = f(x) + hf'(x + \theta h), \; 0 < \theta < 1.$$

(ii) Taylor's Series (*finite form*).

$$f(x+h) = f(x) + hf'(x) + \frac{h^2}{2\,!} f''(x) + \cdots\cdots + \frac{h^{n-1}}{(n-1)\,!} f^{n-1}(x) + R_n.$$

Remainder R_n $= \dfrac{h^n}{n\,!} f^n(x + \theta h), \; 0 < \theta < 1$ (*Lagrange's form*)

or, $\quad = \dfrac{h^n (1-\theta)^{n-1}}{(n-1)\,!} f^n(x + \theta h)$ (*Cauchy's form*)

$$f(x) = f(a) + (x-a) f'(a) + \frac{(x-a)^2}{2\,!} f''(a) + \cdots\cdots + \frac{(x-a)^{n-1}}{(n-1)\,!} f^{n-1}(a) + \frac{(x-a)^n}{n\,!} f^n\{a + \theta(x-a)\}$$

$$0 < \theta < 1.$$

Remainder $R_n = \dfrac{(x-a)^n}{n\,!} f^n\{a + \theta(x-a)\}.$

(iii) Maclaurin's Series (*finite form*).

$$f(x)=f(0)+xf'(0)+\frac{x^2}{2!}f''(0)+$$

$$+\frac{x^{n-1}}{(n-1)!}f^{n-1}(0)+\frac{x^n}{n!}f^n \qquad < 1.$$

(iv) Taylor's Series (*extended to infinity*).

$$f(x+h)=f(x)+hf'(x)+\frac{h^2}{2!}f''(x)+\cdots\cdots \text{ to } \infty.$$

(v) Maclaurin's Series (*extended to infinity*).

$$f(x)=f(0)+xf'(0)+\frac{x^2}{2!}f''(0)+\cdots\cdots \text{ to } \infty.$$

(vi) Expansions of some well-known functions in series.

$$\left.\begin{array}{ll}(a) & e^x=1+\frac{x}{1!}+\frac{x^2}{2!}+\cdots+\frac{x^n}{n!}+\cdots \\ (b) & \sin x=x-\frac{x^3}{3!}+\frac{x^5}{5!}-\cdots+(-1)^n\frac{x^{2n+1}}{(2n+1)!}+\cdots \\ (c) & \cos x=1-\frac{x^2}{2!}+\frac{x^4}{4!}-\cdots+(-1)^n\frac{x^{2n}}{(2n)!}+\cdots \\ (d) & \sinh x=x+\frac{x^3}{3!}+\frac{x^5}{5!}+\cdots+\frac{x^{2n+1}}{(2n+1)!}+\cdots \\ (e) & \cosh x=1+\frac{x^2}{2!}+\frac{x^4}{4!}+\cdots+\frac{x^{2n}}{(2n)!}+\cdots\end{array}\right\}\text{for all values of } x$$

$$(f)\ \log(1+x)=x-\frac{x^2}{2}+\frac{x^3}{3}-\cdots+(-1)^{n-1}\frac{x^n}{n}+\cdots$$

for $-1 < x \leqslant 1$.

$$(g)\ (1+x)^{-1}=1-x+x^2-x^3+\cdots \qquad \text{for } -1 < x < 1.$$

$$(1-x)^{-1}=1+x+x^2+x^3+\cdots \qquad \text{for } -1 < x < 1.$$

$$(1+x)^{-2}=1-2x+3x^2-4x^3+\cdots \qquad \text{for } -1 < x < 1.$$

$$(1-x)^{-\frac{1}{2}}=1+\frac{1}{2}x+\frac{1.3}{2.4}x^2+\frac{1.3.5}{2.4.6}x^3+\cdots$$

for $-1 < x < 1$.

(h) $\tan^{-1}x = x - \dfrac{x^3}{3} + \dfrac{x^5}{5} - \cdots + (-1)^{n-1}\dfrac{x^{2n-1}}{2n-1} + \cdots$

for $-1 \leqslant x \leqslant 1$.

(i) $\sin^{-1}x = x + \dfrac{1}{2}\cdot\dfrac{x^3}{3} + \dfrac{1.3}{2\,4}\cdot\dfrac{x^5}{5} + \cdots$ for $-1 \leqslant x \leqslant 1$.

VII. Maxima and Minima.

(i) If $f'(c)=0$ and $f''(c)$ is *negative*, then $f(x)$ is a *maximum* for $x=c$.

(ii) If $f'(c)=0$ and $f''(c)$ is *positive*, then $f(x)$ is a *minimum* for $x=c$.

(iii) If $f'(c)=f''(c)=\cdots=f^{r-1}(c)=0$ and $f^r(c) \neq 0$, then

(a) if r is *even*, then $f(x)$ is a *maximum* or a *minimum* for $x=c$, according as $f^r(c)$ is *negative* or *positive* ;

(b) if r is *odd*, there is neither a maximum nor a minimum for $f(x)$ at $x=c$.

(iv) *Alternative criterion* for maxima and minima :

(a) $f(x)$ is a maximum if $f'(x+h)$ changes sign from $+$ to $-$, and

(b) $f(x)$ is a minimum if $f'(x+h)$ changes sign from $-$ to $+$,

as h, being numerically infinitely small, changes from $-$ to $+$.

VIII. Indeterminate forms.

(i) **Form** $\dfrac{0}{0}$.

$$\underset{x\to a}{Lt}\frac{\phi(x)}{\psi(x)} = \underset{x\to a}{Lt}\frac{\phi'(x)}{\psi'(x)} = \frac{\phi'(a)}{\psi'(a)}$$ (*L' Hospital's Theorem*)

(ii) **Form** $\dfrac{\infty}{\infty}$. $\underset{x\to a}{Lt}\dfrac{\phi(x)}{\psi(x)} = \underset{x\to a}{Lt}\dfrac{\phi'(x)}{\psi'(x)}$.

IX. Partial Differentiation.

(i) If $u=f(x, y)$, and if f_{yx}, f_{xy} exist and f_{yx} (or f_{xy}) is continuous, then $\frac{\delta^2 u}{\delta x \delta y}=\frac{\delta^2 u}{\delta y \delta x}$.

(ii) If $f(x, y)$ be a homogeneous function of degree n in x, y, then $x\frac{\delta f}{\delta x}+y\frac{\delta f}{\delta y}=nf(x, y)$. [*Euler's Theorem*]

Similarly, $x\frac{\delta f}{\delta x}+y\frac{\delta f}{\delta y}+z\frac{\delta f}{\delta z}=nf(x, y, z)$.

(iii) If $f(x, y)=0$, $\frac{dy}{dx}=-\frac{f_x}{f_y}$ $\quad (f_y \neq 0)$.

(iv) If $u=f(x, y)$ where $x=\phi(t)$, $y=\psi(t)$,

$$\frac{du}{dt}=\frac{\delta u}{\delta x}\frac{dx}{dt}+\frac{\delta u}{\delta y}\frac{dy}{dt}.$$

(v) If $u=f(x, y)$, $du=f_x\,dx+f_y\,dy$.

Similarly, $du=f_x\,dx+f_y\,dy+f_z\,dz$.

(vi) If $u=f(x_1, x_2)$, where $x_1=\phi_1(x, y)$, $x_2=\phi_2(x, y)$,

$$\frac{\delta u}{\delta x}=\frac{\delta u}{\delta x_1}\frac{\delta x_1}{\delta x}+\frac{\delta u}{\delta x_2}\frac{\delta x_2}{\delta x};\ \frac{\delta u}{\delta y}=\frac{\delta u}{\delta x_1}\frac{\delta x_1}{\delta y}+\frac{\delta u}{\delta x_2}\frac{\delta x_2}{\delta y}.$$

X. Tangent and Normal.

(i) *Equation of the tangent*

$$Y-y=\frac{dy}{dx}(X-x),\ \text{or},\ (X-x)f_x+(Y-y)f_y=0.$$

(ii) *Equation of the normal*

$$Y-y=-\frac{dx}{dy}(X-x),\ \text{or},\ \frac{X-x}{f_x}=\frac{Y-y}{f_y}.$$

(iii) *Cartesian sub-tangent* $=y/y_1$; *sub-normal* $=yy_1$
Length of tangent $=y\sqrt{1+y_1^2}/y_1$
Length of normal $=y\sqrt{1+y_1^2}$.

(iv) *Arc-differential (Cartesian)*

$$\frac{dx}{ds} = \cos\psi,\ \frac{dy}{ds} = \sin\psi,\ \frac{dy}{dx} = \tan\psi$$

$$\left(\frac{dx}{ds}\right) + \left(\frac{dy}{ds}\right)^2 = 1\ ;\ \left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 = \left(\frac{ds}{dt}\right)^2$$

$$\frac{ds}{dx} = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\ ;\ \frac{ds}{dy} = \sqrt{1 + \left(\frac{dx}{dy}\right)^2}$$

$$ds^2 = dx^2 + dy^2.$$

(v) *Angle between tangent and radius vector* (ϕ)

$$\tan\phi = \frac{r\,d\theta}{dr}\ ;\ \sin\phi = \frac{r\,d\theta}{ds}\ ;\ \cos\phi = \frac{dr}{ds}$$

$$\psi = \theta + \phi.$$

(vi) *Arc-differential (Polar)*

$$\frac{ds}{d\theta} = \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\ ;\ \frac{ds}{dr} = \sqrt{1 + \left(\frac{rd\theta}{dr}\right)^2}$$

$$ds = \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\,d\theta\ ;\ ds = \sqrt{1 + \left(\frac{rd\theta}{dr}\right)^2}\,dr$$

$$ds^2 = dr^2 + r^2\,d\theta^2.$$

(vii *Polar sub-tangent* $= r^2\dfrac{d\theta}{dr} = -\dfrac{d\theta}{du}$, where $r = \dfrac{1}{u}$.

Polar sub-normal $= \dfrac{dr}{d\theta}$.

Perpendicular from pole on tangent (p)

$$p = r\sin\phi\ ;\ \frac{1}{p^2} = \frac{1}{r^2} + \frac{1}{r^4}\left(\frac{dr}{d\theta}\right)^2 = u^2 + \left(\frac{du}{d\theta}\right)^2.$$

(viii) *Pedal equations of some well-known curves*

(1) Circle $x^2 + y^2 = a^2$ (centre)$\cdots r = p$

(2) Circle $x^2 + y^2 = a^2$ (point on the circumference)$\cdots r^2 = 2ap$

(3) Parabola $y^2 = 4ax$ (focus)$\cdots p^2 = ar$

(4) Ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ (focus)$\cdots\frac{b^2}{p^2}=\frac{2a}{r}-1$

(5) Ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ (centre)$\cdots\frac{a^2b^2}{p^2}+r^2=a^2+b^2$

(6) Hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ (focus)$\cdots\frac{b^2}{p^2}=\frac{2a}{r}+1$

(7) Hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ (centre)$\cdots$

$$\frac{a^2b^2}{p^2}-r^2=a^2-b^2$$

(8) Rect. Hyperbola $x^2-y^2=a^2$ (centre)$\cdots pr=a^2$

(9) Parabola $r=\frac{2a}{1\pm\cos\theta}$ (focus) $\cdots$ $p^2=ar$

(10) Cardioide $r=a\,(1\pm\cos\theta)$ (pole)$\cdots r^3=2ap^2$

(11) Lemniscate $\left.\begin{array}{l} r^2=a^2\cos 2\theta \\ r^2=a^2\sin 2\theta \end{array}\right.$ $\cdots r^3=a^2p$

(12) $\left.\begin{array}{l} r^n=a^n\cos n\theta \\ r^n=a^n\sin n\theta \end{array}\right\}$ (pole)$\cdots r^{n+1}=a^np.$

XI. Curvature.

$$\rho=\frac{ds}{d\psi}\ [\ s=f(\psi)\ ].$$

$$\rho=\frac{(1+y_1{}^2)^{\frac{3}{2}}}{y_2}\ [\ y=f(x)\ ]\ ;\ y_2\neq 0.$$

$$\rho=\frac{(1+x_1{}^2)^{\frac{3}{2}}}{x_2}\ [\ x=f(y)\ ]\ ,\ x_2\neq 0.$$

$$\rho=\frac{(x'^2+y'^2)^{\frac{3}{2}}}{x'y''-y'x''}\ [\ x=\phi(t),\ y=\psi(t)\ ].$$

$$\rho=\frac{(f_x{}^2+f_y{}^2)^{\frac{3}{2}}}{f_{xx}f_y{}^2-2f_{xy}f_xf_y+f_{yy}f_x{}^2}\ [\ f(x,y)=0\ ].$$

$$\rho = \frac{(r^2 + r_1^2)^{\frac{3}{2}}}{r^2 + 2r_1^2 - rr_2} \; [\, r = f(\theta) \,].$$

$$\rho = \frac{(u^2 + u_1^2)^{\frac{3}{2}}}{u^3(u + u_2)} \; [\, u = f(\theta) \,].$$

$$\rho = \frac{r\,dr}{dp} \; [\, p = f(r) \,]. \qquad \rho = p + \frac{d^2 p}{d\psi^2} \; [\, p = f(\psi) \,].$$

$$\rho = Lt\,\frac{x^2}{2y} \; [\text{ at the origin, } x\text{-axis } (y = 0) \text{ being tangent }].$$

$$\rho = Lt\,\frac{y^2}{2x} \; [\text{ at the origin, } y\text{-axis } (x = 0) \text{ being tangent }].$$

$$\rho = \sqrt{a^2 + b^2} \,.\, Lt\,\frac{x^2 + y^2}{ax + by} \; [\text{ at the origin, } ax + by = 0 \text{ being tangent }].$$

Chord of curvature through the pole $= 2\rho \sin \phi$.

Chord of curvature $\begin{cases} \text{parallel to } x\text{-axis} = 2\rho \sin \psi \\ \text{parallel to } y\text{-axis} = 2\rho \cos \psi \end{cases}$

Centre of curvature

$$\bar{x} = x - \frac{y_1(1 + y_1^2)}{y_2}, \quad \bar{y} = y + \frac{1 + y_1^2}{y_2} \qquad [\, y = f(x) \,]$$

$$\bar{x} = x + \frac{1 + x_1^2}{x_2}, \quad \bar{y} = y - \frac{x_1(1 + x_1^2)}{x_2} \qquad [\, x = f(y) \,]$$

$$\bar{x} = x - \rho \sin \psi, \qquad \bar{y} = y + \rho \cos \psi.$$

Radius of curvature of the evolute $= \dfrac{d^2 s}{d\psi^2}$.

DIFFERENTIAL CALCULUS

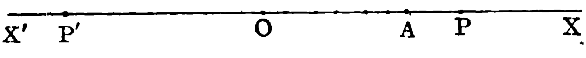
INTRODUCTION

NUMBERS

0·1. Numbers.

The earliest concept of numbers originated from counting, and the first set of numbers which was known to men was the set of *positive integers*. The arithmetical process of subtraction needed an extension to *negative integers*, and *zero* was included as a number. The process of division required a further extension to *rational numbers*, which are defined to be numbers of the form $\dfrac{m}{n}$, where m and n are integers, ultimately prime to each other, n being positive and not equal to zero. It may be noted that terminating decimals, as also recurring decimals, which are expressible in the form m/n fall under this category.

0·2. Geometrical representation of rational numbers ; rational points.



Take an indefinite line $X'OX$ for reference, and a suitable point O on it as origin. A suitable length OA on it being chosen as unit, if we divide OA geometrically into n equal parts, and take a length OP (or OP') equal to m such parts (towards the right of O if m be positive, and towards the left if m be negative), the length OP (or OP'), or the point P (or P', as the case may be) represents the rational number m/n. The point P, representing a rational number, is called a *rational point*.

0·3. Properties of rational numbers.

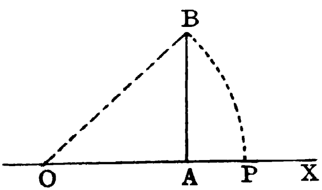
(i) Rational numbers are *well-ordered*. This means that of two unequal rational numbers a and b, either $a > b$ or $a < b$, also if $a > b$ and $b > c$, then $a > c$, etc. In other words, rational numbers are well-arranged in respect of their magnitudes, points representing higher numbers always falling to the right of those representing smaller ones, and vice versa, in their geometrical representation.

(ii) Rational numbers are everywhere *dense*; in other words, between any two rational numbers however close, or within any interval on the axis representing rational numbers, however small, there is an infinite number of rational numbers or points.

This may be easily seen from the fact that however close the two rational numbers a and b may be, $\frac{1}{2}(a+b)$ is a rational number lying between them. Similarly, between a and $\frac{1}{2}(a+b)$, as also between $\frac{1}{2}(a+b)$ and b we can insert rational numbers, and so on. Thus there is an infinite number of rational numbers between a and b.

0·4. Irrational numbers.

Whereas all rational numbers are represented by points on the axis, and though in any interval however small there is an infinite number of rational points, still the converse, that every point on the axis must represent some rational number, is not true;



e.g., $OP \equiv \sqrt{2}$ *is not rational.*

OA being unity, if AB be taken at right angles to OA and equal to it, OB is joined, and on OX, OP be cut off equal to OB, OP represents a number equal to $\sqrt{2}$, which is not rational.

Proof: For if $\sqrt{2} \equiv m/n$, where m, n are integers prime to each other, $m^2 = 2n^2$, showing that m^2 and so m is an even integer (for the square of an odd integer is evidently odd). Let $m = 2m'$, where m' is an integer. Then we get $n^2 = 2m'^2$ and so n is also an even integer. Thus m and n, which have a common factor 2, cannot be prime to one another, thus leading to a contradiction.

Similarly, equations like $x^3 = 7$, $4x^4 + 8 = 21$ etc. cannot be solved in terms of rational numbers alone. Besides radicals, there are other types of number like e, π,... (called *transcendental* numbers) which are not rational.

There are therefore numbers other than rational numbers, which are called irrational numbers, thus leading to a further extension of numbers.

0·5. Relations of irrational numbers to rational numbers ; representation of numbers (rational as well as irrational) **as sections of rational numbers.**

Consider the number $\sqrt{2}$ There is no rational number whose square is 2. The system of rational numbers therefore can be divided into two classes, say L and R, such that all numbers of the L-class have their squares less than 2, and those of the R-class have squares greater than 2. Hence, every number of the R-class $>$ every number of the L-class.

Thus, 1, 1·4, 1·41, 1·414, 1·4142 ... belong to L-class

and 2, 1·5, 1·42, 1·415, 1·4143 .... belong to R-class.

The differences of the corresponding numbers of the two classes are respectively,

1, ·1, ·01, ·001, ·0001......

Proceeding in this manner (by expressing $\sqrt{2}$ in a decimal form, which will lead to an endless decimals not recurring, and choosing the rational numbers of the two classes by stopping at any stage) we can find a member of the L-class and a member of the R-class which differ from one another by as little as we please. Our common-sense notion therefore demands the existence of a number x, and a corresponding point P on the axis, such that P divides the class L from the class R.

But this number x is not rational and belongs to neither of the two classes. Further, x^2 is neither > 2, nor < 2.

For if $x^2 > 2$, let $x^2 = 2 + \varepsilon$. Then however small ε may be, we can get rational numbers of the R-class whose squares being > 2 will differ from 2 by less than ε. Such rational numbers of the R-class will lie to the left of x, and so the assumption that x is the point dividing the two classes is untenable. Similarly $x^2 \nless 2$.

$\therefore$ $x^2 = 2$ or $x = \sqrt{2}$, and being not rational as proved before, it belongs neither to class L, nor to class R. The point P is thus only a point of section of the two classes of rational numbers L and R defined before, not belonging to either class, and representing the irrational number $\sqrt{2}$.

This leads to a new idea of defining *numbers as sections of rational numbers*, as follows :

"If by some means or other we divide all rational numbers into two classes L and R, such that each class contains at least one rational number, every rational number belongs to either L or R, and each number belonging to R-class $>$ every number of the L-class, then we obtain a section of rational numbers which defines a number, rational, or irrational, the particular mode of division defines a particular number by its section."

Three cases may arise : (i) The L-class has a greatest number, but the R-class has no least, *e.g.*, Let all rational numbers > 5 belong to R-class, and the number 5, as also all rational numbers < 5 belong to L-class. The section in this case represents the rational number 5, which belongs here to one of the two classes, namely the L class. (ii) The L-class has no greatest number, but the R-class has a least one ; *e.g.*, All rational numbers $< -3{\cdot}5$ belong to L-class and $-3{\cdot}5$ with all rational numbers greater than this belongs to R-class Here the section represents the rational number $-3{\cdot}5$, and the number itself belongs to R-class. (iii) The L-class has no greatest number and the R-class has no least number, *e.g.*, All rational numbers whose cubes are < 7 belong to L-class, and those whose cubes are > 7 belong to R-class ; there is no rational number, as can be shown, whose cube is equal to 7. The section in this case represents the irrational number $\sqrt[3]{7}$, and belongs to none of the classes L and R which consist of rational numbers only.

If may be noted that the case in which the L-class has a greatest number and R-class has a least number simultaneously is not possible, for otherwise, between these two rational numbers there would be an infinite number of rational numbers as proved before, and they would belong to none of the two classes.

This extension of our conception of numbers as sections of rational numbers gives us a more satisfactory basis of defining all numbers in a uniform way. We no longer think of numbers as isolated members, but as an aggregate of rational numbers divided into sections.

0·6. Real numbers.

All kinds of numbers, rational as well as irrational, positive or negative, including zero, constitute what are called real numbers.

It may be thought that just as from rational numbers, by dividing them into two classes by sections, we get in addition to rational numbers, a new type of numbers, namely irrational numbers, similarly by sections of real numbers again, we may expect a further extension of numbers. But this is not true. In this connection we state the following theorem.

Dedekind's theorem (on sections of real numbers)

If real numbers be divided into two classes L and R in such way that

(i) *every real number belongs to one class or the other,*

(ii) *each class contains at least one number,*

and (iii) *any number of the L-class is less than every number of the R-class,* then *there exists a real number a which effects this section, i.e., which has the property that all numbers less than a belong to L-class, and all numbers greater than a belong to R-class, the number a itself may belong to either class.*

[*For proof*, see *Hardy's Pure Mathematics*]

Thus as sections of real numbers we get real numbers alone (unlike that in case of rational numbers), and not any other new type of numbers.

Thus no further extension of numbers is possible ; and the aggregate of real numbers is complete. The correspondence (one to one) between *all* the points on the line $X'OX$ without exception (called the **linear continuum**) and the system of *all* real numbers rational and irrational (constituting what is called the **arithmetical continuum**) is now perfect.

0·7. Properties of real numbers.

(i) All fundamental laws of algebra which hold for rational numbers (including index laws etc.) remain true for irrational numbers as well.

[*For proof*, see *Bromwich's Infinite Series, Appendix I*]

(ii) Between any two real numbers however close, there exists an infinite number of rational numbers, as also an infinite number of irrational numbers.

One special case of this is Art. 0·3(ii).

0·8. Complex numbers.

In order to fill up the gaps and bring about a uniformity in the theory of equations, as also in all other theories of higher mathematics, it has been found necessary to introduce a class of numbers, called complex numbers. A *complex number* has been defined by modern mathematicians as an *ordered couple of real numbers*, *i.e.*, a pair of real numbers united symbolically in a particular order for the purpose of technical convenience. Thus a complex number is, strictly speaking, not a single number at all, but a pair of real numbers with a proper order If the order is reversed, we get a different complex number A complex number may be expressed in the form $[a, b]$, where a and b are two real numbers. It is also represented, for convenience, in the form $a+ib$, where the symbol i has no meaning by itself, it merely indicates the order in which the real numbers a and b are considered. In defining all ordinary algebraical operations with regard to complex numbers it has been found convenient to associate the symbol i with the property $i^2 \equiv -1$, in which case all operations consistent with the algebra of real numbers may be applied to the case of complex numbers.

For geometrical representation and further introduction into the algebra of complex numbers see Chapter VI, Das and Mukherjees' Higher Trigonometry.

CHAPTER I

FUNCTIONS

1·1. In higher mathematics and various branches of science very often we have to deal with *changeable quantities* which are *inter-related* to one another, and in many such cases we have occasions to investigate how one of these quantities changes with a gradual change in the other. For example, in a given amount of gas enclosed in a cylinder with a movable piston, and kept at a constant temperature, the volume and pressure are interdependent, and a change in one produces a corresponding change in the other; or again, for a falling particle, the height from the ground depends on the time, and changes with it, the area of a circle changes with its radius etc.

In Differential Calculus we deal with the way in which one quantity varies with another when the change in the latter is ultimately very small, or more properly, with the *rate of change* of one quantity with another, as also other allied problems.*

In these investigations we shall be dealing with the relations between pure numbers which represent the magnitudes (with proper signs) of the quantities, and not with the concrete quantities themselves, so that the result will be general in nature, applicable to any pair of interdependent quantities under similar mathematical conditions.

In the following discussions we shall be concerned with the system of *real numbers* only, meaning by real numbers, zero, integers, rational and irrational numbers, positive or negative.

---

* While investigating problems of this type, Newton (in England) and Leibnitz (in Germany) were independently led ot the investigation of the principles of Calculus, towards the close of the seventeenth century. The principles of Calculus, in some form, were also known to the Hindus in India much earlier.

1·2. Preliminary Definitions and Notations.

Aggregate or **Set :** A system of real numbers defined in any way whatever is called an 'aggregate' or 'set' of numbers.

Illustration : The aggregate of positive integers ; the aggregate of all negative rational numbers ; the aggregate of all real numbers positive or negative ; the aggregate of all rational numbers from -3 to $+7$; the aggregate of numbers $\frac{1}{1}, \frac{1}{-2}, \frac{1}{3}, \frac{1}{-4}, \frac{1}{5}, \frac{1}{-6}$, etc.

• **Variable :** Let x be a symbol used during any mathematical investigation, to which may be assigned any numerical value out of a given set of real numbers. Then x is called a 'variable' or a 'real variable', and the totality of the values of x constitutes what is called the **domain** of x.

Illus. In the expression $x\,!$, x may be considered a real variable whose domain is the aggregate of positive integers.

Note. Variables are usually denoted by latter letters of the alphabet, such as x, y, z, u, v, w, ξ, η, ζ, etc.

Continuous Variable : If x assumes successively every numerical value of an aggregate of *all* real numbers from a given number 'a' to another given number 'b', then x is called a 'continuous real variable'.

The **domain** or **interval** (as it will be sometimes called) of x in this case is denoted by (a, b) or $\mathbf{a \leqslant x \leqslant b}$.

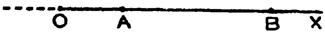
If *a be omitted* from the domain, it is indicated as $a < x \leqslant b$.

In the last case the domain is said to be *open* at the left end, whereas the domain $a \leqslant x \leqslant b$ is said to be *closed*. The interval $a < x < b$ is open at both ends, a and b being both excluded from the domain of possible values of x.

Illus. : In the expression $\sqrt{(5-x)(x+3)}$, x is a continuous real variable whose domain is $-3 \leqslant x \leqslant 5$; again, in $\sqrt{x+2} / \sqrt{7-x}$, the real variable x has the interval $-2 \leqslant x < 7$. In $\sin^{-1}x$, the interval of x is $-1 \leqslant x \leqslant 1$.

The domain of the variable x in any expression containing x, as in the above cases, consists of those values of x for which the expression has a definite real value.

The interval (a, b) is very often graphically represented on the x-axis by means of the length bounded by the two points $A\ (x=a)$ and $B\ (x=b)$. The *length of the interval* (a, b) is obviously $AB = OB - OA = b - a$.



Constant : A symbol which retains the same numerical value throughout a set of mathematical operations is called a constant.

Note. Constants (other than numerical constants like 2, -3, e, π etc.) are usually denoted by the earlier letters of the alphabet, such as a, b, α, β, etc.

Absolute value : By absolute value of a quantity x, as distinguished from its algebraical value, we mean its magnitude or numerical value, taken with a positive sign. It is represented by the notation $|\,x\,|$ which is $=x$, 0 or $-x$ according as $x >=$ or < 0.

From the very definition the following results are apparent, *viz.*,

(i) $|\,\mathbf{a} \pm \mathbf{b}\,| \leqslant |\,\mathbf{a}\,| + |\,\mathbf{b}\,|$ or more generally,

$$|\,a \pm b \pm c \pm \cdots\cdots\,| \leqslant |\,a\,| + |\,b\,| + |\,c\,| + \cdots\cdots$$

(ii) $|\,\mathbf{a} \pm \mathbf{b}\,| \geqslant |\,\mathbf{a}\,| \sim |\,\mathbf{b}\,|$, *i.e.*, $|\;|\,\mathbf{a}\,| - |\,\mathbf{b}\,|\;|$

Illus. : $|-2| = 2$, $|\,6\,| = 6$, $|-2+6| < 2+6$, $|-2-6| = 2+6$
$|-6-2| > 6 \sim 2$, $|\,2-6\,| = 6 \sim 2$, etc.

Note. *Meaning of the symbol* $|\,x-a\,| < \delta$

If $x > a$, $x-a < \delta$, *i.e.*, $x < a+\delta$. If $x < a$, $a-x < \delta$, *i.e.*, $a-\delta < x$. Hence, combining the two, we see that $|\,x-a\,| < \delta$ means $a-\delta < x < a+\delta$. Similarly, $|\,x-a\,| \leqslant \delta$ means $a-\delta \leqslant x \leqslant a+\delta$. Symbol $0 < |\,x-a\,| \leqslant \delta$ means $a-\delta \leqslant x \leqslant a+\delta$, but $x \neq a$.

Thus, $|\,x\,| < \delta$ means $-\delta < x < \delta$.

Functions : *By a function of x defined for a given domain, is understood, a quantity which has a single and definite value for every value of x in its domain*

[See *note 1*]

In other words, "*If x and y be two real variables, so related, that corresponding to every value of x within a defined domain we get a definite value of y, then y is said to be a function of x defined in its domain*"

• In this case, the variable x, to which we may arbitrarily assign different values in the given domain, is referred to as the **independent variable** (or, *argument*), and y is called the **dependent variable** (or, *function*). [See *note 2*]

We shall generally denote functions of x by such symbols as $f(x)$, $\psi(x)$, $F(x)$, $\phi(x)$ etc., where the mathematical forms of these functions may or may not be obtainable.

Note 1. When an expression or equation which defines a function gives two or more values of the function for each value of x, we call the function *multiple-valued*. The definition given above refers to a *single-valued* function with which we are mainly concerned in all mathematical investigations. A multiple-valued function, with proper limitations imposed on its value to be used in any particular investigation, can in general be treated as defining two or more different single-valued functions of x, *e.g.*, $y=\sin^{-1}x\ (-1 \leqslant x \leqslant 1)$ can be broken up into $y=\sin^{-1}x$, where (i) $-\frac{1}{2}\pi \leqslant y \leqslant \frac{1}{2}\pi$, (ii) $\frac{1}{2}\pi \leqslant y \leqslant \frac{3}{2}\pi$, (iii) $\frac{3}{2}\pi \leqslant y \leqslant \frac{5}{2}\pi$ etc. ; again, $y^2=x$ can be broken up into $y=+\sqrt{x}$ and $y=-\sqrt{x}$, and so on.

More generally (without restricting to single-valued functions only) a function of x may be defined as follows :—

If two quantities x and y are so related, that corresponding to values of x, there are values of y, then y is said to be a function of x.

Note 2. If y be a function of the variable x, it will generally be open to us also to regard x as a function of y by virtue of the functional relation between x and y, the proper domain of y being taken into account in this case, because it may so happen that the domain in which y is defined is not the domain in which x is defined. For

example $y = \sqrt{x}$ can be written as $x = y^2$, the domain of x in the former relation being $x \geqslant 0$, and that of y in the latter is the aggregate of all real numbers, positive or negative.

In the latter case y will be the independent variable, x the dependent one

Note 3 *A function may be undefined (i.e., may not have a definite value) for some particular value or values of x in a given interval.* In this connection we may make the following remarks :

Division by zero (symbols a/0, 0/0) meaningless.

The quotient of two finite numbers a and b (*viz*, a/b) is defined as the definite finite number x such that $a = bx$. Now, obviously, in the division, zero value of b is excluded, for if $b=0$, then $a\ (=bx)=0$, and x can be any number Hence, the above definition rules out division by zero. Therefore, forms $a/0$, $0/0$ are meaningless.

The following simple illustration shows how *division by zero leads to fallacious results.*

Suppose, $x = y$, ($x \neq 0$, $y \neq 0$), $\therefore\ x^2 = xy$.

$\therefore\ x^2 - y^2 = xy - y^2$, or, $(x+y)(x-y) = y(x-y)$.

Hence, dividing out by $x-y$, $x+y=y$, *i.e.*, $2y = y$, or, $2=1$.

The fallacy is due to the fact that we divided by $x-y$ which is equal to zero.

Similarly, the assumption $0/0=1$, on the basis that anything divided by itself is 1, leads to fallacious results, as shown below.

$$3 \times \frac{0}{0} = 3 \times 1 = 3\,, \qquad \text{again } 3 \times \frac{0}{0} = \frac{3 \times 0}{0} = \frac{0}{0} = 1. \quad \therefore\ 3 = 1.$$

From the above remarks, it will be apparent that

the function $f(x) = \dfrac{x^2-25}{x-5}$ *is not defined* for $x=5$;

the function $f(x) = \sin\,(1/x)$ is *not defined* for $x=0$; etc.

Note 4. If $f(x)$ denotes a certain function of x, then in case $f(x)$ is given by a mathematical expression involving x, $f(a)$, *i.e.*, the value of the function for $x=a$ may in general (but not always, as explained in note 3 above, and also in (iv) Art. 1·4) be obtained by putting a for x in the expression for $f(x)$.

Thus, if $f(x)=\sin x$, $f(0)=\sin 0=0$;
if $f(x)=x^2-5x+1$, $f(1)=-3$, $f(-1)=7$;
if $f(x)=x^2$, $f(x+h)=(x+h)^2=x^2+2xh+h^2$; etc.
whereas, if $f(x)=x\cos(1/x)$, $f(0)$ is undefined.

1·3. Graphical representation of functions.

Taking the straight line $X'OX$, with origin O on it as usual, to represent the real variable x, the value of the function, y or $f(x)$, may be represented parallel to the line YOY' drawn at right angles to $X'OX$, as in ordinary graphs. Corresponding to every value of x (in the assigned domain) the point is plotted whose ordinate gives the corresponding value of the function. The assemblage of the points, which may or may not form a continuous line, represents the graph of the function.

In drawing the graph it is not necessary to know the exact mathematical relation between x and y (which may or may not be obtainable) It will be sufficient if we know the definite value of y corresponding to every value (at least a large number of values) of x in the defined domain.

The graph at once presents to the eye the way in which the function is related to, and changes with the argument.

1·4. Some remarks on functions.

From the very definition the following points should be clear :

(i) *It is not essential for a function to be expressible by a mathematical form always.* For example, suppose x hours after noon on a certain day, the temperature of a patient is T degrees. Now to each value of x (up to a certain number, depending on our contemplated period of observation), there corresponds a definite value of T. Hence, T is a function of x by definition. But T cannot be expressed analytically by a mathematical expression in terms of x. Nevertheless, we can draw a graph which is the temperature chart of the patient, giving an idea how T changes with the time x. For other examples, see (vii) of the next article.

(ii) In some cases *a function may have different mathematical forms for different ranges of its domain of existence;* for illustration, see (v) of the next article.

(iii) *A function may be undefined for some value or values of the argument,* as has already been remarked and illustrated in note 3, Art. 1·2. Also, every function cannot be defined in every interval, thus, $\sin^{-1}x$ cannot be defined in the interval (2, 3) for $\sin^{-1} 2$ has no meaning, there being no angle whose sine is 2.

(iv) *A function may be defined arbitrarily.* For instance, we may define a function as

$$f(x) = x^2 \text{ when } x < 0,$$

$$f(0) = 3,$$

$$f(x) = \tfrac{1}{2} - x \text{ when } x > 0.$$

The function is thus definitely defined for all real values of x.

(v) The function $\dfrac{x^2 - 25}{x - 5}$ and $x + 5$ are different functions. The former is undefined at $x = 5$, and so its domain of existence is the aggregate of all real numbers excepting 5 for the argument x. The latter exists for all real values of x. Hence, though for other values of x the two functions are equal, there is a point of distinction at $x = 5$.

A third function might be defined as $f(x) = \dfrac{x^2 - 25}{x - 5}$ when $x \neq 5$, and $f(5) = 20$ Then the function is again different from either of the first two. It exists for all real values of x including $x = 5$, but at $x = 5$ its value is different from that of the second function $x + 5$.

If we define a fourth function by saying that $f(x) = \dfrac{x^2 - 25}{x - 5}$ when $x \neq 5$, and $f(5) = 10$, then this function is identical with the function $x + 5$.

1·5. Examples of functions.

Below is given a number of examples of functions of a variety of types, with their graphs in certain cases which will help to form a clear notion about functions and will further elucidate the remarks of the previous article.

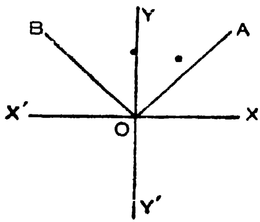
(i) Analytical functions like x^2, $\dfrac{2x^3+7}{x^2+9}$ etc., or more generally, *polynomials in* x of the type $f(x) \equiv a_0x^n + a_1x^{n-1} + \cdots\cdots + a_{n-1}x + a_n$ (where n is a positive integer), or *rational algebraic functions* of the type $P(x) / Q(x)$, where $P(x)$ and $Q(x)$ are polynomials.

The domain of these are generally the set of all real numbers ; in the last case the zeroes of the denominator are excluded, for the function is not defined at these points.

(ii) $f(x) = x$ when $x > 0$

$= 0$ when $x = 0$

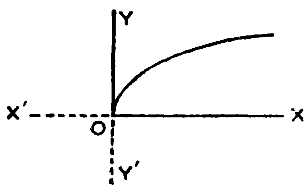
$= -x$ when $x < 0$.



The graph, as shown, consists of two lines OA and OB which bisect the $\angle^s$ XOY, $X'OY$ respectively.

This is also the graph of the function $f(x) = |x|$.

(iii) $f(x) = \sqrt{x}$



$f(x)$ is defined for $x = 0$, and all positive values of x ; graph is a continuous curve (parabola) in the first quadrant.

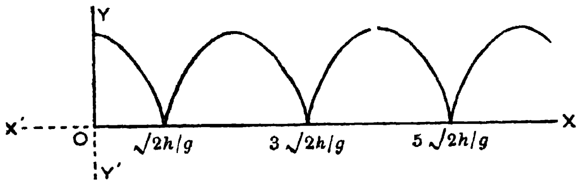
(iv) $f(x) = x\,!$

Or, $f(x) =$ sum of first x terms of $\dfrac{1}{1^2} + \dfrac{1}{2^2} + \dfrac{1}{3^2} + \cdots\cdots$

The functions are defined only for positive integral values of x.

The graph in each case consists of a series of isolated points.

(v) The height y from the ground, at a time x, of a perfectly elastic ball originally dropped from a height h.



Here y is defined for all positive values of x, but expressed by different mathematical terms for different ranges of the values of x.

Thus, denoting the time of fall (from start to first impact), *i.e.*, $\sqrt{(2h/g)}$ by x_1.

$y = h - \frac{1}{2}gx^2$, when $0 \leqslant x \leqslant x_1$ (*i.e.*, before first impact)

$y = (x - x_1)\sqrt{2gh} - \frac{1}{2}g\,(x - x_1)^2$, when $x_1 < x \leqslant 3x_1$ (*i.e.*, between first and second impact)

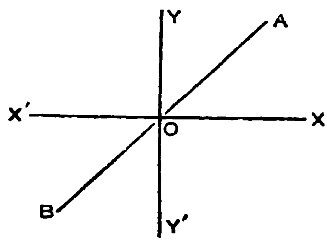
$y = (x - 3x_1)\sqrt{2gh} - \frac{1}{2}g\,(x - 3x_1)^2$, when $3x_1 < x \leqslant 5x_1$ (*i.e.*, between second and third impact) etc.

The graph, as shown, consists of a series of parabolic arcs, on the positive side of the x-axis.

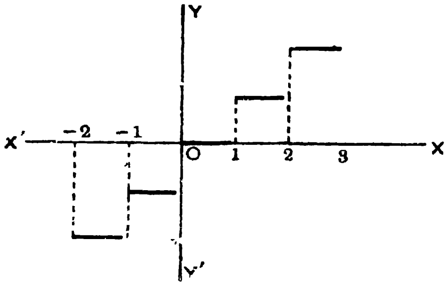
(vi) $y = \dfrac{x^2}{x}$.

For $x \neq 0$, $y = x$,
for $x = 0$, y is not known (undefined).

The graph is that of the straight line $y = x$, with the origin left out.



(vii) $y = [\,x\,]$, where $[\,x\,]$ denotes the greatest integer not exceeding x.



For $\quad 0 \leqslant x < 1,\ y = 0\ ; \quad 1 \leqslant x < 2,\ y = 1\ ;$

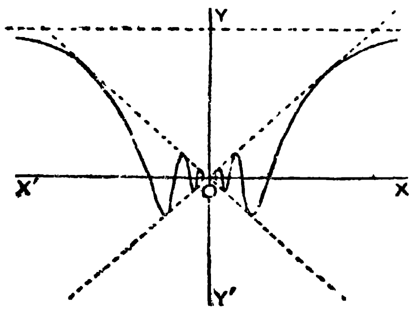
$2 \leqslant x < 3,\ y = 2\ ;\ -1 \leqslant x < 0,\ y = -1\ ;$

$-2 \leqslant x < -1,\ y = -2$ etc.

Thus, the graph consists of parallel segments of line in which the right-hand end-points are left out.

(viii) $y = x \sin (1/x)$.

Here y is not defined for $x = 0$. Thus, the domain of x is the aggregate of all real numbers except 0.



The graph is shown here, which is continuous everywhere excepting at $x=0$, where a point is missing on the graph. Near 0, on either side, the graph has an infinite number of oscillations with gradually diminishing amplitude. The graph is comprised between the lines $y=x$ and $y=-x$.

(ix) Functions like e^x, $\log x$, $\sin x$, $\cos^{-1}x$, etc., which are not algebraic functions are called *Transcendental functions.* For graphs of first two, see § 16·9.

1·6. Bounded functions and their Bounds.

Let $f(x)$ be a function defined in the interval (a, b). If a finite number K can be found such that $f(x) \leqslant K$ for every value of x in the interval, then $f(x)$ is said to be *bounded above* in the interval.

Similarly, if a finite number k exists such that $f(x) \geqslant k$ for every x in the interval, then $f(x)$ is said to be *bounded below.*

If $f(x)$ is both bounded above and below in the interval, then it is said to be simply *bounded.*

If $f(x)$ is bounded above, then it easily follows from Dedekind's Theorem that there exists a definite finite number M such that $M \geqslant f(x)$ for every value of x in the interval, but ε being any pre-assigned positive quantity, however small, there is at least one value of x in the interval for which $f(x) > M-\varepsilon$. This number M is called the *upper bound* of the function in the interval.

In a similar way, if $f(x)$ be bounded below, then there exists a definite finite number m such that $m \leqslant f(x)$ for every x in the interval, but given any pre-assigned positive number ε, however small, there is at least one value of x for which $f(x) < m+\varepsilon$. This number m is called the *lower bound* of $f(x)$ in the interval.

We know, $\sin x$, $\cos x$ are bounded functions in the interval $(-\pi, \pi)$, the upper bounds of both being 1 and their lower bounds being -1.

For $\sin x$, $M=1$; now, taking $\varepsilon=\frac{1}{2}$, we can find at least one value, say $\frac{\pi}{3}$, of x, such that $\sin\frac{\pi}{3} > 1-\frac{1}{2}=\frac{1}{2}$. Other values of x can be obtained from the tables, for which $\sin x > \frac{1}{2}$. The function defined in Ex. (v), § 1·5 is a bounded function in the interval $(0, \infty)$, the upper bound being h and the lower bound being zero.

For the interval $0 \leqslant x < \frac{\pi}{2}$, $\tan x$ has the lower bound zero, but no upper bound. For the interval $-\frac{\pi}{2} < x \leqslant 0$, $\tan x$ has no lower bound, but its upper bound is zero. In the interval $\left(-\frac{\pi}{2} < x < \frac{\pi}{2}\right)$, $\tan x$ has neither lower bound nor upper bound. Thus we see that a function may have different upper and lower bounds in different intervals.

The function $x^2 + 3x + 5$ in the interval $1 \leqslant x \leqslant 2$ lies between 9 and 15; so its upper bound is 15 and lower bound is 9.

The function $f(x) \equiv \sqrt{\dfrac{2+x}{(5-x)(x-3)}}$ is not bounded above in the interval $3 < x < 5$, in which it is defined.

Let $x = 3 + \varepsilon$, where ε is a small positive number.

$$\therefore \quad f(x) = \sqrt{\frac{5+\varepsilon}{(2-\varepsilon)\varepsilon}} > \sqrt{\frac{5+\varepsilon}{2\varepsilon}} > \sqrt{\frac{5}{2\varepsilon}},$$ which can be made greater than any positive quantity by taking ε smaller and smaller. Hence $f(x)$ has no upper bound.

1·7. Monotone Function.

Let x_1, x_2 be *any* two points such that $x_1 < x_2$ in the interval of definition of a function $f(x)$. Then $f(x)$ is said to be *monotonically increasing* if $f(x_1) \leqslant f(x_2)$ and *monotonically decreasing* if $f(x_1) \geqslant f(x_2)$. Thus in $(0, \frac{1}{2}\pi)$, $\sin x$ is a monotonically increasing and $\cos x$ is a monotonically decreasing function.

Sometimes the following definition is used. If for $x_1 < x_2$, $f(x_1) < f(x_2)$, then $f(x)$ is said to be *strictly mono-*

tonically increasing, and if $f(x_1) > f(x_2)$, then $f(x)$ is said to be *strictly monotonically decreasing.*

In the interval $0 \leqslant x < \infty$, the function e^x is a strictly increasing function, since $e^{x_1} < e^{x_2}$ when $x_1 < x_2$.

In the interval $0 \leqslant x < \infty$, the function $f(x) \equiv \dfrac{3x+5}{2x+1}$ is a strictly decreasing function, for $f(x_1) > f(x_2)$ when $x_1 < x_2$.

The example (vii) of § 1·5 is an example of a function defined in the interval (0, 3), which is monotonic increasing but not strictly increasing.

Examples I

1. If $y = 6$ for every value of x, can y be regarded as a function of x ?

2. If $y =$ the number of windows in the house number x on a particular road, is y a function of x ?

3. Given $f(x) = x^2 - 10x + 3$, find $f(0)$ and $f(-2)$.

4. If $f(x) = \sec x + \cos x$, then $f(x) = f(-x)$.

5. If $f(x) = b\dfrac{x-a}{b-a} + a\dfrac{x-b}{a-b}$, then $f(a) + f(b) = f(a+b)$.

6. If $f(x) = x^2 - 3x + 7$, show that

$$\{f(x+h) - f(x)\}/h = 2x - 3 + h.$$

7. Show that

(i) $\dfrac{1-\tan x}{\cos x - \sin x}$ is not defined for $x = \frac{1}{4}\pi$.

(ii) $\sqrt{x^2 - 5x + 6}$ is not defined for $2 < x < 3$.

(iii) $\dfrac{x^2 - 5x + 6}{x^2 - 8x + 12}$ is not defined for $x = 2$; also find $f(-5)$ and $f(6)$ in this case.

8. Draw the graphs of the following functions :

(i) $y=1$ when $x>0$,
$=0$ when $x=0$,
$=-1$ when $x<0$.

(ii) $y=x^2$ for $x \neq 1$,
$=2$ for $x=1$.

(iii) $f(x)=1$ when x is an integer,
$=0$ when x is not an integer.

(iv) $y=\cos(1/x)$. (v) $y=x\cos(1/x)$.

(vi) $y=x-[x]$, where $[x]$ denotes the greatest integer not greater than x.

(vii) $f(x)=\sqrt{-(x-1)^2}$. (viii) $f(x)=\dfrac{|x|}{x}$.

(ix) $f(x)=1-\dfrac{\sin \pi x}{\sin \pi x}$.

(x) $f(x)=\sqrt{x^2}$, where the positive sign of the square root is to be taken.

(xi) $f(x)=0$ when $|x|>1$,
$=1+x$ when $-1 \leqslant x \leqslant 0$,
$=1-x$ when $0<x \leqslant 1$.

9. (i) Show that $f(x)=\sec x$ in the interval $0 \leqslant x < \frac{1}{2}\pi$, has the lower bound 1, and no upper bound.

(ii) Show that $f(x)=2x^2+4x+6$ in the interval $0 \leqslant x \leqslant 1$, has the lower bound 6 and the upper bound 12.

(iii) Show that $f(x)=\left(\dfrac{1-\theta}{1+\theta x}\right)^n$ when $0<\theta<1$ and $-1<x<1$, and n a positive integer, is bounded.

10. (i) Show that $f(x)=\dfrac{x}{x+1}$ is monotone ascending, $x>0$.

(ii) Show that $f(x)=\dfrac{1}{x+1}+\dfrac{1}{x+2}+\cdots+\dfrac{1}{x+n}$, $x>0$ is monotone descending.

(iii) Show that $f(x)=\left(1+\dfrac{1}{x}\right)^x$, $x>0$ is monotone ascending.

11. Given the relation $y^2-6y-x+7=0$, which of the following statements is true ?

(i) The equation defines x as a function of y for all values of y.

(ii) The equation defines y as a function of x for all values of x.

12. A taxi company charges one rupee for one mile or less from start, and at a rate of (i) 50 paise per mile, (ii) 50 paise per mile or any fraction thereof, for additional distance. Express analytically the fare F (in rupees) as a function of the distance d (in miles), and draw the graph of the function.

ANSWERS

1. Yes. **2** Yes. **3.** 3 ; 27. **7.** (iii) $\frac{8}{11}$; does not exist.

11. (i) True. (ii) Not true ; true only for values of $x \geqslant -2$.

12. (i) $\begin{cases} F=1, \text{ for } 0<d\leqslant 1 \\ F=1+\frac{1}{2}(d-1), \text{ for } d>1 \end{cases}$

(ii) $\begin{cases} F=1, \text{ for } 0<d\leqslant 1 \\ F=1+\frac{1}{2}m, \text{ for } m<d\leqslant m+1 \text{ where } m \text{ is a positive integer.} \end{cases}$

CHAPTER II

LIMIT

2·1. The whole structure of Differential Calculus is based upon a concept of most outstanding importance, *viz*, *limit.* We give below some fundamental definitions and theorems regarding limits.

2·2. By the expressions 'the variable x approaches the constant number a' or simply '*x tends to the value a*', we mean that x assumes successively values whose numerical difference from a, *i e*, $|x-a|$, is gradually less and less, and can ultimately be taken to be less than any small quantity we can name or imagine (*i e*, less than any pre-assigned positive quantity, however small), and we denote this by the symbol $\mathbf{x \to a}$.

Here the successive values of x may be greater than, as well as less than a.

If the variable x, remaining always greater than a, approaches a, such that ultimately $x-a$ is less than any pre-assigned positive quantity, however small, (but $x \neq a$ actually) then we say that *x approaches or tends to 'a' from the right,* and denote it by the symbol $\mathbf{x \to a+0}$ or simply by $x \to a+$.

Similarly, when x is less than a always, and $a-x$ is ultimately less than any pre-assigned positive quantity, however small, we say *x tends to 'a' from the left,* and denote it by $\mathbf{x \to a-0}$, or simply by $x \to a-$.

Illus: When the successive values of x are 1·9, 1·99, 1·999,... we say $x \to 2-0$, and when the successive values of x are 2·1, 2·01, 2·001,..., we say $x \to 2+0$. If the successive values of x are $2+1$, $2-\frac{1}{2}$, $2+\frac{1}{3}$, $2-\frac{1}{4}$, $2+\frac{1}{5}$, .. we say $x \to 2$.

2·3. Limit of a function.

$\mathbf{\underset{x \to a}{Lt}\ f(x)}$: When x approaches a constant quantity a from either side (but $\neq a$), if there exists a definite finite number l towards which $f(x)$ approaches*, such that the numerical difference of $f(x)$ and l can be made as small as we please (*i.e.*, less than any pre-assigned positive quantity, however small) by taking x sufficiently close to a, then l is defined as the limit of $f(x)$ as x tends to a. This is symbolically written as $\underset{x \to a}{Lt} f(x) = l$.

Mathematically speaking $\underset{x \to a}{\mathbf{Lt}}\ \mathbf{f(x)} = \boldsymbol{l}$, *provided, given any pre-assigned positive quantity ε, however small, we can determine another positive quantity δ (depending on ε) such that* $|\mathbf{f(x)} - \boldsymbol{l}| < \varepsilon$ *for all values of x satisfying* $0 < |x - a| \leqslant \delta$, *i.e., whenever* $\mathbf{a} - \boldsymbol{\delta} \leqslant \mathbf{x} \leqslant \mathbf{a} + \boldsymbol{\delta}$, but $x \neq a$.

Ex. (i) $\underset{x \to 3}{Lt} \dfrac{x^2 - 9}{x - 3} = 6$. For if $x = 3 + \delta_1$, whether δ_1 be positive or negative, $\dfrac{x^2 - 9}{x - 3} = \dfrac{(x-3)(x+3)}{x-3} = \dfrac{\delta_1(6 + \delta_1)}{\delta_1} = 6 + \delta_1$, and by taking δ_1 numerically small enough, the difference of $\dfrac{x^2 - 9}{x - 3}$ and 6 can be made as small as we like. It may be noted here that however small δ_1 may be, since $\delta_1 \neq 0$, we can cancel the factor $x - 3$ *i.e.*, δ_1 between the numerator and denominator in this case. Hence, $\underset{x \to 3}{Lt} \dfrac{x^2 - 9}{x - 3} = 6$. But when $x = 3$, the function $\dfrac{x^2 - 9}{x - 3}$ is non-existent or undefined, for we cannot cancel the factor $x - 3$, which is equal to zero, in that case. Thus, writing $f(x) = \dfrac{x^2 - 9}{x - 3}$, $\underset{x \to 3}{Lt} f(x) = 6$, whereas $f(3)$ does not exist or is undefined.

Ex. (ii) $\underset{x \to 0}{Lt}\, x \sin \dfrac{1}{x} = 0$. For $\sin \dfrac{1}{x}$, whatever small value x may have provided it is not exactly equal to zero, is a finite quantity lying between $+1$ and -1, and so by taking x numerically small enough (*i.e.*, sufficiently near to zero), we can make $x \sin \dfrac{1}{x}$ numerically as

*As a particular case $f(x)$ may remain always equal to l when x is sufficiently close to a.

small as we like, *i.e.*, $\left| x \sin \frac{1}{x} - 0 \right|$ is less than any assignable quantity. Hence the limit is zero.

Here also the value of $x \sin \frac{1}{x}$ when x is exactly equal to zero is non-existent.

Ex. (iii) $\underset{x \to -1}{Lt} \frac{x^2-7}{x+3} = -3$. For writing $x = -1 \pm \delta$, we can show that the numerical difference of $\frac{x^2-7}{x+3}$ and -3 can be made as small as we like by taking δ small enough.

In this case the value of $\frac{x^2-7}{x+3}$ when x is exactly $= -1$ is also available, and that is also equal to -3.

$\underset{x \to a+0}{\textbf{Lt}} \mathbf{f(x)}$ **:** The limit of a function $f(x)$ as x approaches the value a from the right (*i.e.*, from bigger values), is that quantity l_1, (if one such exists), towards which $f(x)$ approaches, and from which the numerical difference of $f(x)$ can be made as small as we please by making x approach a sufficiently closely, all the time keeping it greater than a. It is called the **Right-hand limit** of $f(x)$ as x tends to a, and is written as $\underset{x \to a+0}{Lt} f(x) = l_1$.

Mathematically, $\underset{x \to a+0}{Lt} f(x) = l_1$, *provided, given any preassigned positive quantity* ε, *however small, we can determine a positive quantity* δ, *such that* $| f(x) - l_1 | < \varepsilon$ *whenever* $0 < x - a \leqslant \delta$, *i.e.*, $a < x \leqslant a + \delta$.

$\underset{x \to a+0}{Lt} f(x)$ is sometimes denoted by the symbol $f(a+0)$.

Ex. (iv) $\underset{x \to 2+0}{Lt} \left\{ \frac{1}{5 + e^{\frac{1}{x-2}}} \right\} = 0$, as can be shown by writing $x = 2 + \delta$, where δ is positive, and then making δ arbitrarily small when the denominator becomes arbitrarily large.

Similarly, $\underset{x \to 0-0}{Lt} f(x) = -1$; because when x remaining less than zero, becomes arbitrarily near to 0, $f(x)$ always remains equal to -1.

Since, $\underset{x \to 0+0}{Lt} f(x) \neq \underset{x \to 0-0}{Lt} f(x)$,

$\therefore$ $\underset{x \to 0}{Lt} f(x)$ does not exist; but by definition, $f(0) = 0$ here.

(iii) *$f(a)$ and $\underset{x \to a}{Lt} f(x)$ both exist but are unequal.*

Let $f(x) = 0$, for $x \neq 0$
$= 1$ for $x = 0$.

As in (ii), it can be easily shown here that

$$\underset{x \to 0+0}{Lt} f(x) = 0 = \underset{x \to 0-0}{Lt} f(x).$$

$\therefore$ $\underset{x \to 0}{Lt} f(x) = 0$. But by definition, $f(0) = 1$.

(iv) *$f(a)$ and $\underset{x \to a}{Lt} f(x)$ both exist and are equal.*

This is illustrated by Ex. (iii) of Art. 2·3.

(v) *Neither $f(a)$ nor $\underset{x \to a}{Lt} f(x)$ exists.*

Let $f(x) = \cos(1/x)$.

Here, $\underset{x \to 0}{Lt} \cos(1/x)$ does not exist [See *Ex. 1, Art. 2·10*] and $f(0)$ does not exist, as it would involve division by zero, and is otherwise undefined.

2·5. Symbols $+\infty$ and $-\infty$.

If a variable x, assuming positive values only, increases without limit (*i.e.*, ultimately becomes and remains greater than any pre-assigned positive number, however large), we say that x *tends to infinity*, and write it as $\mathbf{x \to \infty}$.

Similarly, if a variable x, assuming negative values only, increases numerically without limit (*i.e.*, $-x$ ultimately becomes and remains greater than any pre-assigned positive number, however large), we say that *x tends to minus infinity* and write it as $\mathbf{x \to -\infty}$.

Note. It should be borne in mind that there is no number such as ∞ or $-\infty$ towards which x approaches. The symbols are used only to indicate that the numerical value of x increases without limit.

2·6. Function tending to infinity : $\underset{x \to a}{\text{Lt}} f(x) = \pm \infty$.

As x approaches a either from the right or left, if $f(x)$ tends to infinity with the same sign in both cases, then we say that as x tends to a, $f(x)$ tends to infinity,* (or loosely, the limit of $f(x)$ is infinite), positive or negative as the case may be, and write it as $\underset{x \to a}{Lt} f(x) = \infty$ or $-\infty$.

If however, as x approaches a from both sides, $f(x)$ tends to infinity with different signs, we say, '$f(x)$ does not possess any limit as x tends to a'.

The formal definitions are as follows :

If corresponding to any pre-assigned positive quantity N, however large, we can determine a positive quantity δ, such that $f(x) > N$ whenever $0 < |x-a| \leqslant \delta$, we say

$$\underset{x \to a}{Lt} f(x) = \infty.$$

If in the above circumstances, $-f(x) > N$ whenever $0 < |x-a| \leqslant \delta$, we say $\underset{x \to a}{Lt} f(x) = -\infty$.

Similarly, we may define the cases,

$$\underset{x \to a+0}{Lt} f(x) = \infty, \qquad \underset{x \to a-0}{Lt} f(x) = \infty,$$

$$\underset{x \to a+0}{Lt} f(x) = -\infty, \qquad \underset{x \to a-0}{Lt} f(x) = -\infty.$$

*According to some modern writers, this is described as '$f(x)$ becoming infinitely large', and infinite limit is not recognised as a limit.

Illus : $\underset{x\to 0+0}{Lt}\ \dfrac{1}{x^2}=\infty,\ \underset{x\to 0-0}{Lt}\ \dfrac{1}{x^2}=\infty, \quad \therefore\ \underset{x\to 0}{Lt}\ \dfrac{1}{x^2}=\infty.$

$\underset{x\to 0+0}{Lt}\ \dfrac{1}{x}=\infty,\ \underset{x\to 0-0}{Lt}\ \dfrac{1}{x}=-\infty, \quad \therefore\ \underset{x\to 0}{Lt}\ \dfrac{1}{x}$ does not exist.

In either case however $f(0)$ does not exist.

2·7. Limit of a function as the variable tends to infinity : Lt f(x).

As x, remaining positive, becomes larger and larger, if there exists a definite finite number l towards which $f(x)$ continually approaches, such that the numerical difference of $f(x)$ and l can be made as small as we please by taking x large enough, we say $\underset{x\to\infty}{Lt}\ f(x)=l$.

Mathematically, $\underset{x\to\infty}{\mathbf{Lt}}\ \mathbf{f(x)=}\boldsymbol{l}$, *provided, given any pre-assigned positive quantity* ε, *however small, we can determine a positive quantity* M, *such that* $|f(x)-l|<\varepsilon$ *for all values of* $x \gtrsim M$.

Similarly, $\underset{x\to-\infty}{\mathbf{Lt}}\ \mathbf{f(x)=}\boldsymbol{l'}$ provided, given any pre-assigned positive quantity ε, however small, we can determine a positive quantity M, such that $|f(x)-l'|<\varepsilon$ whenever $-x>M$.

In a similar way, we may define the cases,

$$\underset{x\to\infty}{Lt}\ f(x)=\infty,\quad \underset{x\to\infty}{Lt}\ f(x)=-\infty,\quad \underset{x\to-\infty}{Lt}\ f(x)=\infty, \text{ etc.}$$

Illus : $\underset{x\to\infty}{Lt}\ \dfrac{1}{x}=0, \qquad \underset{x\to-\infty}{Lt}\ \dfrac{1}{x^2}=0,$

$\underset{x\to\infty}{Lt}\ e^{\frac{1}{x}}=1, \qquad \underset{x\to-\infty}{Lt}\ x^2=\infty$, etc.

2·8. Fundamental Theorems on Limit.

We give below some fundamental theorems on limit which are of frequent use.

If $\underset{x \to a}{\mathrm{Lt}}\ f(x) = l$, and $\underset{x \to a}{\mathrm{Lt}}\ \Phi(x) = l'$, where l and l' are finite quantities, then

(i) $\underset{x \to a}{\mathrm{Lt}}\ \{f(x) \pm \Phi(x)\} = l \pm l'$.

(ii) $\underset{x \to a}{\mathrm{Lt}}\ \{f(x) \times \Phi(x)\} = ll'$.

(iii) $\underset{x \to a}{\mathrm{Lt}}\ \left\{\dfrac{f(x)}{\Phi(x)}\right\} = \dfrac{l}{l'}$, provided $l' \neq 0$.

(iv) $\underset{x \to a}{\mathrm{Lt}}\ F\{f(x)\} = F\{\underset{x \to a}{\mathrm{Lt}}\ f(x)\}$, *i.e.*, $= F(l)$, where $F(u)$ is a function of u which is continuous* for $u = l$.

(v) If $\phi(x) < f(x) < \psi(x)$ in a certain neighbourhood of the point 'a' and $\underset{x \to a}{Lt}\ \phi(x) = l$ and $\underset{x \to a}{Lt}\ \psi(x) = l$, then $\underset{x \to a}{Lt}\ f(x)$ exists and is equal to l.

In particular, if $|f(x)| < |g(x)|$, *i.e.*, $f(x)$ lies between $-g(x)$ and $g(x)$, and if $\underset{x \to a}{Lt}\ g(x) = 0$, then $\underset{x \to a}{Lt}\ f(x) = 0$.

For proofs of these, see Appendix

Note. The first two theorems may be extended to any finite number of functions. In language, the first three theorems may be stated as follows :

(*i*) *The limit of a sum or difference of any finite number of functions is equal to the sum or difference of the limits of the functions taken separately.*

(*ii*) *The limit of the product of a finite number of functions is equal to the product of their limits taken separately.*

(*iii*) *The limit of the quotient of two functions is equal to the quotient of their limits, provided the limit of the denominator is not zero.*

*See next chapter.

As examples of (iv), we get

$$\underset{x \to a}{Lt} \log f(x) = \log \{\underset{x \to a}{Lt} f(x)\} = \log l, \text{ provided } l > 0.$$

$$\underset{x \to a}{Lt} e^{f(x)} = e^{\underset{x \to a}{Lt} f(x)} = e^{l},$$

$$\underset{x \to a}{Lt} \{f(x)\}^n = \{\underset{x \to a}{Lt} f(x)\}^n = l^n, \text{ etc.}$$

2·9. Some Important Limits.

(i) $\underset{x \to 0}{\text{Lt}} \dfrac{\sin x}{x}$ 1, where x is expressed in radian measure.

From elementary trigonometry*, we know that if x be the radian measure of any positive acute angle i e., $0 < x < \frac{1}{2}\pi$,

then $\sin x < x < \tan x$, or, $\cos x < \dfrac{\sin x}{x} < 1$. (1)

$$0 < 1 - \frac{\sin x}{x} < 1 - \cos x, \text{ i.e., } < 2 \sin^2 \tfrac{1}{2}x.$$

But $\qquad 2 \sin^2 \frac{1}{2}x < 2(\frac{1}{2}x)^2 < \frac{1}{2}x^2$.

Hence, $\qquad 0 < 1 - \dfrac{\sin x}{x} < \frac{1}{2}x^2$.

Now, since $x^2 \to 0$ as $x \to 0+0$, we get

$$\underset{x \to 0+0}{Lt} \left(1 - \frac{\sin x}{x}\right) = 0, \text{ i.e., } \underset{x \to 0+0}{Lt} \frac{\sin x}{x} = 1.$$

Alternatively, noting that $\cos x \to 1$ as $x \to 0$, we can conclude directly from (1) that $Lt\,(\sin x/x) = 1$.

When $-\frac{1}{2}\pi < x < 0$, putting $x = -z$, we get $0 < z < \frac{1}{2}\pi$.

Also, $\qquad \dfrac{\sin x}{x} = \dfrac{\sin(-z)}{-z} = \dfrac{\sin z}{z}$.

*See Das and Mukherjees' Intermediate Trigonometry.

Hence, $\underset{x\to 0-0}{Lt} \frac{\sin x}{x} = \underset{z\to 0+0}{Lt} \frac{\sin z}{z} = 1.$

Hence the result.

(ii) (*a*) $\underset{n\to+\infty}{\mathbf{Lt}} \left(1+\frac{1}{n}\right)^n = e$, ($n\to\infty$ through positive integral values).

(*b*) $\underset{x\to\pm\infty}{\mathbf{Lt}} \left(1+\frac{1}{x}\right)^x = e.$

For a complete proof, see Appendix.

Cor. $\underset{x\to 0}{\mathbf{Lt}} (1+x)^{\frac{1}{x}} = e.$

For a complete proof, see Appendix

(iii) $\underset{x\to 0}{\mathbf{Lt}} \frac{1}{x} \log(1+x) = 1.$

Proof. We have

$$\underset{x\to 0}{Lt} \frac{1}{x} \log(1+x) = \underset{x\to 0}{Lt} \log(1+x)^{\frac{1}{x}}$$

$$\log\left\{\underset{x\to 0}{Lt} (1+x)^{\frac{1}{x}}\right\} \text{ [by § 2·8 (iv)]} = \log e = 1.$$

[*by (ii) above*]

(iv) $\underset{x\to 0}{\mathbf{Lt}} \frac{e^x-1}{x} = 1.$

Proof. Put $e^x = 1+z$.

Then, $x = \log(1+z)$, and as $x\to 0$, $z\to 0$.

Thus, $\underset{x\to 0}{Lt} \frac{e^x-1}{x} = \underset{z\to 0}{Lt} \frac{z}{\log(1+z)}$

$$= \underset{z\to 0}{Lt} \left\{1\Big/\frac{1}{z}\log(1+z)\right\}$$

$$= 1\Big/\underset{z\to 0}{Lt}\left\{\frac{1}{z}\log(1+z)\right\}$$

$$= \tfrac{1}{1} = 1. \quad [\text{ by (iii) }]$$

(v) $\mathbf{\underset{x \to a}{Lt} \dfrac{x^n - a^n}{x - a} = na^{n-1}}$

for all rational values of n, provided a is positive.

CASE I. *When n is a positive integer.*

By actual division, we have

$$\frac{x^n - a^n}{x - a} = x^{n-1} + x^{n-2}a + x^{n-3}a^2 + \cdots\cdots + a^{n-1}.$$

$$\therefore \quad \text{reqd. limit} = \underset{x \to a}{Lt}\ (x^{n-1} + x^{n-2}a + \cdots + a^{n-1}) = na^{n-1},$$

since, the limit of each of the n terms as $x \to a$, is a^{n-1}, and the limit of the sum of a finite number of terms is equal to the sum of their limits (*Art. 2·8*)

CASE II. *When n is a negative integer.*

Suppose, $n = -m$, where m is a positive integer and $a \neq 0$.

$$\frac{x^n - a^n}{x - a} = \frac{x^{-m} - a^{-m}}{x - a} = -\frac{1}{x^m a^m} \cdot \frac{x^m - a^m}{x - a}.$$

Now, as $x \to a$, limiting value of $\dfrac{1}{x^m a^m} = \dfrac{1}{a^m a^m} = \dfrac{1}{a^{2m}}$

and as $x \to a$, limiting value of $\dfrac{x^m - a^m}{x - a} = ma^{m-1}$ by Case I.

$$\therefore \quad \underset{x \to a}{Lt} \frac{x^n - a^n}{x - a} = -\frac{1}{a^{2m}}\, ma^{m-1} = -ma^{-m-1} = na^{n-1}.$$

CASE III. *When n is a rational fraction.*

Suppose, $n = p/q$, when q is a positive integer and p any integer, positive or negative. Let us put $x^{1/q} = y$ and $a^{1/q} = b$.

$$\therefore \quad \frac{x^n - a^n}{x - a} = \frac{x^{p/q} - a^{p/q}}{x - a} = \frac{y^p - b^p}{y^q - b^q} = \frac{(y^p - b^p)/(y - b)}{(y^q - b^q)/(y - b)}.$$

Now, as $x \to a$, $x^{1/q} \to a^{1/q}$. $\therefore y \to b$. Again, as $y \to b$, the limiting value of the numerator of the right side $= pb^{p-1}$ (*by Cases I and II*) and that of the denominator $= qb^{q-1}$ (*by Case I*).

$$\therefore \quad \underset{x \to a}{Lt} \frac{x^n - a^n}{x-a} = \frac{p}{q}\frac{b^{p-1}}{b^{q-1}} = \frac{p}{q} b^{p-q} = \frac{p}{q} a^{\frac{p}{q}-1} = na^{n-1}.$$

When $n=0$, the limit is $\underset{x \to a}{Lt} \dfrac{0}{x-a} = 0$.

(vi) $\underset{x \to 0}{\mathbf{Lt}} \dfrac{(1+x)^n - 1}{x} = n.$

Proof. We have

$$\underset{x \to 0}{Lt} \frac{(1+x)^n - 1}{x} = \underset{x \to 0}{Lt} \left\{ \frac{(1+x)^n - 1}{\log(1+x)} \times \frac{\log(1+x)}{x} \right\}$$

$$= \underset{x \to 0}{Lt} \frac{(1+x)^n - 1}{\log(1+x)} \times \underset{x \to 0}{Lt} \frac{\log(1+x)}{x}.$$

Now, put $(1+x)^n = 1+z$. Then $n \log(1+x) = \log(1+z)$. Hence, as $x \to 0$, $\log(1+z) \to 0$ and so $z \to 0$.

$$\text{Thus,} \quad \underset{x \to 0}{Lt} \frac{(1+x)^n - 1}{\log(1+x)} = \underset{z \to 0}{Lt} \frac{nz}{\log(1+z)}$$

$$= \underset{z \to 0}{Lt}\ n \Big/ \frac{1}{z} \log(1+z)$$

$$= n \Big/ \underset{z \to 0}{Lt} \left\{ \frac{1}{z} \log(1+z) \right\}$$

$$= \frac{n}{1} = n \text{ [by (iii)]}.$$

$$\text{Also,} \quad \underset{x \to 0}{Lt} \frac{\log(1+x)}{x} = 1 \text{ [by (iii)]}$$

$$\text{Hence,} \quad \underset{x \to 0}{Lt} \frac{(1+x)^n - 1}{x} = n \times 1 = n.$$

This result also follows by replacing x by $x+1$ and a by 1 in (v) above.

2·10. Condition for the existence of a limit. *(Cauchy)* *The necessary and sufficient condition that the limit $\underset{x \to a}{Lt} f(x)$ exists and is finite, is that corresponding to any pre-assigned positive number ε however small (but not equal to zero), we can find a positive number δ, such that x_1 and x_2 being any two quantities satisfying $0 < |x - a| \leqslant \delta$;* $|f(x_1) - f(x_2)| < \varepsilon$.

For proof see Appendix.

Note. In some cases even if we may not know the value of a limit beforehand, we can determine by the above test whether a limit exists or not. Illustrations of this are given below.

Ex. 1. *Show that* $\underset{x \to 0}{Lt} \cos \frac{1}{x}$ *does not exist.*

In order that the limit may exist, it must be possible to find a positive number δ, such that x_1 and x_2 satisfying $0 < |x| \leqslant \delta$

$$\cos \frac{1}{x_1} - \cos \frac{1}{x_2} < \epsilon,$$

where ϵ is any pre-assigned positive quantity.

Now, whatever δ we may choose, if we take $x_1 = 1/2n\pi$ and $x_2 = 1/(2n+1)\pi$, by taking n a sufficiently large positive integer, both x_1 and x_2 will satisfy $0 < | x | \leqslant \delta$.

But in this case, $| \cos 1/x_1 - \cos 1/x_2 | = | \cos 2n\pi - \cos (2n+1)\pi |$ $= 2$, a finite quantity, and is not less than any chosen ϵ.

Thus, the necessary condition is not satisfied, and so the required limit does not exist.

Here, the right-hand limit as also the left-hand limit are both non-existent.

Ex. 2. *Show that* $\underset{x \to 0}{Lt} \frac{1}{2 + e^{1/x}}$ *does not exist.*

Here, taking $x_1=-1/n$, $x_2=1/n$, whatever δ we may choose, by taking n a sufficiently large positive integer we can make x_1 and x_2 both satisfy $0<|x|\leq\delta$. But in this case,

$$\left|\frac{1}{2+e^{1/x_1}}-\frac{1}{2+e^{1/x_2}}\right|=\left|\frac{1}{2+e^{-n}}-\frac{1}{2+e^{n}}\right|$$

$$=\frac{1}{2+e^{-n}}-\frac{1}{2+e^{n}}>\frac{1}{2+e^{-1}}-\frac{1}{2+e}$$

which is a finite quantity and so cannot be less than any chosen ϵ however small.

Thus, the necessary condition being not satisfied, the limit in question does not exist.

Here, the right-hand limit exists and $=0$, and the left-hand limit exists and $=\frac{1}{2}$.

2·11. Illustrative Examples.

Ex. 1. *Find the value of* $\underset{x\to 2}{Lt}\ x^2$.

By taking successive values of x which always remaining less than 2, tend to 2, *viz*, $x=1·9$, 1·99, 1·999, .. we see that x^2 has the values 3·61, 3·9601, 3·996001,... which tend to 4, and we can make the difference between 4 and x^2 smaller than any positive number however small by taking x sufficiently near to 2. Hence, the left-hand limit is 4.

Similarly, by taking values of x, which always remaining greater than 2, approach 2, *viz.*, $x=2·1$, 2·01, 2·001,... we see that x^2 has the values 4·41, 4·0401, 4·004001, .. which continually approach 4. Hence, as before, the right-hand limit is 4.

Hence, value of the required limit is 4.

Note. Exactly in the same way we can show that $\underset{x\to a}{Lt}\ x^n=a^n$, where n is an integer or a rational fraction (except when $a=0$ and n is negative).

Ex. 2. *Show that* (i) $\underset{\theta\to 0}{Lt}\ \sin\theta=0$; (ii) $\underset{\theta\to 0}{Lt}\ \cos\theta=1$.

(iii) $\underset{\theta\to a}{Lt}\ \sin\theta=\sin a$; (iv) $\underset{\theta\to a}{Lt}\ \cos\theta=\cos a$.

(i) Since, from the definition of sine of a real angle θ in trigonometry, with the help of a figure, it may be easily seen that $|\sin \theta - 0|$ *i.e.*, $|\sin \theta|$ can be made less than any positive number ϵ however small by making $|\theta|$ arbitrarily small, it follows that $\underset{\theta \to 0}{Lt} \sin \theta = 0$.

(ii) $\underset{\theta \to 0}{Lt} (1 - \cos \theta) = \underset{\theta \to 0}{Lt}\ 2 \sin^2 \tfrac{1}{2}\theta = 2 \times \underset{\theta \to 0}{Lt} (\sin \tfrac{1}{2}\theta \times \sin \tfrac{1}{2}\theta)$

$$= 2 \times 0 \ [\ by\ (i)\ ] = 0.$$

$\therefore \qquad \underset{\theta \to 0}{Lt} \cos \theta = 1.$

(iii) $\sin \theta - \sin \alpha = 2 \sin \tfrac{1}{2}(\theta - \alpha) \cos \tfrac{1}{2}(\theta + \alpha)$.

As $\theta \to \alpha$, $\tfrac{1}{2}(\theta - \alpha) \to 0$, $\qquad \therefore \underset{\theta \to \alpha}{Lt} \sin \tfrac{1}{2}(\theta - \alpha) = 0$.

Also $|\cos \tfrac{1}{2}(\theta + \alpha)| \leqslant 1$, $\qquad \therefore \underset{\theta \to \alpha}{Lt} (\sin \theta - \sin \alpha) = 0$,

i.e., $\underset{\theta \to \alpha}{Lt} \sin \theta = \sin \alpha$.

(iv) Since, $\cos \theta - \cos \alpha = 2 \sin \tfrac{1}{2}(\alpha - \theta) \sin \tfrac{1}{2}(\theta + \alpha)$, it follows as in (iii) that $\underset{\theta \to \alpha}{Lt} (\cos \theta - \cos \alpha) = 0$, *i.e.*, $\underset{\theta \to \alpha}{Lt} \cos \theta = \cos \alpha$.

Ex. 3. *Apply* (δ, ϵ) *definition of limit to illustrate that*

$$\underset{x \to 4}{Lt} (2x - 2) = 6.$$

Let us choose $\epsilon = {\cdot}01$.

Then, $|(2x-2) - 6| < {\cdot}01$ if $|2x - 8| < {\cdot}01$, *i.e.*, if $|x - 4| < {\cdot}005$, *i.e.*, $\delta = {\cdot}005$. Similarly, if $\epsilon = {\cdot}001$, $\delta = {\cdot}0005$; and so on.

Thus, δ depends upon ϵ, *i.e.*, the nearer $(2x - 2)$ is to 6, the nearer x is to 4. We have,

$$|(2x-2)-6| < {\cdot}01 \qquad \text{if } 0 < |x-4| < {\cdot}005$$

$$|(2x-2)-6| < {\cdot}001 \qquad \text{if } 0 < |x-4| < {\cdot}0005$$

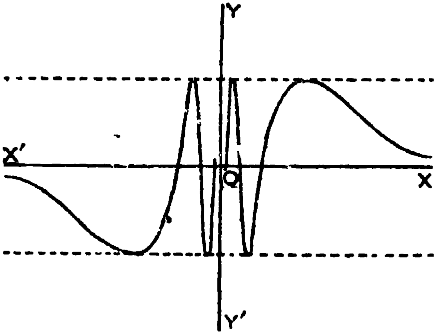
and generally, $|(2x-2)-6| < \epsilon \qquad$ if $0 < |x-4| < \tfrac{1}{2}\epsilon$.

Hence, 6 is the limit of $2x - 2$ as $x \to 4$.

Ex. 4. *Draw the graph of* $\sin (1/x)$ *and show that neither the right-hand limit nor the left-hand limit exists as* x *tends to zero.*

When $x = 0$, $\sin 1/x$ is meaningless and hence, its value is not known.

For all other values of x, $\sin(1/x)$ exists and may take any value from -1 to 1. Thus, the graph is a continuous curve with a break at $x=0$ and is comprised between the lines $x=1$ and $x=-1$.



As $x \to 0+0$ by passing successively through values $2/n\pi$ where n is a positive integer which can be made as large as we like, $\sin(1/x)$ passes through values $0, -1, 0, 1$, etc. taking intermediate values at intermediate points. Now it is evident that these values are taken more frequently as x comes nearer to 0 and so $\sin(1/x)$ does not approach any fixed value as $x \to 0+0$, but oscillates through all values between -1 and $+1$ *i.e.*, the function has no right-hand limit. Since, when x is negative, $\sin(1/x) = -\sin(1/z)$, where $z\,(=-x)$ is positive, the function behaves exactly in the same way when $x \to 0-0$. Hence, the left-hand limit also does not exist for the function.

Note. Hence, it follows that $\underset{x\to 0}{Lt} \sin(1/x)$ also does not exist.

Ex. 5. *Give an example to illustrate the following limit-inequality:*

If $\underset{x\to a}{Lt}\ \phi(x) = A$ *and* $\underset{x\to a}{Lt}\ \psi(x) = B$ *and if* $\phi(x) < \psi(x)$ *in a certain neighbourhood of a except a, then* $A \leq B$.*

Suppose, $\quad \phi(x) = 5 + x^2$; $\psi(x) = 5 + 3x^2$.

$\therefore \quad \underset{x\to 0}{Lt}\ \phi(x) = 5 = \underset{x\to 0}{Lt}\ \psi(x)$. $\qquad$ But $\phi(x) < \psi(x)$ if $x \neq 0$.

*For proof of this important theorem see Appendix.

Thus, the limits of the two functions are equal, even though $\phi(x) < \psi(x)$ for all values of x on which the limits depend.

If however, $\phi(x)=5+x^2$, $\psi(x)=7+3x^2$,

then of course, $\underset{x\to 0}{Lt}\ \phi(x) < \underset{x\to 0}{Lt}\ \psi(x)$.

Ex. 6. *Evaluate* $\underset{x\to 0}{Lt}\ \frac{1}{x}\{\sqrt{(1+x)}-\sqrt{(1-x)}\}$.

As it stands, theorem (iii) of Art. 2·8 is not applicable, since the denominator x is zero as $x\to 0$. But it can be easily transformed into a form in which the theorem is applicable.

Multiplying numerator and denominator by $\sqrt{(1+x)}+\sqrt{(1-x)}$, the

$$\text{required limit} = \underset{x\to 0}{Lt}\ \frac{2x}{x\{\sqrt{(1+x)}+\sqrt{(1-x)}\}}$$

$$= \underset{x\to 0}{Lt}\ \frac{2}{\sqrt{(1+x)}+\sqrt{(1-x)}} = \frac{2}{2} = 1,$$

since $\underset{x\to 0}{Lt}\ \sqrt{(1+x)} = \underset{y\to 1}{Lt}\ \sqrt{y} = 1$ (*putting* $1+x=y$),

and similarly $\underset{x\to 0}{Lt}\ \sqrt{(1-x)}=1$.

Ex. 7. If $-1 < x < 1$, then $\underset{n\to\infty}{Lt}\ x^n = 0$. (*$n$ is a positive integer*)

Let us first consider the case when $0 < x < 1$.

Put $x=1-p$, so that $0 < p < 1$. Since, $(1-p)(1+p)=1-p^2$, which is less than 1, we have $1-p < 1/(1+p)$.

$$\therefore\quad x^n = (1-p)^n < \frac{1}{(1+p)^n} < \frac{1}{1+np} < \frac{1}{np}.$$

$\therefore$ x^n can be made less than any given positive number ϵ by taking n large enough (*i.e.*, taking $n > 1/\epsilon p$) ; but x^n is positive.

$\therefore$ $Lt\ x^n = 0$ when $n\to\infty$.

Since, $(-x)^n = (-1)^n x^n$ the result also holds for $-1 < x < 0$,

when $x=0$, $x^n=0$ for every positive value of n.

Hence, $Lt\ x^n = 0$ when $n\to\infty$.

Note. When $x > 1$, putting x for $1+p$ in the inequality $(1+p)^n > 1+np > np$, it can be shown that $x^n > k$, where k is any positive number however large, for all $n > k/p$.

Hence, it follows that for **$x > 1$, $\underset{n\to\infty}{Lt}\ x^n = \infty$.**

Ex. 8. *Prove that (n being a positive integer)*

(i) $\underset{n\to\infty}{\text{Lt}}\ nx^n = 0$ when $|x| < 1$.

(ii) $\underset{n\to\infty}{\text{Lt}}\ \dfrac{x^n}{n} = 0$ when $|x| \leq 1$

$= \infty$ when $x > 1$.

(iii) $\underset{n\to\infty}{\text{Lt}}\ \dfrac{x^n}{n!} = 0$ for all values of x.

(iv) $\underset{n\to\infty}{\text{Lt}}\ \dfrac{m(m-1)(m-2)\cdots(m-n+1)}{n!}\, x^n = 0$ when $|x| < 1$.

For proof see Appendix.

Examples II

1. Evaluate the following limits :

(i) $\underset{x\to1}{Lt}\ \dfrac{x^2+2x-2}{2x+2}$. (ii) $\underset{x\to1}{Lt}\ \dfrac{x^2-3x+2}{x^2-4x+3}$.

(iii) $\underset{x\to a}{Lt}\ \dfrac{a-\sqrt{a^2-x^2}}{x^2}$. (iv) $\underset{x\to0}{Lt}\ \dfrac{\sqrt{1+2x}-\sqrt{1-3x}}{x}$.

2. Find the value of

$$\underset{x\to0}{Lt}\ \frac{a_0x^n+a_1x^{n-1}+\cdots+a_n}{b_0x^n+b_1x^{n-1}+\cdots+b_n} \qquad (b_n \neq 0).$$

3. Do the following limits exist ? If so, find their values :

(i) $\underset{x\to\pi}{Lt}\ \dfrac{1}{\pi-x}$. (ii) $\underset{x\to\pi}{Lt}\ \dfrac{\sin x}{\pi-x}$.

4. Find the values of :

(i) $\underset{x\to0}{Lt}\ \dfrac{\tan x}{x}$. (ii) $\underset{x\to0}{Lt}\ \dfrac{\sin(x^2)}{x}$.

(iii) $\underset{x\to0}{Lt}\ \dfrac{1-\cos x}{x}$. (iv) $\underset{x\to0}{Lt}\ \dfrac{1-\cos x}{x^2}$

(v) $\underset{x \to 0}{Lt} \dfrac{x^2 \sin(1/x)}{\sin x}$. (vi) $\underset{x \to 0}{Lt} \dfrac{\operatorname{cosec} x - \cot x}{x}$.

(vii) $\underset{x \to 0}{Lt} \dfrac{\sin x^{\circ}}{x}$. (viii) $\underset{x \to 0}{Lt} \dfrac{\sin^{-1} x}{x}$.

(ix) $\underset{x \to 0}{Lt} \dfrac{\tan^{-1} x}{x}$. (x) $\underset{x \to \infty}{Lt} \dfrac{\sin x}{x}$.

(xi) $\underset{x \to \infty}{Lt} \dfrac{\sin x}{x + \cos x}$. (xii) $\underset{x \to \infty}{Lt} \dfrac{x+1}{x^2+1}$.

(xiii) $\underset{x \to 0}{Lt} \left(\dfrac{1}{\sin x} - \dfrac{1}{\tan x} \right)$.

5. A function $f(x)$ is defined as follows :

$$\begin{aligned} f(x) &= x \quad \text{when } x > 0 \\ &= 0 \quad \text{when } x = 0 \\ &= -x \quad \text{when } x < 0. \end{aligned}$$

Find the value of $\underset{x \to 0}{Lt} f(x)$.

6. A function $\phi(x)$ is defined as follows :

$$\begin{aligned} \phi(x) &= x^2 \quad \text{when } x < 1 \\ &= 2{\cdot}5 \quad \text{when } x = 1 \\ &= x^2 + 2 \quad \text{when } x > 1. \end{aligned}$$

Does $\underset{x \to 1}{Lt} \phi(x)$ exist ?

7. Do the following limits exist ?

(i) $\underset{x \to 2}{Lt} [x]$, where $[x]$ denotes the integral part of x.

(ii) $\underset{x \to 1}{Lt} \{x^2 + \sqrt{x-1}\}$.

(iii) $\underset{x \to \frac{1}{2}\pi}{Lt} \dfrac{e^{\tan x} - 1}{e^{\tan x} + 1}$.

8. Given $f(x) = ax^2 + bx + c$, show that

$$\underset{h \to 0}{Lt} \left\{ \frac{f(x+h) - f(x)}{h} \right\} = 2ax + b.$$

9. Given $f(x) = |x|$, show that

$\underset{h \to 0}{Lt} \{f(h) - f(0)\}/h$ does not exist.

10. If $\phi(x) = \{(x+2)^2 - 4\}/x$, then

$\underset{x \to 0}{Lt} \phi(x) = 4$, although $\phi(0)$ does not exist.

11. Show that $\underset{x \to 2}{Lt} \dfrac{2x^2 - 8}{x - 2} = 8.$

Applying (δ, ϵ) definition, find δ if $\epsilon = \cdot 1$.

12. (i) Is $\underset{x \to a}{Lt} \dfrac{x^2}{x-a} - \underset{x \to a}{Lt} \dfrac{a^2}{x-a} = \underset{x \to a}{Lt} \dfrac{x^2 - a^2}{x-a}$?

(ii) Is $\underset{x \to a}{Lt} (x^2 - a^2) \times \underset{x \to a}{Lt} \dfrac{1}{x-a} = \underset{x \to a}{Lt} \left\{ (x^2 - a^2) \times \dfrac{1}{x-a} \right\}$?

13. Evaluate

(i) $\underset{n \to \infty}{Lt} \left\{ \dfrac{1}{n^2} + \dfrac{2}{n^2} + \dfrac{3}{n^2} + \cdots + \dfrac{n}{n^2} \right\}$.

(ii) $\underset{n \to \infty}{Lt} \dfrac{1^2 + 2^2 + 3^2 + \cdots + n^2}{n^3}$.

14. Does $\underset{x \to 0}{Lt} f(x)$ exist when

(i) $f(x) = \{2^{1/x} + 2^x + 1/2^x\}$?

(ii) $f(x) = \left\{ \sin \dfrac{1}{x} + x \sin \dfrac{1}{x} + x^2 \sin \dfrac{1}{x} \right\}$?

15. Evaluate

(i) $\underset{n \to \infty}{Lt} x^n$. (ii) $\underset{n \to \infty}{Lt} \dfrac{x^n}{x^n + 1}$. [*C. H. 1957*]

(iii) $\underset{n\to\infty}{Lt} \dfrac{x^n f(x)+g(x)}{x^n+1}$. [*C. H. 1956*]

(iv) $\underset{n\to\infty}{Lt} \dfrac{x^n-1}{x^n+1}$.

16. Find the value of $\underset{n\to\infty}{Lt} \dfrac{2}{\pi}$ arc tan nx.

17. Evaluate

(i) $\underset{n\to\infty}{Lt}$ sin $n\pi x$.

(ii) $\underset{n\to\infty}{Lt} \dfrac{1}{1+n\sin^2 n\pi x}$. [*C. H. 1957*]

18. (i) Prove that

$$\underset{x\to 0}{Lt} \tan^{-1}(a/x^2) = -\tfrac{1}{2}\pi,\ 0 \text{ or } \tfrac{1}{2}\pi \text{ according as}$$

a is negative, zero or positive.

(ii) Draw the graph of the function $f(x)$, where

$$f(x) = \underset{t\to 0}{Lt}\left(\frac{2x}{\pi}\tan^{-1}\frac{x}{t^2}\right).$$

19. If $f(x) = \underset{n\to\infty}{Lt} \dfrac{1}{1+x^{2n}}$, show that $f(x)=1$, $\frac{1}{2}$ or 0 according as $|x| <, =,$ or > 1. [*C. H. 1950*]

Draw the graph of $f(x)$ in this case.

20. Given the function $y=f(x)$ defined as follows: $f(x)=0$ when $x^2>1$, $f(x)=1$ when $x^2<1$, $f(x)=\frac{1}{2}$ when $x^2=1$.

Using the idea of a limit, show that the above function can be represented by

$f(x) = \underset{n\to\infty}{Lt} \dfrac{1}{1+x^{2n}}$, for all values of x. [*C. P. 1949*]

ANSWERS

1. (i) $\frac{1}{4}$. (ii) $\frac{1}{2}$. (iii) $1/2a$. (iv) $\frac{5}{2}$. **2.** a_n/b_n.

3. (i) Does not exist. (ii) 1. **4.** (i) 1. (ii) 0.

(iii) 0. (iv) $\frac{1}{2}$. (v) 0 (vi) $\frac{1}{2}$. (vii) $\frac{\pi}{180}$.

(viii) 1. (ix) 1. (x) 0. (xi) 0. (xii) 0.

(xiii) 0. **5.** 0. **6.** Does not exist. **7.** (i) Does not exist.

(ii) Does not exist (iii) Does not exist.

11. ·05. **12.** (i) No. (ii) No. **13.** (i) $\frac{1}{2}$. (ii) $\frac{1}{3}$.

14. (i) No. (ii) No.

15. (i) $+\infty$ when $x > 1$, 0 when $-1 < x < 1$, 1 when $x=1$, no limit exists when $x \leq -1$

(ii) 0 when $-1 < x < 1$; $\frac{1}{2}$ when $x=1$; 1 when $x < -1$ or > 1. not defined when $x=-1$.

(iii) $f(x)$ when $|x| > 1$; $g(x)$ when $|x| < 1$; $\frac{1}{2}[f(x)+g(x)]$ when $x=1$; undefined when $x=-1$.

(iv) -1 when $-1 < x < 1$; 0 when $x=1$; 1 when $|x| > 1$.

16. 1 when $x > 0$; 0 when $x=0$; -1 when $x < 0$.

17. (i) 0 when x is an integer; no limit exists if x is not an integer.
(ii) 0.

CHAPTER III

CONTINUITY

3·1. We have a common-sense idea of what a continuous curve is. For instance, in Art. 1·5, the curves of examples (ii), (iii), (v) are continuous, while those of (vi) and (vii) are discontinuous, the curve in (vi) having a point of discontinuity at the origin O. A function $f(x)$ is commonly said to be continuous provided its graph is a continuous curve, and if there is any discontinuity or break at any point on the curve, the function is said to be discontinuous for the corresponding value of x. The general notions of continuity of a function $f(x)$ for any value of the variable x require that the function should be finite at the point, and for a very small change in x, the change in the value of $f(x)$ should also be small, or in other words, as we approach the particular value of x from either side, the function should also approach the corresponding value of $f(x)$, and ultimately coincide with it at the point. If $f(x)$ be non-existent at a point, so that the corresponding point on the graph is missing, or else, if the value of $f(x)$ suddenly jumps as x passes from one side to the other of the particular value, or $f(x)$ becomes infinitely large at a point, then the function is discontinuous there.

We proceed below to give a formal mathematical definition of continuity.

3·2. Continuity.

A function $f(x)$ is said to be continuous for $x=a$ provided $\underset{x \to a}{Lt}\ f(x)$ exists, is finite, and is equal to $f(a)$.

In other words, for $f(x)$ to be continuous at $x=a$,

$$\underset{x \to a+0}{Lt} f(x) = \underset{x \to a-0}{Lt} f(x) = f(a)$$

or briefly, $\mathbf{f(a+0) = f(a-0) = f(a)}$.

This may also be written in the form $\underset{h\to 0}{\mathrm{Lt}}\, f(a+h)=f(a)$.

If $f(x)$ be continuous for every value of x in the interval (a, b), it is said to be *continuous throughout the interval.*

A function which is not continuous at a point is said to have a *discontinuity* at that point.

Ex. (i) $f(x)=x^2$ is continuous for any value a of x,
for $\underset{x\to a}{Lt}\ x^2=a^2$.

(ii) $f(x)=\cos\dfrac{1}{x}$ is discontinuous at $x=0$, since $\underset{x\to 0}{Lt}\cos\dfrac{1}{x}$ does not exist. [See *Ex. 1*, § *2·10*]

(iii) $f(x)=\dfrac{1}{x^2}$ is discontinuous at $x=0$, for $\underset{x\to 0}{Lt}\left(\dfrac{1}{x^2}\right)$ is not finite.

(iv) If $f(x)=x\sin\dfrac{1}{x}$ when $x\neq 0$, and $f(0)=0$, then $f(x)$ is continuous at $x=0$, for $\underset{x\to 0}{Lt}\ x\sin\dfrac{1}{x}=0$.
[See *Ex.* (*iii*), § *2·3*]

(v) If $f(x)=\dfrac{1}{5+e^{1/(x-2)}}$ when $x\neq 2$ and $f(2)=\frac{1}{2}$, then $f(x)$ is discontinuous at $x=2$, since $\underset{x\to 2+0}{Lt} f(x)\neq \underset{x\to 2-0}{Lt} f(x)$ here, so that $\underset{x\to 2}{Lt} f(x)$ does not exist.

(vi) $f(x)=e^{-(a-x)^{-2}}$ is discontinuous at $x=a$, since though $\underset{x\to a+0}{Lt} f(x)=\underset{x\to a-0}{Lt} f(x)=0$, *i. e.*, $\underset{x\to a}{Lt} f(x)$ exists and $=0$, $f(a)$ is undefined.

Corresponding to the analytical definition of limit, we have the following *analytical definition* of continuity of a function at a point :

The function $f(x)$ is continuous at $x=a$, provided $f(a)$ exists and given any pre-assigned positive quantity ϵ, *however small, we can determine a positive quantity δ, such that $|f(x)-f(a)| < \varepsilon$ for all values of x satisfying $a-\delta \leqslant x \leqslant a+\delta$.*

3·3. Different classes of Discontinuity.

(A) If $f(a+0) \neq f(a-0)$, then $f(x)$ is said to have an *ordinary discontinuity* at $x=a$. In this case, $f(a)$ may or may not exist, or if it exists, it may be equal to one of $f(a+0)$ and $f(a-0)$, or may be equal to neither.

To these is to be added the case where only one of $f(a+0)$ and $f(a-0)$ is existent, and $f(a)$ exists, but is not equal to that.

Illus: $f(x)=\left(2+e^{\frac{1}{x}}\right)^{-1}$ has an ordinary discontinuity at $x=0$, for $\underset{x \to 0+0}{Lt} f(x)=0$, and $\underset{x \to 0-0}{Lt} f(x)=\frac{1}{2}$.

Note. Continuity on one side.

In case where $f(x)$ is undefined on one side of a (say for $x < a$), if $f(a+0)$ exists and is equal to $f(a)$ (which also exists and is finite) we say, as a special case, that $f(x)$ is continuous at $x=a$.

(B) If $f(a+0)=f(a-0) \neq f(a)$, or $f(a)$ is not defined then $f(x)$ is said to have a *removable discontinuity* at $x=a$.

Illus: $f(x) = (x^2-a^2)/(x-a)$ has a removable discontinuity at $x=a$, for $f(a)$ is undefined here, though $\underset{x \to a}{Lt} f(x)$ exists, and $=2a$.

Again, if $f(x)=1$ when $x=a$, and $f(x)=e^{-(x-a)^{-2}}$ when $x \neq a$, $f(x)$ has a removable discontinuity at a, for $\underset{x \to a}{Lt} f(x)=0$, whereas $f(a)=1$ as defined.

It may be noted that a function which has a removable discontinuity at a point can be made continuous there by suitably defining the function at the particular point only.

The two classes of discontinuity (A) and (B) are termed *simple discontinuities*.

(C) If one or both of $f(a+0)$ and $f(a-0)$ tend to $+\infty$ or $-\infty$, then $f(x)$ is said to have an *infinite discontinuity* at a. Here, $f(a)$ may or may not exist.

Illus: $f(x)=e^{-\frac{1}{x-a}}$ has an infinite discontinuity at $x=a$, since, $f(a-0)\to\infty$, $f(x)=\dfrac{3x^2}{(x-2)^2}$ has an infinite discontinuity at $x=2$.

(D) Any point of discontinuity which is not a point of simple discontinuity, nor an infinity, is called a point of *oscillatory discontinuity*. At such a point the function may *oscillate finitely* or *oscillate infinitely*, and does not tend to a limit or to $+\infty$ or $-\infty$.

Illus: $f(x)=\sin\dfrac{1}{x}$ oscillates finitely at $x=0$

$f(x)=\dfrac{1}{x-a}\sin\dfrac{1}{x-a}$ oscillates infinitely at $x=a$ $\quad$ $Lt\ x^n(x=-1)$ oscillates finitely, and $Lt\ x^n(x<-1)$ oscillates infinitely as $n\to\infty$.

3·4. Some properties of continuous functions.

(i) *The sum or difference of two continuous functions is a continuous function ;*

i.e., if $f(x)$ and $\phi(x)$ are both continuous at $x=a$, then $f(x)\pm\phi(x)$ is continuous at $x=a$.

For in this case, by definition of continuity, $\underset{x\to a}{Lt} f(x)$ exists, and $=f(a)$, as also $\underset{x\to a}{Lt}\ \phi(x)=\phi(a)$.

$$\text{Hence, } \underset{x\to a}{Lt}\ \{f(x)\pm\phi(x)\}=\underset{x\to a}{Lt} f(x)\pm\underset{x\to a}{Lt}\ \phi(x),$$
$$=f(a)\pm\phi(a), \qquad \text{[See § } 2\cdot 8(i)\text{]}$$

whence, by definition, $f(x)\pm\phi(x)$ is continuous at $x=a$.

Note 1. The result may be extended to the case of any finite number of functions.

Note 2. If $f(x)$ be continuous at $x=a$, and $\phi(x)$ is not, then $f(x)\pm\phi(x)$ is discontinuous at $x=a$, and behaves like $\phi(x)$.

(*ii*) *Product of two continuous functions is a continuous function ;*

i e., $f(x)$ and $\phi(x)$ being continuous at $x=a$, $f(x)\times\phi(x)$ is continuous there.

Proof is exactly similar to that in the above case, depending on the corresponding limit theorem [§ *2·8(ii)*].

Note. This result may also be extended to any finite number of functions.

(*iii*) *Quotient of two continuous functions is a continuous function, provided the denominator is not zero anywhere for the range of values considered ;*

i e., if $f(x)$ and $\phi(x)$ be both continuous at $x=a$, and $\phi(x)\neq 0$, then $f(x)/\phi(x)$ is continuous there.

Proof depends on the corresponding limit theorem [§ *2·8(iii)*].

(*iv*) *If $f(x)$ be continuous at $x=a$, and $f(a)\neq 0$, then in the neighbourhood of $x=a$, $f(x)$ has the same sign as that of $f(a)$, i.e., we can get a positive quantity δ such that $f(x)$ preserves the same sign as that of $f(a)$ for every value of x in the interval $a-\delta < x < a+\delta$.*

Let $f(x)=\sin x$, $a=\frac{1}{2}\pi$; then $f(a)=1$ and hence $\neq 0$ and *positive* Let us take $\delta=\frac{1}{4}\pi$. Then in the interval $\frac{1}{2}\pi-\frac{1}{4}\pi < x < \frac{1}{2}\pi+\frac{1}{4}\pi$ *i.e.*, $\frac{1}{4}\pi < x < \frac{3}{4}\pi$, $f(x)$ is always *positive.*

(*v*) *If $f(x)$ be continuous throughout the interval (a, b), and if $f(a)$ and $f(b)$ be of opposite signs, then there is at least one value, say ξ, of x within the interval for which $f(\xi)=0$.*

Let $f(x)=\cos x$, $a=0$, $b=\pi$. Then $f(a)=1$, $f(b)=-1$. Now $\cos x=0$ if $x=\frac{1}{2}\pi$, which obviously lies in the interval

$(0, \pi)$, and so here $\xi = \frac{1}{2}\pi$. Similarly, if we take $a=0$, $b=3\pi$, we get another value of ξ, *viz*, $\frac{5}{2}\pi$, besides $\frac{1}{2}\pi$.

(*vi*) *If $f(x)$ be continuous throughout the interval (a, b) and if $f(a) \neq f(b)$, then $f(x)$ assumes every value between $f(a)$ and $f(b)$ at least once in the interval.*

Let $f(x)=x^2$, $a=0$, $b=1$; then $f(a)=0$, $f(b)=1$. Let c be any number between 0 and 1. Then $f(x)=x^2=c$, if $x=+\sqrt{c}$, which evidently lies in $(0, 1)$.

Let $f(x)=\sin x$, $a=0$, $b=\dfrac{5\pi}{2}$; then $f(a)=0$, $f(b)=1$, so $f(a) \neq f(b)$. Let c be any number lying in $(0, 1)$. Then $\sin x = c$, if $x = n\pi + (-1)^n \sin^{-1} c$, $n=0, \pm 1, \pm 2 \ldots$ Now, for $n=0, 1, 2$, only, x lies in the interval $\left(0, \dfrac{5\pi}{2}\right)$. That is, when $x=\sin^{-1}c$, or $\pi-\sin^{-1}c$, or $2\pi+\sin^{-1}c$, we have $f(x)=c$. Thus $f(x)$ assumes the value c at least once (here 3 times and in the previous example, once only).

(*vii*) *A function which is continuous throughout a closed interval is bounded therein.*

The function $f(x) \equiv \sin x$, is continuous in the closed interval $0 \leqslant x \leqslant \pi$, and has the upper bound at $x=\frac{1}{2}\pi$ and lower bound at $x=0$ or π, and hence it is bounded.

(*viii*) *A continuous function in an interval actually attains its upper and lower bounds, at least once each, in the interval.*

The function $f(x) \equiv \sin x$, is continuous in the interval $0 \leqslant x \leqslant \pi$. Its upper bound 1 is attained at the point $x=\frac{1}{2}\pi$ and lower bound 0 is attained at the points $x=0$ and $x=\pi$. Thus, $f(x)$ attains its upper and lower bounds at least once each (here the upper bound is attained once, whereas the lower bound is attained twice).

(*ix*) *A function $f(x)$, continuous in a closed interval (a, b), attains every intermediate value between its upper and lower bounds in the interval, at least once.*

Let $f(x) \equiv x^2$, $a = -1$, $b = 2$, then the upper bound of $f(x)$ is 4 and its lower bound is 0. Let c be any number in $(0, 4)$. Now, if $0 \leqslant c \leqslant 1$, then $f(x) = x^2 = c$ if $x = \pm\sqrt{c}$, which lie in $(-1, 2)$; and if $1 < c \leqslant 4$, then $f(x) = x^2 = c$, if $x = \pm\sqrt{c}$, of which only $+\sqrt{c}$ lies in the interval $(-1, 2)$. Thus $f(x)$ attains the value c at least once.

[*For formal proofs of (iv)-(ix) see Appendix*]

3·5. Continuity of some Elementary Functions.

(*i*) *Function* x^n, where n is any rational number.

We know that $\underset{x \to a}{Lt}\ x^n = a^n$, for all values of n, except when $a = 0$ and n is negative [See *Note, Ex. 1*, § *2 11*].

Hence, x^n is continuous for all values of x when n is positive, and continuous for all values of x except 0 when n is negative.

When n is negative and $= -m$ say, where m is positive, $x^n = x^{-m} = 1/x^m$ which either does not tend to a limit or $\to \infty$ as $x \to 0$.

(*ii*) *Polynomials.*

Since the polynomial $a_0 x^n + a_1 x^{n-1} + \cdots + a_n$ is the sum of a finite number of positive integral powers of x (each multiplied by a constant) each of which is continuous for all values of x, the polynomial itself [*by* § *3·4(i)*], is continuous for all values of x.

(*iii*) *Rational Algebraic Functions.*

Rational algebraic functions like $\dfrac{a_0x^n + a_1x^{n-1} + \cdots + a_n}{b_0x^m + b_1x^{m-1} + \cdots + b_m}$ being the quotient of two polynomials which are continuous for all values of x, are continuous for all values of x except those which make the denominator zero [*by* § *3·4(iii)*].

(*iv*) *Trigonometric Functions.*

Since, the limiting values of $\sin x$ and $\cos x$ when $x \to a$, where a has any value, are $\sin a$ and $\cos a$ [See *Ex. 2*, § *2·11*], it follows that $\sin x$ and $\cos x$ are continuous for all values of x.

Since, $\tan x = \sin x/\cos x$, $\tan x$ is continuous for all values of x except those which make $\cos x$ zero, *i.e.*, except for $x = (2n+1)\frac{1}{2}\pi$. Similarly, $\sec x$ is continuous for all values of x except for $x = (2n+1)\frac{1}{2}\pi$ and $\cot x$ and $\operatorname{cosec} x$ are continuous for all values of x, except when $x = 0$ or any multiple of π when $\sin x = 0$.

(*v*) *Inverse Circular Functions.*

Inverse circular functions being many-valued, we make a convention of defining their domain in such a way as to make them single-valued. Throughout the book we shall suppose (unless otherwise stated) that $\sin^{-1}x$, $\tan^{-1}x$, $\cot^{-1}x$, $\operatorname{cosec}^{-1}x$ lie between $-\frac{1}{2}\pi$ and $\frac{1}{2}\pi$ (both values inclusive) and $\cos^{-1}x$, $\sec^{-1}x$ lie between 0 and π (both values inclusive), which are the principal values of these inverse functions. It should be noted however that $\sin^{-1}x$ and $\cos^{-1}x$ have no existence *outside* the *closed* interval $(-1, 1)$ of x, and $\operatorname{cosec}^{-1}x$ and $\sec^{-1}x$ have no existence *inside* the *open* interval $(-1, 1)$. All the inverse circular functions are continuous for all values for which they exist ; this follows immediately from the continuity of the corresponding circular functions.

(vi) *Function* e^x.

Corresponding to the positive number ϵ, however small, we can choose n sufficiently large such that $(1+\epsilon)^n > e$, [$\because$ $(1+\epsilon)^n > 1+n\epsilon$,* and e is finite]. Thus, $e^{\frac{1}{n}} - 1 < \epsilon$. Hence, if $0 < x < 1/n$, $e^x - 1 < e^{\frac{1}{n}} - 1 < \epsilon$, and $\therefore \underset{x \to 0+}{Lt} (e^x - 1) = 0$, or, $\underset{x \to 0+}{Lt} e^x = 1$. If x be negative, putting $x = -y$, $\underset{x \to 0-}{Lt} e^x = \underset{y \to 0+}{Lt} 1/e^y = 1$. Hence, $\underset{x \to 0}{Lt} e^x = 1$. $\therefore \underset{x \to c}{Lt} e^{x-c} = 1$, *i.e.*, $\underset{x \to c}{Lt} e^x = e^c$. $\therefore$ e^x is continuous at any point $x = c$.

(vii) *Function* $\log x$, $x > 0$.

It should be noted that $\log x$ is defined only for values of $x > 0$.

Let $\log x = y$ and $\log (x+h) = y+k$. Then $e^y = x$ and $e^{y+k} = x+h$; $\therefore$ $h = e^{y+k} - e^y$. As e^y is a continuous function of y, $e^{y+k} \to e^y$, *i.e.*, $h \to 0$ as $k \to 0$. Thus, $\{\log (x+h) - \log x\} \to 0$ as $k \to 0$, *i.e.*, as $h \to 0$. Hence, $\log x$ is continuous

3·6. Illustrative Examples.

Ex. 1. *A function is defined as follows*

$$f(x) = x \text{ when } x > 0,\ f(0) = 0,\ f(x) = -x \text{ when } x < 0.$$

Prove that the function is continuous at $x = 0$.

Here, $\underset{x \to 0+0}{Lt} f(x) = \underset{x \to 0+0}{Lt} x = 0$ and $\underset{x \to 0-0}{Lt} f(x) = \underset{x \to 0-0}{Lt} (-x) = 0$.

Thus, $\underset{x \to 0+0}{Lt} f(x) = \underset{x \to 0-0}{Lt} f(x) = f(0) = 0$ here.

Hence, $f(x)$ is continuous at $x = 0$.

For its graph, see figure of § 1·5 (ii).

Ex. 2. *A function* $f(x)$ *is defined as follows* :

$$f(x) = x \sin \frac{1}{x} \quad \text{for } x \neq 0$$
$$= 0 \quad \text{for } x = 0.$$

Show that $f(x)$ *is continuous at* $x = 0$.

* See *Appendix*.

Since, $|\sin(1/x)| \leqslant 1$, by making $|x| < \epsilon$

we can make $|x \sin(1/x)| < \epsilon$,

where ϵ is any pre-assigned positive quantity, however small.

Hence, $\underset{x \to 0}{Lt}\ x \sin \frac{1}{x} = 0$. Also, $f(0) = 0$, as defined.

Thus, $\underset{x \to 0}{Lt}\ f(x) = f(0)$. $\therefore$ $f(x)$ is continuous at $x = 0$.

For its graph, see figure of § 1·5 (viii).

Note. It should be noted that the function $x \sin(1/x)$ is continuous for all values of x, except for $x = 0$; because when $x = 0$, $x \sin(1/x)$ is meaningless. In the above example, the discontinuity of $x \sin(1/x)$ at $x = 0$ has been *removed by definition of* $f(0)$.

Ex. 3. *A function $f(x)$ is defined as follows*

$$f(x) = \tfrac{1}{2} - x \text{ when } 0 < x < \tfrac{1}{2}$$
$$= \tfrac{1}{2} \text{ when } x = \tfrac{1}{2}$$
$$= \tfrac{3}{2} - x \text{ when } \tfrac{1}{2} < x < 1.$$

Show that $f(x)$ is discontinuous at $x = \frac{1}{2}$

Here, $\underset{x \to \frac{1}{2} - 0}{Lt} f(x) = \underset{x \to \frac{1}{2} - 0}{Lt} (\frac{1}{2} - x) = \frac{1}{2} - \frac{1}{2} = 0.$

$\underset{x \to \frac{1}{2} + 0}{Lt} f(x) = \underset{x \to \frac{1}{2} + 0}{Lt} (\frac{3}{2} - x) = \frac{3}{2} - \frac{1}{2} = 1.$

Since, $\underset{x \to \frac{1}{2}}{Lt} f(x)$ does not exist, hence $f(x)$ is discontinuous at $x = \frac{1}{2}$.

Ex. 4. *A function $f(x)$ is defined in* $(0, 3)$ *in the following way* :

$$f(x) = x^2 \quad \text{when } 0 < x < 1$$
$$= x \quad \text{when } 1 \leqslant x < 2$$
$$= \tfrac{1}{4}x^3 \quad \text{when } 2 \leqslant x < 3.$$

Show that $f(x)$ is continuous at $x = 1$ and $x = 2$. [*C. P. 1941*]

When $x = 1$, $f(x) = x$. $\therefore f(1) = 1$.

$\underset{x \to 1 - 0}{Lt} f(x) = \underset{x \to 1 - 0}{Lt} x^2 = 1$; also, $\underset{x \to 1 + 0}{Lt} f(x) = \underset{x \to 1 + 0}{Lt} x = 1.$

Hence, $\underset{x \to 1 - 0}{Lt} f(x) = \underset{x \to 1 + 0}{Lt} f(x) = f(1)$ here.

$\therefore$ $f(x)$ is continuous at $x = 1$.

Similarly, it can be shown (from the definition of the function in the relevant ranges), that $\underset{x\to 2-0}{Lt} f(x) = \underset{x\to 2+0}{Lt} f(x) = f(2) = 2.$

Hence, $f(x)$ is continuous at $x=2$.

Examples III

1. A function $f(x)$ is defined as follows :

$$f(x) = x^2 \text{ when } x \neq 1,\ f(x) = 2 \text{ when } x = 1.$$

Is $f(x)$ continuous at $x=1$?

2. Are the following functions continuous at the origin ?

(i) $f(x) = \sin(1/x)$ when $x \neq 0$, $f(0) = 0$.

(ii) $f(x) = x\cos(1/x)$ when $x \neq 0$, $f(0) = 0$.

(iii) $f(x) = x\cos(1/x)$ when $x \neq 0$, $f(0) = 1$.

(iv) $f(x) = \dfrac{\sin\dfrac{1}{x}}{\dfrac{1}{x}}$ when $x \neq 0$

$= 1$ when $x = 0$.

(v) $f(x) = \sin x \cos\dfrac{1}{x}$ when $x \neq 0$

$= 0$ when $x = 0$.

3. A function $\phi(x)$ is defined as follows :

$$\begin{aligned} \phi(x) &= x^2 && \text{when } x < 1 \\ &= 2{\cdot}5 && \text{when } x = 1 \\ &= x^2 + 2 && \text{when } x > 1. \end{aligned}$$

Is $\phi(x)$ continuous at $x=1$?

4. A function $f(x)$ is defined in the following way :

$$\begin{aligned} f(x) &= -x && \text{when } x \leqslant 0 \\ &= x && \text{when } 0 < x < 1 \\ &= 2 - x && \text{when } x \geqslant 1. \end{aligned}$$

Show that it is continuous at $x=0$ and $x=1$.

[*C. P. 1942*]

5. A function $f(x)$ is defined as follows :

$$f(x)=1,\ 0 \text{ or } -1 \text{ according as } x > = \text{ or } < 0.$$

Show that it is discontinuous at $x=0$.

6. The function $f(x)=\dfrac{x^2-16}{x-4}$ is undefined at $x=4$.

What value must be assigned to $f(4)$, if $f(x)$ is to be continuous at $x=4$?

7. Determine whether the following functions are continuous at $x=0$.

(i) $f(x)=(x^4+x^3+2x^2)/\sin x,\ f(0)=0.$

(ii) $f(x)=(x^4+4x^3+2x)/\sin x,\ f(0)=0.$

8. Find the points of discontinuity of the following functions :

(i) $\dfrac{x^3+2x+5}{x^2-8x+12}$. (ii) $\dfrac{x^3+2x+5}{x^2-8x+16}$.

9. A function $f(x)$ is defined as follows :

$$f(x)= \begin{cases} 3+2x & \text{for } -\frac{3}{2}\leqslant x<0 \\ 3-2x & \text{for } 0\leqslant x<\frac{3}{2} \\ =-3-2x & \text{for } x \geqslant \frac{3}{2}. \end{cases}$$

Show that $f(x)$ is continuous at $x=0$ and discontinuous at $x=\frac{3}{2}$.

10. Given the function $y=f(x)$ defined as follows : $f(x)=0$ when $x^2>1$, $f(x)=2$ when $x^2<1$, $f(x)=\frac{1}{2}$ when $x^2=1$. Draw a diagram of the function and discuss from the diagram that, except at points $x=1$ and $x=-1$, the function is continuous. Discuss also why the function is discontinuous at these two points although it has a value for every value of x. [*C. P. 1949*]

ANSWERS

1. No. 2. (i) No. (ii) Yes. (iii) No. (iv) No. (v) Yes. 3. No. 6. 8. 7. (i) Continuous. (ii) Discontinuous. 8. (i) 6, 2. (ii) 4.

CHAPTER IV

DIFFERENTIATION

4·1. Increment.

The increment of a variable in changing from one value to another is the *difference* obtained by subtracting the first value from the second. An increment of x is denoted by Δx (read as delta x) or h. Evidently, increment may be positive or negative according as the variable in changing, increases or decreases.

If in $y=f(x)$, the independent variable x takes an increment Δx (or h), then Δy (or k) denotes the corresponding increment of y, *i.e.*, of $f(x)$, and we have

$$y+\Delta y=f(x+\Delta x), \quad i.e., \quad \Delta y=f(x+\Delta x)-f(x)$$

or, $$y+k=f(x+h), \quad i.e., \quad k=f(x+h)-f(x).$$

Illustration : Let $y=x^2$.

Suppose, x increases from 2 to 2·1, *i.e.*, $\Delta x=·1$;

then y increases from 4 to 4·41, *i.e.*, $\Delta y=·41$.

Suppose, x decreases from 2 to 1·9, *i.e.*, $\Delta x=-·1$;

then y decreases from 4 to 3·61, *i.e.*, $\Delta y=-·39$.

Increments are always reckoned from the arbitrarily fixed initial value of the independent variable x.

If y decreases as x increases, or the reverse, then Δx and Δy will have opposite signs.

From a fixed initial value 2 of x, if x increases successively to 2·1, 2·01, 2·001, etc., then although the corresponding increment Δx (=·1, ·01, ·001...) and Δy (=·41, ·0401, ·004001,...) are getting smaller, their ratio, *i.e.*, $\Delta y/\Delta x$ being 4·1, 4·01, 4·001, ... is approaching a definite number 4, thus, illustrating the fact that the ratio can be brought as near to 4 as we please by making Δx approach zero. Thus, the ratio of the increments $\Delta y/\Delta x$ has a definite finite limit 4 as $\Delta x \to 0$, and consequently, $\Delta y \to 0$.

4·2. Differential Coefficient (or Derivative).

Let $y = f(x)$ be a finite and single-valued function defined in any interval of x, and assume x to have any particular value in the interval. Let Δx (or h) be the increment of x, and let Δy (or k) $= f(x+\Delta x) - f(x)$ be the corresponding increment of y. If the ratio $\Delta y/\Delta x$ of these increments tends to a definite finite limit as Δx tends to zero, then this limit is called the *differential coefficient* (or *derivative*) of $f(x)$ (or y) for the particular value of x, and is denoted by $f'(x)$, $\frac{d}{dx}\{f(x)\}$, $\frac{dy}{dx}$, or, $D\{f(x)\}$. $\frac{dy}{dx}$, $\underset{\Delta x \to 0}{Lt} \frac{\Delta y}{\Delta x}$, $\frac{\Delta y}{\Delta x}$

Thus, symbolically, *the differential coefficient of y $[=f(x)]$ with respect to x (for any particular value of x) is*

$$f'(x) \text{ or } \frac{\mathbf{dy}}{\mathbf{dx}} = \underset{\Delta x \to 0}{Lt} \frac{\Delta y}{\Delta x} = \underset{\Delta x \to 0}{Lt} \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

or, $$= \underset{h \to 0}{\mathbf{Lt}} \frac{\mathbf{f(x+h) - f(x)}}{\mathbf{h}},$$ *provided this limit exists.*

If, as $\Delta x \to 0$, $\Delta y/\Delta x \to +\infty$ or $-\infty$, then also we say that the derivative exists, and $= +\infty$ or $-\infty$.

Note 1. The process of finding the differential coefficient is called *differentiation*, and we are said to *differentiate* $f(x)$ and sometimes *differentiate* $f(x)$ *with respect to* x, to emphasise that x is the independent variable.

Note 2. $\frac{dy}{dx}$ stands here for the symbol $\frac{d}{dx}(y)$, a limiting process and hence must not be regarded as a function $dy \div dx$, although for convenience of printing, it may sometimes be written as dy/dx.

Note 3. The differential coefficient of $f(x)$ for any particular value a of x, is often denoted by $f'(a)$. Thus, from definition, $f'(a) = \underset{h \to 0}{Lt} \frac{f(a+h)-f(a)}{h}$, provided this limit exists.

Note 4. *If $f'(a)$ is finite, $f(x)$ must be continuous at $x = a$.*

$$f'(a) = \underset{h \to 0}{Lt} \frac{f(a+h)-f(a)}{h}.$$

We can write $f(a+h) - f(a) = \frac{f(a+h)-f(a)}{h} \times h.$

$$\therefore \quad \underset{h \to 0}{Lt} f(a+h) - f(a) = \underset{h \to 0}{Lt} \left\{ \frac{f(a+h)-f(a)}{h} \times h \right\}$$

$$= \underset{h \to 0}{Lt} \frac{f(a+h)-f(a)}{h} \times \underset{h \to 0}{Lt} h$$

$$= f'(a) \times 0$$

$$= 0, \text{ since } f'(a) \text{ is finite.}$$

$$\therefore \quad \underset{h \to 0}{Lt} f(a+h) = f(a).$$

$\therefore$ from the definition of continuity, it follows that $f(x)$ is continuous at $x = a$.

Hence, *for the differential coefficient of $f(x)$ to exist finitely for any value of x, the function $f(x)$ must be continuous at the point.*

The *converse however is not always true*, *i.e.*, if a function be continuous at any point, it is not necessarily true that a finite derivative of the function for that value of x should exist. For illustration, see § 4·5.

Again, a function $f(x)$, though *discontinuous* at a point, may have an *infinite derivative* at a point. [See *Ex. 7(ii), Examples IV(A)*]

Note 5. The right-hand limit $\underset{h \to 0+0}{Lt} \frac{f(x+h)-f(x)}{h}$ for any particular value of x, when it exists, is called the *right-hand derivative of* $f(x)$ at that point and is denoted by $Rf'(x)$. Similarly, the left-hand limit $\underset{h \to 0-0}{Lt} \frac{f(x+h)-f(x)}{h}$ or, $\underset{h \to 0+0}{Lt} \frac{f(x-h)-f(x)}{-h}$ when it exists, is called the *left-hand derivative of* $f(x)$ at x, denoted by $Lf'(x)$. When these two derivatives both exist, and are equal, it is then only that the derivative of $f(x)$ exists at x. When however the left-hand and right-hand derivatives of $f(x)$ at x are unequal, or one or both are non-existent, then $f(x)$ is said to have no proper derivative at x.

Thus, though $f'(x)$ may not exist at a point, one or both of the right-hand and left-hand derivatives may exist (the two being unequal in the latter case).

For illustration, see § 4·5, Ex. 4.

4·3. Differential coefficient in some standard cases.

• (i) Differential coefficient of $\mathbf{x^n}$.

Let $f(x) = x^n$.

Then from definition, $f'(x) = \underset{h\to 0}{Lt} \dfrac{(x+h)^n - x^n}{h}$.

Now, writing X for $x+h$, so that $h = X - x$, and noting that when $h \to 0$, $X \to x$, we get

$$f'(x) = \underset{X\to x}{Lt} \frac{X^n - x^n}{X - x} = nx^{n-1} \text{ for all rational values of } n.$$

[See § 2·9(v)]

Thus, $\dfrac{\mathbf{d}}{\mathbf{dx}}(\mathbf{x^n}) = \mathbf{nx^{n-1}}$, for all rational values of n.

Otherwise

$$f'(x) = \underset{h\to 0}{Lt} \frac{(x+h)^n - x^n}{h} = \underset{h\to 0}{Lt}\ x^{n-1}.\frac{(1+h/x)^n - 1}{h/x}$$

[supposing $x \neq 0$]

$$= x^{n-1} \underset{z\to 0}{Lt} \frac{(1+z)^n - 1}{z} \quad [\text{ putting } z = h/x\ ]$$

$$= nx^{n-1}. \qquad [\text{ See § 2·9(}iv\text{) }]$$

The result can also be derived for any rational value of n [$\neq 0$] from the well-known inequality *

$$nX^{n-1}(X-x) \gtrless X^n - x^n \gtrless nx^{n-1}(X-x),$$

$$\left[\begin{array}{l}\text{upper sign if } n > 1 \text{ or } < 0 \\ \text{and lower if } \quad 0 < n < 1\end{array}\right]$$

whence $nX^{n-1} \gtrless \dfrac{X^n - x^n}{X-x} \gtrless nx^{n-1}$.

* See any text-book on Higher Algebra (*e.g.*, See § 10, Chap. XIV, *Barnard & Child*).

Now putting $X = x + h$, and letting $h \to 0$, we get

$$\underset{h\to 0}{Lt} \frac{(x+h)^n - x^n}{h} = nx^{n-1}$$ [*by* § *2·8(v)*], since both extremes tend to the same limit nx^{n-1}.

When n is a positive integer, the result can also be proved as follows :

$$\begin{aligned} f'(x) &= \underset{h\to 0}{Lt} \frac{(x+h)^n - x^n}{h} \\ &= \underset{h\to 0}{Lt} \frac{\{x^n + nx^{n-1}h + \frac{1}{2}n(n-1)x^{n-2}h^2 + \cdots + h^n\} - x^n}{h} \\ &\qquad\qquad \text{(By Binomial Theorem)} \\ &= \underset{h\to 0}{Lt} \{nx^{n-1} + \tfrac{1}{2}n(n-1)x^{n-2}h + \cdots + h^{n-1}\} \\ &= nx^{n-1}. \end{aligned}$$

When n is not a positive integer, for an alternative proof, see Ex. 1, § 4·13. See also Ex. 2, § 4·13 for the case when n has any real value, not necessarily rational.

Cor. $\dfrac{d}{dx}(x) = 1$, $\dfrac{d}{dx}(\sqrt{x}) = \dfrac{1}{2\sqrt{x}}$, $\dfrac{d}{dx}\left(\dfrac{1}{x^n}\right) = -\dfrac{n}{x^{n+1}}$.

Note. It is to be noted that in the above formula we tacitly assume those values of x as do not make x^n or x^{n-1} meaningless ; *e.g., if n be a fraction of even denominator, zero and negative values of x are excluded and if n − 1 be negative, zero value for x is excluded.*

Following the definition it may be seen in particular, that if $f(x) = x^n$, $f'(0) = 0$ when $n > 1$, $f'(0) = 1$ when $n = 1$, and $f'(0)$ is non-existent if $n < 1$.

(ii) Differential coefficient of e^x.

Let $f(x) = e^x$. Then from definition,

$$f'(x) = \underset{h\to 0}{Lt} \frac{e^{x+h} - e^x}{h} = \underset{h\to 0}{Lt}\, e^x \cdot \frac{e^h - 1}{h} = e^x,$$

since, $\underset{h\to 0}{Lt} (e^h - 1)/h = 1$. [See § *2·9 (iv)*]

Thus, $\dfrac{d}{dx}(e^x) = e^x$.

(iii) Differential coefficient of $\mathbf{a^x}$.

Let $\quad f(x)=a^x$.

Then, $f'(x)=\underset{h\to 0}{Lt}\dfrac{a^{x+h}-a^x}{h}=a^x\cdot\underset{h\to 0}{Lt}\dfrac{a^h-1}{h}$.

Now, $\underset{h\to 0}{Lt}\dfrac{a^h-1}{h}=\underset{h\to 0}{Lt}\dfrac{e^{h\log a}-1}{h\log a}\cdot\log a$

$$=\underset{h'\to 0}{Lt}\frac{e^{h'}-1}{h'}\cdot\log a\ [\text{ where } h'=h\log a\ ]$$

$$=\log a,\ \because\ \underset{h'\to 0}{Lt}\frac{e^{h'}-1}{h'}=1.$$

[See § *2·9 (iv)*]

$\therefore\quad f'(x)=a^x\log a.$

Thus, $\quad \dfrac{\mathbf{d}}{\mathbf{dx}}(\mathbf{a^x})=\mathbf{a^x\log_e a}.$

(iv) Differential coefficient of **log x.**

Let $\quad f(x)=\log x$.

Then, $f'(x)=\underset{h\to 0}{Lt}\dfrac{\log(x+h)-\log x}{h}=\underset{h\to 0}{Lt}\dfrac{1}{h}\cdot\log\dfrac{x+h}{x}$

$$=\underset{h\to 0}{Lt}\frac{1}{x}\cdot\frac{x}{h}\log\left(1+\frac{h}{x}\right)$$

$$=\frac{1}{x}\underset{z\to 0}{Lt}\frac{1}{z}\log(1+z),\left[\text{ where } z=\frac{h}{x}\right]$$

$$=\frac{1}{x}.$$

[See § *2·9 (iii)*]

Thus, $\quad \dfrac{\mathbf{d}}{\mathbf{dx}}(\mathbf{\log x})=\dfrac{\mathbf{1}}{\mathbf{x}}.$

Cor. Proceeding exactly as above it can be easily shown that

$$\frac{\mathbf{d}}{\mathbf{dx}}(\mathbf{\log_a x})=\frac{\mathbf{1}}{\mathbf{x}}\mathbf{\log_a e}.$$

(v) Differential coefficient of **sin x.**

Let $f(x) = \sin x.$

$$\text{Then, } f'(x) = \underset{h \to 0}{Lt} \frac{\sin(x+h) - \sin x}{h}$$

$$= \underset{h \to 0}{Lt} \frac{2 \sin \frac{1}{2}h \cos(x + \frac{1}{2}h)}{h}$$

$$= \underset{h \to 0}{Lt} \frac{\sin \frac{1}{2}h}{\frac{1}{2}h} \cdot \cos(x + \tfrac{1}{2}h) = \cos x,$$

because as $h \to 0$, $\cos x$ being a continuous function of x, $\cos(x + \frac{1}{2}h) \to \cos x$, also by § *2·9(i)*, $\underset{h \to 0}{Lt} (\sin \frac{1}{2}h / \frac{1}{2}h) = 1.$

Thus, $$\mathbf{\frac{d}{dx}(\sin x) = \cos x.}$$

(vi) Differential coefficient of **cos x.**

Let $f(x) = \cos x.$

$$\text{Then, } f'(x) = \underset{h \to 0}{Lt} \frac{\cos(x+h) - \cos x}{h}$$

$$= \underset{h \to 0}{Lt} \frac{-2 \sin \frac{1}{2}h \sin(x + \frac{1}{2}h)}{h}$$

$$= \underset{h \to 0}{Lt} -\sin(x + \tfrac{1}{2}h) \cdot \frac{\sin \frac{1}{2}h}{\frac{1}{2}h}$$

$$= -\sin x \; [\textit{ as in (v) }].$$

Thus, $$\mathbf{\frac{d}{dx}(\cos x) = -\sin x.}$$

Note. It should be noted that in finding the above differential coefficients of sin x and cos x, we tacitly assume that *x is in radian measure,* because we make use of the limit $\sin \frac{1}{2}h / \frac{1}{2}h = 1$ as $h \to 0$, which is true when h is in radian. Hence, the above results require modification when x is given in any other measure.

(vii) Differential coefficient of **tan x.**

Let $f(x) = \tan x$.

Then, $$f'(x) = \underset{h\to 0}{Lt} \frac{\tan(x+h) - \tan x}{h}$$

$$= \underset{h\to 0}{Lt} \frac{1}{h}\left\{\frac{\sin(x+h)}{\cos(x+h)} - \frac{\sin x}{\cos x}\right\}$$

$$= \underset{h\to 0}{Lt} \frac{\sin(x+h-x)}{h\cos(x+h)\cos x}$$

$$= \underset{h\to 0}{Lt} \frac{\sin h}{h} \cdot \frac{1}{\cos(x+h)\cos x}$$

$$= \frac{1}{\cos^2 x}, \quad [\, x \neq \tfrac{1}{2}(2n+1)\pi \,]$$

$\because$ $\underset{h\to 0}{Lt} (\sin h/h) = 1$, and $\underset{h\to 0}{Lt} \cos(x+h) = \cos x$.

Thus, $$\frac{\mathbf{d}}{\mathbf{dx}}(\mathbf{tan\ x}) = \mathbf{sec^2 x}. \quad [\, x \neq \tfrac{1}{2}(2n+1)\pi \,]$$

(viii) Exactly in a similar way, we can get

$$\frac{\mathbf{d}}{\mathbf{dx}}(\mathbf{cot\ x}) = -\mathbf{cosec^2 x}. \quad [\, x \neq n\pi \,]$$

(ix) Differential coefficient of **sec x.**

$$\frac{d}{dx}(\sec x) = \underset{h\to 0}{Lt} \frac{\sec(x+h) - \sec x}{h}$$

$$= \underset{h\to 0}{Lt} \frac{1}{h}\left\{\frac{1}{\cos(x+h)} - \frac{1}{\cos x}\right\}$$

$$= \underset{h\to 0}{Lt} \frac{\cos x - \cos(x+h)}{h\cos(x+h)\cos x}$$

$$= \underset{h\to 0}{Lt} \frac{2\sin\tfrac{1}{2}h\sin(x+\tfrac{1}{2}h)}{h\cos(x+h)\cos x}$$

$$= \underset{h\to 0}{Lt}\ \frac{\sin \frac{1}{2}h}{\frac{1}{2}h}\cdot \sin\ (x+\tfrac{1}{2}h)\cdot \frac{1}{\cos\ (x+h)\cos x}$$

$$= 1.\sin x \cdot \frac{1}{\cos^2 x} = \tan x \sec x,$$

since $\underset{h\to 0}{Lt}\ (\sin \frac{1}{2}h/\frac{1}{2}h)=1,\ \underset{h\to 0}{Lt}\ \sin\ (x+\frac{1}{2}h)=\sin x$

and $\underset{h\to 0}{Lt}\ \cos\ (x+h)=\cos x.$

Thus, $\dfrac{\mathbf{d}}{\mathbf{dx}}\ \mathbf{(sec\ x) = sec\ x\ tan\ x.}$ $[\ x \neq \frac{1}{2}(2n+1)\pi\ ]$

(x) Proceeding exactly in a similar way, we get

$$\frac{\mathbf{d}}{\mathbf{dx}}\ \mathbf{(cosec\ x) = -\,cosec\ x\ cot\ x.} \quad [\ x \neq n\pi\ ]$$

Note. For an alternative method of differentiating $\tan x$, $\cot x$, $\sec x$ and $\operatorname{cosec} x$ from a knowledge of the derivatives of $\sin x$ and $\cos x$, see § 4·4, Theorem V.

4·4. Fundamental Theorems on Differentiation.

In the following theorems we assume that $\phi(x)$ and $\psi(x)$ are continuous, and $\phi'(x)$ and $\psi'(x)$ exist.

Theorem I. *The differential coefficient of a constant is zero,*

i.e., $\dfrac{\mathbf{d}}{\mathbf{dx}}\ \mathbf{(c) = 0,}$ where c is a constant.

Let $f(x)=c$ for every value of x.

Then, $f'(x) = \underset{h\to 0}{Lt}\ \dfrac{f(x+h)-f(x)}{h}$

$$= \underset{h\to 0}{Lt}\ \frac{c-c}{h} = \underset{h\to 0}{Lt}\ \frac{0}{h} = 0.$$

Theorem II. *The differential coefficient of the product of a constant and a function is the product of the constant and the differential coefficient of the function,*

i.e., $\dfrac{\mathbf{d}}{\mathbf{dx}}\{\mathbf{c}\phi(\mathbf{x})\} = \mathbf{c}\dfrac{\mathbf{d}}{\mathbf{dx}}\,\phi(\mathbf{x})$, where c is a constant.

For, $$\frac{d}{dx}\{c\phi(x)\} = \underset{h\to0}{Lt}\,\frac{c\phi(x+h)-c\phi(x)}{h}$$
$$= c.\underset{h\to0}{Lt}\,\frac{\phi(x+h)-\phi(x)}{h} = c\phi'(x).$$

Theorem III. *The differential coefficient of the sum or difference of two functions is the sum or difference of their derivatives,*

i.e., $\dfrac{\mathbf{d}}{\mathbf{dx}}\{\Phi(\mathbf{x}) \pm \Psi(\mathbf{x})\} = \Phi'(\mathbf{x}) \pm \Psi'(\mathbf{x})$.

Let $f(x) = \phi(x) + \psi(x)$.

Then, $f(x+h) = \phi(x+h) + \psi(x+h)$.

Now, $$f'(x) = \underset{h\to0}{Lt}\,\frac{f(x+h)-f(x)}{h}$$
$$= \underset{h\to0}{Lt}\,\frac{\{\phi(x+h)+\psi(x+h)\}-\{\phi(x)+\psi(x)\}}{h}$$
$$= \underset{h\to0}{Lt}\left[\frac{\phi(x+h)-\phi(x)}{h} + \frac{\psi(x+h)-\psi(x)}{h}\right]$$
$$= \underset{h\to0}{Lt}\,\frac{\phi(x+h)-\phi(x)}{h} + \underset{h\to0}{Lt}\,\frac{\psi(x+h)-\psi(x)}{h}$$
$$= \phi'(x) + \psi'(x).$$

Similarly, if $f(x) = \phi(x) - \psi(x)$, then $f'(x) = \phi'(x) - \psi'(x)$.

Note. The above result can be easily generalized to the case of the *sum or difference of any finite number of functions.*

Illus : If $f(x) = e^x - 4\sin x + x^2 + 5$, then $f'(x) = e^x - 4\cos x + 2x$.

(3) If $f(x) = \text{cosec } x = \dfrac{1}{\sin x}$,

then $f'(x) = \dfrac{0.\sin x - \cos x.1}{\sin^2 x} = -\dfrac{\cos x}{\sin^2 x} = -\text{cosec } x \cot x.$

4·5. Illustrative Examples.

Ex. 1. *Find from first principles the derivative of $\sqrt{x}$, $(x > 0)$.*

Let $f(x) = \sqrt{x}$.

$\therefore$ $$f'(x) = \underset{h\to 0}{Lt} \frac{\sqrt{(x+h)} - \sqrt{x}}{h}, \text{ [by definition]}$$

$$= \underset{h\to 0}{Lt} \frac{(x+h)-x}{h\{\sqrt{(x+h)}+\sqrt{x}\}} = \underset{h\to 0}{Lt} \frac{1}{\sqrt{(x+h)}+\sqrt{x}} = \frac{1}{2\sqrt{x}}.$$

Ex. 2. *Find from first principles the differential coefficient of $\tan^{-1}x$.*

Let $\tan^{-1}x = y$ and $\tan^{-1}(x+h) = y+k$.

Then, as $h \to 0$, $k \to 0$. Also $x = \tan y$, $x+h = \tan(y+k)$.

$\therefore$ $h = (x+h) - x = \tan(y+k) - \tan y$.

$\therefore$ $$\frac{d}{dx}(\tan^{-1}x) = \underset{h\to 0}{Lt} \frac{\tan^{-1}(x+h) - \tan^{-1}x}{h}$$

$$= \underset{k\to 0}{Lt} \frac{k}{\tan(y+k) - \tan y}$$

$$= \underset{k\to 0}{Lt} \frac{k}{\sin k} \cdot \cos(y+k) \cos y$$

$$= \cos^2 y = \frac{1}{\sec^2 y} = \frac{1}{1+\tan^2 y} = \frac{1}{1+x^2}.$$

Note. In a similar way we can work out the derivatives of the other inverse circular functions from first principles.

These have however been worked out by a different method in § 4·8.

Ex. 3. *Find from definition the differential coefficient of log cos x.* [*C. P. 1933*]

Let us put $\cos x = u$, $\cos(x+h) = u+k$.

$\therefore$ $k = \cos(x+h) - \cos x$ and so, when $h \to 0$, $k \to 0$.

$$\frac{d}{dx}(\log\cos x)=\underset{h\to 0}{Lt}\ \frac{\log\cos(x+h)-\log\cos x}{h}$$

$$=\underset{h\to 0}{Lt}\ \frac{\log(u+k)-\log u}{k}\cdot\frac{k}{h}$$

$$=\underset{h\to 0}{Lt}\ \frac{\log(1+k/u)}{k/u}\cdot\frac{1}{u}\cdot\frac{k}{h}.$$

As $k\to 0$, $k/u\to 0$. $\therefore$ limit of 1st factor $=1$. [See § *2·9 (iii)*]

Again, $\dfrac{k}{h}=\dfrac{\cos(x+h)-\cos x}{h}=-\dfrac{\sin(x+\frac{1}{2}h).\sin\frac{1}{2}h}{\frac{1}{2}h}.$

$\therefore$ as $k\to 0$ i.e., $h\to 0$, $k/h\to -\sin x$. Also, $u=\cos x$.

$\therefore$ $\dfrac{d}{dx}(\log\cos x)=\dfrac{-\sin x}{\cos x}=-\tan x.$

Note. Differentiation from 'first principles' or 'definition' means that we are to find out the derivative without assuming any of the rules of differentiation, or the derivative of any standard function, but we are permitted to use fundamental rules of limiting operations (§ *2·8*) and the standard limit results (§ *2·9*).

Ex. 4. *A function is defined in the following way :*

$f(x)=|x|$, i.e., $f(x)=x$, 0, or, $-x$, according as $x >\ = or < 0$; show that $f'(0)$ does not exist.

$$f'(0)=\underset{h\to 0}{Lt}\ \frac{f(0+h)-f(0)}{h}=\underset{h\to 0}{Lt}\ \frac{f(h)}{h}.$$

Now, $\underset{h\to 0+0}{Lt}\ \dfrac{f(h)}{h}=\underset{h\to 0+0}{Lt}\ \dfrac{h}{h}=1,$

and $\underset{h\to 0-0}{Lt}\ \dfrac{f(h)}{h}=\underset{h\to 0-0}{Lt}\ \dfrac{-h}{h}=-1.$

Since the right-hand derivative is not equal to the left-hand derivative, the derivative at $x=0$ does not exist.

Ex. 5. *A function is defined in the following way :*

$$f(x)=x\ \sin\frac{1}{x}\ \textit{for}\ x\neq 0,\ f(0)=0.$$

Show that $f'(0)$ does not exist.

$$f'(0) = \underset{h \to 0}{Lt} \frac{f(0+h) - f(0)}{h} = \underset{h \to 0}{Lt} \frac{h \sin(1/h)}{h}$$

$$= \underset{h \to 0}{Lt} \sin(1/h), \text{ which does not exist.}$$

[See § *2'11, Ex. 4*]

$\therefore$ $f'(0)$ does not exist.

Note. In both the examples 4 and 5, $f(x)$ is continuous at $x=0$ (See § *3'6, Ex. 1 and Ex. 2*) but $f(x)$ does not possess derivative at $x=0$.

Ex. 6. *If* $f(x) = x^2 \sin(1/x)$ *when* $x \neq 0$, *and* $f(0)=0$, *find* $f'(0)$.

$$f'(0) = \underset{h \to 0}{Lt} \frac{f(0+h) - f(0)}{h} = \underset{h \to 0}{Lt} \frac{1}{h}\left\{ h^2 \sin\frac{1}{h} - 0 \right\}$$

$$= \underset{h \to 0}{Lt} \left(h \sin\frac{1}{h} \right) = 0.$$

[$\because$ when h is not exactly zero, $\sin \dfrac{1}{h}$ is finite, not exceeding 1 numerically.]

Ex. 7. *Find from first principles the derivative of* x^x, ($x > 0$).

Let $f(x) = x^x = e^{x \log x}$

$$\therefore \quad f'(x) = \underset{h \to 0}{Lt} \frac{e^{(x+h)\log(x+h)} - e^{x \log x}}{h}$$

$$= \underset{h \to 0}{Lt}\ e^{x \log x}\ \frac{e^{(x+h)\log(x+h) - x\log x} - 1}{h}$$

$$= e^{x \log x} \underset{h \to 0}{Lt} \frac{e^z - 1}{z} \cdot \frac{z}{h}$$

where $z = (x+h)\log(x+h) - x \log x$

and hence $z \to 0$, as $h \to 0$.

$$\therefore \quad f'(x) = x^x \underset{z \to 0}{Lt} \frac{e^z - 1}{z} \cdot \underset{h \to 0}{Lt} \frac{z}{h} = x^x \underset{h \to 0}{Lt} \frac{z}{h}, \quad \because \underset{z \to 0}{Lt} \frac{e^z - 1}{z} = 1.$$

Now, $$\underset{h \to 0}{Lt} \frac{z}{h} = \underset{h \to 0}{Lt} \frac{x\{\log(x+h) - \log x\} + h \log(x+h)}{h}$$

$$= \underset{h \to 0}{Lt} \frac{x}{h} \log\left(1 + \frac{h}{x}\right) + \underset{h \to 0}{Lt} \log(x+h)$$

$$= \underset{k \to 0}{Lt} \frac{1}{k} \log(1+k) + \log x = 1 + \log x,$$

where k being $h/x \to 0$, as $h \to 0$.

$$\therefore \qquad f'(x) = x^x (1 + \log x).$$

Examples IV(A)

Find from *first principles* the derivatives of (*Ex. 1-5*):

1. (i) x^3+2x. (ii) x^4+6. (iii) $1/x$ ($x \neq 0$).
(iv) $1/\sqrt{x}$ ($x>0$). (v) $\sqrt[3]{x}$. (vi) $\sqrt{x^2+a^2}$.
(vii) $x+\sqrt{x^2+1}$.

2. (i) $e^{\sqrt{x}}$. (ii) $e^{\sin x}$. (iii) 2^{x^2}. (iv) e^x/x.

3. (i) $\log_{10}x$. [*C. P. 1941*] (ii) $x \log x$.
(iii) $\log \sin (x/a)$. [*C. P. 1930*] (iv) $\log \sec x$.

4. (i) $a \sin (x/a)$. [*C. P. 1937*] (ii) $\sin^2 x$.
(iii) $\sin x^2$. (iv) $\sin^{-1}x$. (v) $\sqrt{\tan x}$.
(vi) $(\sin x)/x$. (vii) $x^2 \tan x$.

5. (i) $e^{\cos x}$ at $x=0$. (ii) $\log \cos x$ at $x=0$.

6. (i) $f(x)=x^2 \cos (1/x)$ for $x \neq 0$, $f(0)=0$.
Find $f'(0)$.

(ii) $f(x)=x$ for $0 \leqslant x \leqslant \frac{1}{2}$; $f(x)=1-x$ for $\frac{1}{2}<x\leqslant 1$. Does $f'(\frac{1}{2})$ exist ?

7. (i)
$$\begin{aligned} f(x) &= 3+2x \text{ for } -\tfrac{3}{2}<x\leqslant 0 \\ &= 3-2x \text{ for } \quad 0<x<\tfrac{3}{2} \end{aligned}$$

Show that $f(x)$ is continuous at $x=0$ but $f'(0)$ does not exist. [*C. P. 1943*]

(ii) $f(x)=0$ when $0\leqslant x<\frac{1}{2}$; $f(\frac{1}{2})=1$; $f(x)=2$ when $\frac{1}{2}<x\leqslant 1$.

Prove that although $f(x)$ is discontinuous at $x=\frac{1}{2}$, $f'(\frac{1}{2})$ exists and its value is infinite.

8.
$$\begin{aligned} f(x) &= 1 && \text{for } x<0 \\ &= 1+\sin x && \text{for } 0\leqslant x<\tfrac{1}{2}\pi \\ &= 2+(x-\tfrac{1}{2}\pi)^2 && \text{for } \tfrac{1}{2}\pi \leqslant x\,; \end{aligned}$$

show that $f'(x)$ exists at $x=\frac{1}{2}\pi$ but does not exist at $x=0$.

9. $f(x) = 5x - 4 \quad \text{for } 0 < x \leqslant 1$

$\quad = 4x^2 - 3x \quad \text{for } 1 < x < 2$

$\quad = 3x + 4 \quad \text{for } x \geqslant 2.$

Discuss the continuity of $f(x)$ for $x = 1$ and 2, and the existence of $f'(x)$ for these values.

10. (i) $f(x) = x \quad \text{for } 0 < x < 1$

$\quad = 2 - x \quad \text{for } 1 \leqslant x \leqslant 2$

$\quad = x - \frac{1}{2}x^2 \quad \text{for } x > 2.$

Is $f(x)$ continuous at $x = 1$ and 2 ? Does $f'(x)$ exist for these values ?

(ii) $\phi(x) = \frac{1}{2}(b^2 - a^2), \quad 0 \leqslant x \leqslant a$

$\quad = \frac{1}{2}b^2 - \frac{1}{6}x^2 - \frac{1}{3}(a^3/x), \; a < x \leqslant b$

$\quad = \frac{1}{3}(b^3 - a^3)/x \quad x > b\,;$

show that $\phi'(x)$ is continuous for every positive value of x.

[*C. P. 1944*]

Find the differential coefficients of the following with respect to x (*Ex. 11-13*) ·

11. (i) $3x^5 + 7x^4 - 2x^2 - x + 6.$ (ii) $(x^2 - 3)^3.$

(iii) $1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!}.$ (iv) $(x + 2)(x + 1)^2.$

(v) $(3x^6 + 4x^2 - 2)/x^3.$ (vi) $(1 + x)^3/x.$

(vii) $6x^{-2} - 3x^{-1} + 4.$ (viii) $4x^{-\frac{3}{4}} + 6x^{\frac{1}{2}} + 2.$

(ix) $2x^4 + 5x^2 - 4 - \frac{1}{x} + \frac{2}{x^2} + \frac{3}{x^3}.$

(x) $\sqrt{x} + 2\sqrt{(x^2)} + 3\sqrt{(x^3)} + 4\sqrt{(x^4)} + 5\sqrt{(x^5)}.$

(xi) $x\sqrt{x} + x^2\sqrt{x} + \frac{x^2}{\sqrt{x}} - \sqrt{x} + \frac{1}{\sqrt{x}}.$

(xii) $\dfrac{5x^3}{\sqrt[5]{x^2}} - \dfrac{3x}{\sqrt[3]{x^4}} + \dfrac{7x}{\sqrt[7]{x^2}} + 12\dfrac{\sqrt[4]{x}}{\sqrt[3]{x}}$.

(xiii) $2 \sin x - \frac{1}{3} \log x - \frac{1}{2}e^x - 6 \tan x - 7 \operatorname{cosec} x$.

(xiv) $\log_a x + \log x^a + e^{\log x} + \log e^x + e^{1+x}$.

12. (i) $x^n e^x$. (ii) $x^2 \log x$. (iii) $x^2 \log x^2$.

(iv) $e^x \sin x$. (v) $2^x \sin x$. (vi) $10^x.x^{10}$.

(vii) $\cos^2 x$. (viii) $\sec x \tan x$. (ix) $(x^2+1) \sin x$.

(x) $(3x-7)(3-7x)$ (xi) $(x^2+7)(x^3+10)$.

(xii) $(\sin x + \sec x + \tan x)(\operatorname{cosec} x + \cos x + \cot x)$.

(xiii) $\operatorname{cosec}^3 x$. (xiv) $x \tan x \log x$.

(xv) $\sqrt{x}.e^x \sec x$. (xvi) $(1+x)(1+2x)(1+3x)$.

(xvii) $x(1-x)(1-x^2)$. (xviii) $x \sec x \log (xe^x)$.

(xix) $x \cot x \log (x^x).e^x$.

(xx) $\cot x \times \log x \times 10^x \times \sqrt{x}$.

13. (i) $\dfrac{\sin x}{\cos x}$. (ii) $\dfrac{1}{\cos x}$. (iii) $\dfrac{\sin x}{x}$.

(iv) $\dfrac{x^4}{\sin x}$. [*C. P. 1940*] (v) $\dfrac{\cot x}{e^x}$.

(vi) $\dfrac{x^n}{\log x}$. (vii) $\dfrac{x}{e^x - 1}$. (viii) $\dfrac{1+x}{1-x}$.

(ix) $\dfrac{1+x^2}{1-x^2}$. (x) $\dfrac{1+\sqrt{x}}{1-\sqrt{x}}$. (xi) $\dfrac{1+\sin x}{1-\sin x}$.

(xii) $\dfrac{1-\cos x}{1+\cos x}$. (xiii) $\dfrac{\sin x + \cos x}{\sqrt{1+\sin 2x}}$. (xiv) $\dfrac{\cos x - \cos 2x}{1-\cos x}$.

(xv) $x \cdot \dfrac{e^x + e^{3x}}{e^x + e^{-x}}$. (xvi) $\dfrac{\tan x}{x} \cdot \log \left(\dfrac{e^x}{x^x}\right)$.

(xvii) $\dfrac{\sin x + \cos x}{\sin x - \cos x}$. (xviii) $\dfrac{\cot x + \operatorname{cosec} x}{\cot x - \operatorname{cosec} x}$.

(xix) $\dfrac{1+x+x^2}{1-x+x^2}$. (xx) $\dfrac{x^3-2+x^{-3}}{x-2+x^{-1}}$.

(xxi) $\dfrac{\tan x}{x} \cdot e^x \log x$. (xxii) $\dfrac{\sin x - \cos x}{\sin x + \cos x} x^2 e^x$.

14. If $y = \sqrt{2x} - \sqrt{\dfrac{2}{x}} + \dfrac{x+4}{4-x}$, find $\dfrac{dy}{dx}$ for $x=2$.

15. If $f(x) = \dfrac{x^3 - 8x^2 + 13x - 6}{x^2 - 11x + 10}$, find the values of x for which $f'(x) = 0$.

Is there any value of x for which $f'(x)$ is non-existent ?

16. From the relation

$$1 + x + x^2 + \cdots + x^n = \frac{1-x^{n+1}}{1-x}$$

deduce the sum of the series

$$1 + 2x + 3x^2 + \cdots + nx^{n-1},$$

and hence, show that

$$1 + 2x + 3x^2 + \cdots \text{ to } \infty = (1-x)^{-2},\ 0 < |x| < 1.$$

17. If
$$\begin{aligned} f(x) &= 1 + x && \text{for } x < 0 \\ &= 1 && \text{for } 0 \leqslant x \leqslant 1 \\ &= 2x^2 + 4x + 5 && \text{for } x > 1, \end{aligned}$$

find $f'(x)$ for all values of x for which it exists

Does $\underset{x\to 0}{Lt} f'(x)$ exist ? Does $f'(x)$ exist ?

18. (i) If $f(x) = -\frac{1}{2}x^2$ for $x \leqslant 0$

and $f(x) = x^n \sin(1/x)$ for $x > 0$,

find whether $f'(0)$ exists for $n = 1$ and 2.

(ii) If $f(x) = [x]$ where $[x]$ denotes the greatest integer not exceeding x, find $f'(x)$ and draw its graph.

ANSWERS

1. (i) $3x^2+2$. (ii) $4x^3$. (iii) $-1/x^2$. (iv) $-\frac{1}{2}x^{-\frac{3}{2}}$.

(v) $\frac{1}{3}x^{-\frac{2}{3}}$. (vi) $\dfrac{x}{\sqrt{(x^2+a^2)}}$. (vii) $\dfrac{\{x+\sqrt{(x^2+1)}\}}{\sqrt{(x^2+1)}}$.

2. (i) $e^{\sqrt{x}}/2\sqrt{x}$. (ii) $e^{\sin x}.\cos x$. (iii) $2^{x^2}.\log_e 2.2x$.

(iv) $(xe^x-e^x)/x^2$. **3.** (i) $x^{-1}\log_{10} e$. (ii) $1+\log x$.

(iii) $a^{-1}\cot(x/a)$. (iv) $\tan x$ **4.** (i) $\cos(x/a)$.

(ii) $\sin 2x$. (iii) $2x\cos x^2$. (iv) $1/\sqrt{(1-x^2)}$.

(v) $\dfrac{\sec^2 x}{2\sqrt{(\tan x)}}$. (vi) $\dfrac{x\cos x-\sin x}{x^2}$. (vii) $2x\tan x+x^2\sec^2 x$.

5. (i) 0. (ii) 0. **6.** (i) 0. (ii) No.

9. Continuous for $x=1$ and 2, but $f'(x)$ exists for $x=1$ and does not exist for $x=2$. **10.** (i) Continuous at $x=1$ and 2 ; $f'(x)$ does not exist for $x=1$, but exists at $x=2$.

11. (i) $15x^4+28x^3-4x-1$. (ii) $6x^5-36x^3+54x$. (iii) $1+x+\dfrac{x^2}{2!}+\dfrac{x^3}{3!}$.

(iv) $3x^2+8x+5$. (v) $9x^2-4x^{-2}+6x^{-4}$. (vi) $-x^{-2}+3+2x$.

(vii) $-12x^{-3}+3x^{-2}$. (viii) $-3x^{-\frac{7}{4}}+3x^{-\frac{1}{2}}$.

(ix) $8x^3+10x+x^{-2}-4x^{-3}-9x^{-4}$. (x) $\dfrac{1}{2\sqrt{x}}+2+\dfrac{9}{2}x^{\frac{1}{2}}+8x+\dfrac{25}{2}x^{\frac{3}{2}}$.

(xi) $\frac{3}{2}x^{\frac{1}{2}}+\frac{5}{2}x^{\frac{3}{2}}+\frac{3}{2}x^{\frac{1}{2}}-\frac{1}{2}x^{-\frac{1}{2}}-\frac{1}{2}x^{-\frac{3}{2}}$. (xii) $13x^{\frac{8}{5}}+x^{-\frac{4}{3}}+5x^{-\frac{2}{7}}-x^{-\frac{13}{12}}$.

(xiii) $2\cos x-\frac{1}{5}x^{-1}-\frac{1}{2}e^x-6\sec^2 x+7\operatorname{cosec} x\cot x$.

(xiv) $x^{-1}\log_a e+ax^{-1}+2+e^{1+x}$.

12. (i) $(x^n+nx^{n-1})e^x$. (ii) $x+2x\log x$.

(iii) $2(x+2x\log x)$. (iv) $e^x(\cos x+\sin x)$.

(v) $2^x(\cos x+\sin x.\log 2)$. (vi) $10^x(10x^9+x^{10}\log 10)$.

(vii) $-\sin 2x$. (viii) $\sec x(\sec^2 x+\tan^2 x)$.

(ix) $(x^2+1)\cos x+2x\sin x$. (x) $-42x+58$.

(xi) $5x^4-21x^2+20x$. (xii) $-(\sin x+\sec x+\tan x)$ $\times$ $(\operatorname{cosec} x\cot x+\sin x+\operatorname{cosec}^2 x)$ $+$ $(\operatorname{cosec} x+\cos x+\cot x)$ $\times$ $(\cos x+\sec x\tan x+\sec^2 x)$. (xiii) $-3\operatorname{cosec}^3 x\cot x$.

(xiv) $\tan x\log x+x\log x\sec^2 x+\tan x$.

(xv) $e^x\sec x\,(1+2x+2x\tan x)/2\sqrt{x}$. (xvi) $18x^2+22x+6$.

(xvii) $4x^3-3x^2-2x+1$. (xviii) $\sec x\,\{1+x+(1+x\tan x)(x+\log x)\}$.

(xix) $e^x\,\{x\cot x\,(1+2\log x+x\log x)-x^2\operatorname{cosec}^2 x\log x\}$.

(xx) $10^x\cot x\,[(-2\sqrt{x}\operatorname{cosec} 2x+\sqrt{x}\log 10+\frac{1}{2}x^{-\frac{1}{2}})\log x+x^{-\frac{1}{2}}]$.

13. (i) $\sec^2 x$. (ii) $\sec x\tan x$. (iii) $(x\cos x-\sin x)/x^2$.

(iv) $x^3\,(4\sin x-x\cos x)/\sin^2 x$.

(v) $-e^{-x}\,(\operatorname{cosec}^2 x+\cot x)$. (vi) $x^{n-1}\,(n\log x-1)/(\log x)^2$.

(vii) $\dfrac{e^x(1-x)-1}{(e^x-1)^2}$. (viii) $\dfrac{2}{(1-x)^2}$. (ix) $\dfrac{4x}{(1-x^2)^2}$.

(x) $\dfrac{1}{\sqrt{x}(1-\sqrt{x})^2}$. (xi) $\dfrac{2\cos x}{(1-\sin x)^2}$. (xii) $\dfrac{2\sin x}{(1+\cos x)^2}$.

(xiii) 0. (xiv) $-2\sin x$. (xv) $e^{2x}\,(1+2x)$.

(xvi) $-x^{-1}\tan x+(1-\log x)\sec^2 x$. (xvii) $\dfrac{-2}{(\sin x-\cos x)^2}$.

(xviii) $\dfrac{2\operatorname{cosec} x}{(\cot x-\operatorname{cosec} x)^2}$. (xix) $\dfrac{2(1-x^2)}{(1-x+x^2)^2}$.

(xx) $2(x+1-x^{-2}-x^{-3})$.

(xxi) $\dfrac{e^x}{x^2}\,[\{x\sec^2 x+(x-1)\tan x\}\log x+\tan x\,]$.

(xxii) $e^x\left\{\dfrac{2x^2-(x^2+2x)\cos 2x}{(\sin x+\cos x)^2}\right\}$.

14. $2\frac{2}{3}$. **15.** 4, 16 ; non-existent at 1, 10.

16. $\dfrac{1-(n+1)x^n+nx^{n+1}}{(1-x)^2}$.

17. 1 if $x<0$, 0 if $0<x<1$, $4(x+1)$ if $x>1$; No ; No.

18. (i) No, Yes. (ii) $f'(x)=0$ for all values of x except zero and integral values, for which it does not exist.

4·6. Differentiation of a Function of a Function.

Let $y=f(v)$, where $v=\phi(x)$, and $f(v)$ and $\phi(x)$ are continuous. Thus y is also a continuous function of x.*

Let $f'(v)$ and $\phi'(x)$ exist, and be finite.

Assume $v+\Delta v=\phi(x+\Delta x)$ and $y+\Delta y=f(v+\Delta v)$.

It is evident that when $\Delta x\to 0$, $\Delta v\to 0$, and as $\Delta v\to 0$, $\Delta y\to 0$.

Now, $$\frac{\Delta y}{\Delta x}=\frac{\Delta y}{\Delta v}\cdot\frac{\Delta v}{\Delta x}. \qquad [\Delta v\neq 0]$$

$$\therefore \quad \underset{\Delta x\to 0}{Lt}\frac{\Delta y}{\Delta x}=\underset{\Delta v\to 0}{Lt}\frac{\Delta y}{\Delta v}\cdot\underset{\Delta x\to 0}{Lt}\frac{\Delta v}{\Delta x}$$

$$\text{i.e.,}\quad \mathbf{\frac{dy}{dx}=\frac{dy}{dv}\cdot\frac{dv}{dx}}$$

If $\Delta v=0$, then $\Delta y=0$. [Otherwise $\dfrac{dy}{dx}$ i.e., $f'(v)$ would not be finite]. $\therefore$ $\Delta y/\Delta x=0$ [$\because$ $\Delta x\neq 0$].

Hence, $\dfrac{dy}{dx}=\underset{\Delta x\to 0}{Lt}\dfrac{\Delta y}{\Delta x}=0$. Similarly, $\dfrac{dv}{dx}=0$.

Hence, the above relation is true in this case also.

Illus: Suppose $y=\sin x^2$; then we can write
$y=\sin v$, where $v=x^2$.

$$\therefore \quad \frac{dy}{dx}=\frac{dy}{dv}\cdot\frac{dv}{dx}=\cos v\,.\,2x=2x\cos x^2.$$

The above rule can easily be generalized.

Thus, if $y=f(v)$, where $v=\phi(w)$, and $w=\psi(x)$,

then $\dfrac{dy}{dx}=\dfrac{dy}{dv}\cdot\dfrac{dv}{dw}\cdot\dfrac{dw}{dx}$; and so on

4·7. $\mathbf{\dfrac{dy}{dx}\times\dfrac{dx}{dy}=1}$, *i.e.*, $\mathbf{\dfrac{dy}{dx}=1\Big/\dfrac{dx}{dy}}$, provided neither derivative is zero.

* Proof depends on the corresponding limit theorem, see § 2·8 (iv).

Suppose $y=f(x)$, where $f(x)$ is continuous. From this, in most cases, we can treat x as a function of y.

Let $y+\Delta y=f(x+\Delta x)$.

It is evident that when $\Delta y \to 0$, $\Delta x \to 0$.

Now, $\dfrac{\Delta y}{\Delta x} \times \dfrac{\Delta x}{\Delta y}=1$. $\therefore$ $\dfrac{\Delta y}{\Delta x}=1\Big/\dfrac{\Delta x}{\Delta y}$.

$$\therefore \quad \underset{\Delta x\to 0}{Lt}\frac{\Delta y}{\Delta x}=\underset{\Delta y\to 0}{Lt}\left\{1\Big/\frac{\Delta x}{\Delta y}\right\},$$

$$i.e., \quad \frac{dy}{dx}=1\Big/\frac{dx}{dy}, \text{ or, } \frac{dy}{dx}\times\frac{dx}{dy}=1.$$

4·8. Differential Coefficients of Inverse Circular Functions.

(i) Let $y=\sin^{-1}x$. $[\,|x|\leqslant 1\,]$*. $\therefore$ $x=\sin y$.

$$\therefore \quad \frac{dx}{dy}=\cos y=\sqrt{1-\sin^2 y}=\sqrt{1-x^2}.$$

$$\therefore \quad \text{for } x\neq 1, \text{ or } -1, \ \frac{dy}{dx}=1\Big/\frac{dx}{dy}=\frac{1}{\sqrt{1-x^2}}.$$

Thus, $\dfrac{\mathbf{d}}{\mathbf{dx}}(\mathbf{\sin^{-1}x})=\dfrac{\mathbf{1}}{\sqrt{\mathbf{1-x^2}}}$. $[\,-1<x<1\,]$

(ii) Let $y=\cos^{-1}x$. $[\,|x|\leqslant 1\,]$* $\therefore$ $x=\cos y$.

$$\therefore \quad \frac{dx}{dy}=-\sin y=-\sqrt{1-\cos^2 y}=-\sqrt{1-x^2}.$$

$$\therefore \quad \text{for } x\neq 1, \text{ or, } -1, \ \frac{dy}{dx}=1\Big/\frac{dx}{dy}=-\frac{1}{\sqrt{1-x^2}}.$$

Thus, $\dfrac{\mathbf{d}}{\mathbf{dx}}(\mathbf{\cos^{-1}x})=-\dfrac{\mathbf{1}}{\sqrt{\mathbf{1-x^2}}}$. $[\,-1<x<1\,]$

Note. This also follows immediately from the relation
$\cos^{-1}x=\frac{1}{2}\pi-\sin^{-1}x$.

* The domain for which y exists.

(iii) Let $y = \tan^{-1}x$. $\quad\therefore\quad x = \tan y$.

$$\therefore \quad \frac{dx}{dy} = \sec^2 y = 1 + \tan^2 y = 1 + x^2.$$

$$\therefore \quad \frac{dy}{dx} = 1\Big/\frac{dx}{dy} = \frac{1}{1+x^2}.$$

Thus, $\dfrac{\mathbf{d}}{\mathbf{dx}}(\mathbf{\tan^{-1}x}) = \dfrac{\mathbf{1}}{\mathbf{1+x^2}}$.

(iv) Let $y = \cot^{-1}x$. $\quad\therefore\quad x = \cot y$.

$$\therefore \quad \frac{dx}{dy} = -\operatorname{cosec}^2 y = -(1 + \cot^2 y) = -(1 + x^2).$$

$$\therefore \quad \frac{dy}{dx} = 1\Big/\frac{dx}{dy} = -\frac{1}{1+x^2}.$$

Thus, $\dfrac{\mathbf{d}}{\mathbf{dx}}(\mathbf{\cot^{-1}x}) = -\dfrac{\mathbf{1}}{\mathbf{1+x^2}}$.

Note. This also follows at once from the relation

$$\cot^{-1}x = \tfrac{1}{2}\pi - \tan^{-1}x.$$

(v) Let $y = \sec^{-1}x$. $[\ |x| \geqslant 1\ ]$* $\quad\therefore\quad x = \sec y$.

$$\therefore \quad \frac{dx}{dy} = \sec y \tan y = \sec y\sqrt{\sec^2 y - 1} = x\sqrt{x^2-1}.$$

$$\therefore \quad \text{for } x \neq 1, \text{ or, } -1,\ \frac{dy}{dx} = 1\Big/\frac{dx}{dy} = \frac{1}{x\sqrt{x^2-1}}.$$

Thus, $\dfrac{\mathbf{d}}{\mathbf{dx}}(\mathbf{\sec^{-1}x}) = \dfrac{\mathbf{1}}{\mathbf{x\sqrt{x^2-1}}}$. $\quad [\ |x| > 1\ ]$

(vi) Let $y = \operatorname{cosec}^{-1}x$. $[\ |x| \geqslant 1\ ]$* $\quad\therefore\quad x = \operatorname{cosec} y$.

$$\therefore \quad \frac{dx}{dy} = -\operatorname{cosec} y \cot y$$

$$= -\operatorname{cosec} y\sqrt{\operatorname{cosec}^2 y - 1} = -x\sqrt{x^2-1}.$$

$$\therefore \quad \text{for } x \neq 1, \text{ or, } -1,\ \frac{dy}{dx} = 1\Big/\frac{dx}{dy} = -\frac{1}{x\sqrt{x^2-1}}.$$

* **For which y exists.**

Thus, $\dfrac{\mathbf{d}}{\mathbf{dx}}(\mathbf{cosec^{-1}x}) = -\dfrac{\mathbf{1}}{\mathbf{x\sqrt{x^2-1}}}$. $[\,|x|>1\,]$

Note. This also follows at once from the relation

$$\operatorname{cosec}^{-1}x = \tfrac{1}{2}\pi - \sec^{-1}x.$$

4·9. Derivatives of Hyperbolic Functions*.

$$\frac{\mathbf{d}}{\mathbf{dx}}(\mathbf{sinh\ x}) = \frac{d}{dx}\left(\frac{e^x - e^{-x}}{2}\right) = \frac{e^x + e^{-x}}{2} = \mathbf{cosh\ x.}$$

$$\frac{\mathbf{d}}{\mathbf{dx}}(\mathbf{cosh\ x}) = \frac{d}{dx}\left(\frac{e^x + e^{-x}}{2}\right) = \frac{e^x - e^{-x}}{2} = \mathbf{sinh\ x.}$$

$$\frac{\mathbf{d}}{\mathbf{dx}}(\mathbf{tanh\ x}) = \frac{d}{dx}\left(\frac{\sinh x}{\cosh x}\right) = \frac{\cosh^2 x - \sinh^2 x}{\cosh^2 x}$$

$$= \frac{1}{\cosh^2 x} = \mathbf{sech^2 x.}$$

Similarly, $\dfrac{\mathbf{d}}{\mathbf{dx}}(\mathbf{coth\ x}) = -\mathbf{cosech^2 x.}$

$$\frac{\mathbf{d}}{\mathbf{dx}}(\mathbf{sech\ x}) = \frac{d}{dx}\left(\frac{1}{\cosh x}\right) = \frac{0 \times \cosh x - \sinh x}{\cosh^2 x}$$

$$= -\frac{\sinh x}{\cosh^2 x} = -\mathbf{sech\ x\ tanh\ x.}$$

Similarly, $\dfrac{\mathbf{d}}{\mathbf{dx}}(\mathbf{cosech\ x}) = -\mathbf{cosech\ x\ coth\ x.}$

Let $\quad y = \sinh^{-1}x. \qquad \therefore \quad x = \sinh y.$

$$\therefore \quad \frac{dx}{dy} = \cosh y = \sqrt{1 + \sinh^2 y} = \sqrt{1 + x^2}.$$

Since, $\dfrac{dy}{dx} = 1\Big/\dfrac{dx}{dy} = \dfrac{1}{\sqrt{1+x^2}}$.

$$\frac{d}{dx}(\sinh^{-1}x) = \frac{1}{\sqrt{1+x^2}}.$$

* For the definitions and properties of Hyperbolic Functions, see Authors' *Higher Trigonometry, Chapter XII.*

Similarly, $\dfrac{d}{dx}(\cosh^{-1}x) = \dfrac{1}{\sqrt{x^2-1}}$. $\quad (x > 1)$

$$\frac{d}{dx}(\tanh^{-1}x) = \frac{1}{1-x^2}, \quad (x < 1)$$

$$\frac{d}{dx}(\coth^{-1}x) = -\frac{1}{x^2-1}, \quad (x > 1)$$

$$\frac{d}{dx}(\text{cosech}^{-1}x) = -\frac{1}{x\sqrt{x^2+1}}$$

$$\frac{d}{dx}(\text{sech}^{-1}x) = -\frac{1}{x\sqrt{1-x^2}}. \quad (x < 1)$$

The derivatives of inverse hyperbolic functions can also be obtained by differentiating their values *viz.*,

$$\sinh^{-1}x = \log(x + \sqrt{x^2+1})\ ;\ \cosh^{-1}x = \log(x + \sqrt{x^2-1}).$$

$$\tanh^{-1}x = \frac{1}{2}\log\frac{1+x}{1-x}, \qquad \coth^{-1}x = \frac{1}{2}\log\frac{x+1}{x-1},$$

$$\text{cosech}^{-1}x = \log\frac{1+\sqrt{1+x^2}}{x}\ ;\ \text{sech}^{-1}x = \log\frac{1+\sqrt{1-x^2}}{x}.$$

4·10. Logarithmic Differentiation.

If we have a function raised to a power which is also a function, or if we have the product of a number of functions, to differentiate such expressions it would be convenient first to take logarithm of the expression and then differentiate. Such a process is called the *logarithmic differentiation.*

(i) Let $y = \{f(x)\}^{\phi(x)}$; to find $\dfrac{dy}{dx}$.

Here, $\log y = \phi(x) \,.\, \log f(x)$.

Differentiating both sides with respect to x,

$$\frac{1}{y}\frac{dy}{dx} = \phi(x)\cdot\frac{1}{f(x)}\cdot f'(x) + \phi'(x)\,.\,\log f(x).$$

$$\therefore \quad \frac{dy}{dx} = \{f(x)\}^{\phi(x)}\left[\phi(x)\cdot\frac{f'(x)}{f(x)} + \phi'(x)\,.\,\log f(x)\right].$$

(ii) Let $y=f_1(x)\times f_2(x)\ldots\ldots f_n(x)$; to find $\dfrac{dy}{dx}$.

Here, $\log y=\log f_1(x)+\log f_2(x)+\cdots+\log f_n(x)$.

Differentiate each side with respect to x.

$$\therefore \quad \frac{1}{y}\frac{dy}{dx}=\frac{f'_1(x)}{f_1(x)}+\frac{f'_2(x)}{f_2(x)}+\ldots\ldots+\frac{f'_n(x)}{f_n(x)}.$$

Now, multiplying left side by y and right side by $f_1(x), f_2(x)\ldots f_n(x)$, we get

$$\frac{dy}{dx}=f'_1(x)\,.\,f_2(x)\,.\,f_3(x)\ldots f_n(x)+f'_2(x)\;f_1(x)\,.\,f_3(x)\ldots f_n(x)+\ldots$$

Hence, *differential coefficient of the product of a finite number of functions is found by multiplying the differential coefficient of each function taken separately by the product of all the remaining functions and adding up the results thus formed*, as already obtained otherwise.

[See § 4 4, *Theorem IV, Note*]

4·11. Implicit Functions.

In many cases it may be inconvenient or even impossible to solve a given equation of the form $f(x, y)=0$ for y in terms of x. However, the equation may define y as a function of x. In such cases, y is said to be an implicit function of x. If y be a differentiable function of x, then $\dfrac{dy}{dx}$ may be obtained as follows :

Differentiate each term in the equation with respect to x, regarding y as an unknown function of x having a derivative $\dfrac{dy}{dx}$, and then solve the resulting equation for $\dfrac{dy}{dx}$.

Illus. : Find $\dfrac{dy}{dx}$ if $x^3-xy^2+3y^2+2=0$.

Differentiating each term with respect to x,

$$3x^2+\left(-x\,.\,2y\frac{dy}{dx}-y^2\,.\,1\right)+6y\frac{dy}{dx}=0.$$

$$\therefore \quad (6y-2xy)\frac{dy}{dx}=y^2-3x^2. \qquad \therefore \quad \frac{dy}{dx}=\frac{y^2-3x^2}{6y-2xy}.$$

4·12. Parametric Equations.

Sometimes in the equation of a curve, x and y are expressed in terms of a third variable known as a ***parameter***. In such cases, to find $\dfrac{dy}{dx}$ it is not essential to eliminate the parameter and express y in terms of x. We may proceed as follows :

Let $\quad x=\phi(t),\ y=\psi(t)$.

Then t may be regarded as a function of x and also y is a function of t.

$$\text{Now,}\quad \frac{\mathbf{dy}}{\mathbf{dx}}=\frac{dy}{dt}\cdot\frac{dt}{dx}=\frac{\mathbf{dy}}{\mathbf{dt}}\Big/\frac{\mathbf{dx}}{\mathbf{dt}}\ \left(\frac{dx}{dt}\neq 0\right)$$

[*By* § *4·6 and 4·7*]

For *illustration* see § *4·13, Ex. 6.*

4·13. Illustrative Examples.

Ex. 1. (*a*) *Show that* $\dfrac{d}{dx}(x^n)=nx^{n-1}$ *when* n *is a positive integer, by the product rule.* [*Note, Theo. IV, Art. 4·4*]

Let $f(x)=x^n=x.x\,..x$ (n factors).

$\therefore\quad f'(x)=x\,x\ldots$ to $(n-1)$ factors $+x.x\,..$ to $(n-1)$ factors
$\qquad +x.x\ldots$ to $(n-1)$ factors
$\qquad +\cdots\cdots$ to n terms
$\qquad =nx^{n-1}$.

(*b*) *Assuming that* $\dfrac{d}{dx}x^n=nx^{n-1}$ *when* n *is a positive integer, show that the same result is true when* n *is a negative integer, or a rational fraction, positive or negative.*

When n is a negative integer, suppose $n=-m$, when m is a positive integer.

$$\therefore \quad \frac{d}{dx} x^n = \frac{d}{dx} x^{-m}$$

$$= \frac{d}{dx} \frac{1}{x^m} = \frac{0 \times x^m - mx^{m-1} \times 1}{x^{2m}} = -mx^{-m-1} = nx^{n-1}.$$

Next let us suppose n is a rational fraction positive or negative, and let $n = p/q$, where q is a positive integer and p any integer positive or negative.

Then, $y = x^n = x^{p/q}$; let $z = x^{1/q}$, $\therefore$ $x = z^q$ and $y = z^p$. Then,

$$\frac{dy}{dx} = \frac{dy}{dz} \bigg/ \frac{dx}{dz} = \frac{p}{q} z^{p-q} = nx^{(p-q)/q} = nx^{p/q-1} = nx^{n-1}.$$

Ex. 2. *Assuming that* $\frac{d}{dx}\{e^{\phi(x)}\} = e^{\phi(x)}\phi'(x)$ *for all real values of* x, *deduce that* $\frac{d}{dx}(x^n) = nx^{n-1}$ *for all real values of* x.

Let $\quad y = x^n = e^{n \log x}$.

Then, $\frac{dy}{dx} = \frac{d}{dx}(e^{n \log x}) = e^{n \log x} \cdot \frac{d}{dx}(n \log x)$

$$= x^n . n . \frac{1}{x} = nx^{n-1}.$$

Ex. 3. *Find the differential coefficient of* $\sin^2 (\log \sec x)$.

Let $\quad y = \{\sin (\log \sec x)\}^2$

$= u^2$ where $u = \sin (\log \sec x) = \sin v$, where $v = \log \sec x$

$= \log w$, where $w = \sec x$.

$$\therefore \quad \frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dv} \cdot \frac{dv}{dw} \cdot \frac{dw}{dx} = 2u . \cos v \frac{1}{w} \cdot \sec x \tan x$$

$= 2 \sin (\log \sec x) \cos (\log \sec x) . \tan x$

$= \sin (2 \log \sec x) \tan x.$

Ex. 4. *Differentiate* $(\sec x)^{\tan x}$.

Let $\quad y = (\sec x)^{\tan x}$. $\quad \therefore \quad \log y = \tan x \log \sec x$.

Differentiating both sides with respect to x,

$$\frac{1}{y} \frac{dy}{dx} = \tan x \cdot \frac{1}{\sec x} \cdot \sec x \tan x + \sec^2 x \log \sec x$$

$$= \tan^2 x + \sec^2 x \log \sec x.$$

$$\therefore \quad \frac{dy}{dx} = (\sec x)^{\tan x} \{\tan^2 x + \sec^2 x \log \sec x\}.$$

Note. Writing the given function as $e^{\tan x \log \sec x}$, we may proceed to differentiate it.

Ex. 5. *Find* $\frac{dy}{dx}$, *if* $y = \sqrt{\frac{(x-1)(x-2)}{(x-3)(x-4)}}$.

Taking logarithm of both sides,

$$\log y = \tfrac{1}{2} [\log (x-1) + \log (x-2) - \log (x-3) - \log (x-4)].$$

Differentiating both sides with respect to x,

$$\frac{1}{y}\frac{dy}{dx} = \frac{1}{2}\left[\frac{1}{x-1} + \frac{1}{x-2} - \frac{1}{x-3} - \frac{1}{x-4}\right]$$

$$= -\frac{2x^2 - 10x + 11}{(x-1)(x-2)(x-3)(x-4)}.$$

$$\therefore \quad \frac{dy}{dx} = -\frac{2x^2 - 10x + 11}{(x-1)^{\frac{1}{2}}(x-2)^{\frac{1}{2}}(x-3)^{\frac{3}{2}}(x-4)^{\frac{3}{2}}}.$$

Ex. 6. *Find* $\frac{dy}{dx}$, *if* $x = a(\theta - \sin \theta)$, $y = a(1 + \cos \theta)$.

$$\frac{dy}{dx} = \frac{dy}{d\theta}\Big/\frac{dx}{d\theta} = -\frac{a \sin \theta}{a(1 - \cos \theta)} = -\frac{2 \sin \frac{1}{2}\theta \cos \frac{1}{2}\theta}{2 \sin^2 \frac{1}{2}\theta} = -\cot \tfrac{1}{2}\theta.$$

Ex. 7. *Find* $\frac{dy}{dx}$, *if* $y = \tan^{-1} \frac{\sqrt{(1 + \sin x)} - \sqrt{(1 - \sin x)}}{\sqrt{(1 + \sin x)} + \sqrt{(1 - \sin x)}}$.

On rationalizing the denominator,

$$y = \tan^{-1} \frac{1 - \cos x}{\sin x} = \tan^{-1} \frac{2 \sin^2 \frac{1}{2}x}{2 \sin \frac{1}{2}x \cos \frac{1}{2}x}$$

$$= \tan^{-1} \tan \tfrac{1}{2}x = \tfrac{1}{2}x.$$

$$\therefore \quad \frac{dy}{dx} = \tfrac{1}{2}.$$

Note. Sometimes an algebraical or trigonometrical transformation as shown in this example considerably shortens the work. The next example also illustrates the same method.

Ex. 8. *If* $y = \tan^{-1} \frac{\sqrt{1 + x^2} - 1}{x}$, *find* $\frac{dy}{dx}$.

Putting $x=\tan\theta$, $\dfrac{\sqrt{1+x^2}-1}{x}=\dfrac{\sec\theta-1}{\tan\theta}=\dfrac{1-\cos\theta}{\sin\theta}$

$$=\frac{2\sin^2\frac{1}{2}\theta}{2\sin\frac{1}{2}\theta\cos\frac{1}{2}\theta}=\tan\tfrac{1}{2}\theta.$$

Hence, $y=\tan^{-1}\tan\frac{1}{2}\theta=\frac{1}{2}\theta=\frac{1}{2}\tan^{-1}x.$

$$\therefore\quad \frac{dy}{dx}=\frac{1}{2}\cdot\frac{1}{1+x^2}.$$

Ex. 9. *Differentiate* $\sin x$ *with respect to* x^2.

Let $y=\sin x$, $z=x^2$.

$$\therefore\quad \frac{dy}{dz}=\frac{dy}{dx}\frac{dx}{dz}=\frac{dy}{dx}\bigg/\frac{dz}{dx}=\frac{\cos x}{2x}.$$

Note. This is an example of the differential coefficient of a function of x with respect to another function of x.

Ex. 10. *If* $\sin y=x\sin(a+y)$, *prove that*

$$\frac{dy}{dx}=\frac{\sin^2(a+y)}{\sin a}.$$

From the given relation, we have

$$x=\frac{\sin y}{\sin(a+y)} \qquad \cdots \quad (1)$$

Hence x is a function of y.

$\therefore$ differentiating both sides of (1) with respect to y,

$$\frac{dx}{dy}=\frac{\sin(a+y)\cos y-\sin y\cos(a+y)}{\sin^2(a+y)}$$

$$=\frac{\{\sin(a+y)-y\}}{\sin^2(a+y)}=\frac{\sin a}{\sin^2(a+y)}.$$

Since by Art. 4.7, $\dfrac{dy}{dx}=1\bigg/\dfrac{dx}{dy}$, the required result follows.

Ex. 11. *Find the derivative of* $\Delta(x)$, *where*

$$\Delta(x)=\begin{matrix} f_1(x) & \phi_1(x) & \psi_1(x) \\ f_2(x) & \phi_2(x) & \psi_2(x) \\ f_3(x) & \phi_3(x) & \psi_3(x) \end{matrix}$$

and $f_1(x)$, $f_2(x)$, $f_3(x)$, $\phi_1(x)$ *etc. are different functions of* x.

First Method :

$$\Delta(x+h)-\Delta(x)=\begin{vmatrix} f_1(x+h) & \phi_1(x+h) & \psi_1(x+h) \\ f_2(x+h) & \phi_2(x+h) & \psi_2(x+h) \\ f_3(x+h) & \phi_3(x+h) & \psi_3(x+h) \end{vmatrix}$$

$$\begin{vmatrix} f_1(x) & \phi_1(x) & \psi_1(x) \\ f_2(x) & \phi_2(x) & \psi_2(x) \\ f_3(x) & \phi_3(x) & \psi_3(x) \end{vmatrix}$$

$$\begin{vmatrix} f_1(x+h)-f_1(x) & \phi_1(x+h)-\phi_1(x) & \psi_1(x+h)-\psi_1(x) \\ f_2(x+h) & \phi_2(x+h) & \psi_2(x+h) \\ f_3(x+h) & \phi_3(x+h) & \psi_3(x+h) \end{vmatrix}$$

$$+\begin{vmatrix} f_1(x) & \phi_1(x) & \psi_1(x) \\ f_2(x+h)-f_2(x) & \phi_2(x+h)-\phi_2(x) & \psi_2(x+h)-\psi_2(x) \\ f_3(x+h) & \phi_3(x+h) & \psi_3(x+h) \end{vmatrix}$$

$$+\begin{vmatrix} f_1(x) & \phi_1(x) & \psi_1(x) \\ f_2(x) & \phi_2(x) & \psi_2(x) \\ f_3(x+h)-f_3(x) & \phi_3(x+h)-\phi_3(x) & \psi_3(x+h)-\psi_3(x) \end{vmatrix} \qquad (1)$$

[The right side on simplification can be easily shown to be equal to left side.]

Dividing (1) throughout by h and letting $h \to 0$, we get

$$\Delta'(x)=\begin{vmatrix} f'_1(x) & \phi'_1(x) & \psi'_1(x) \\ f_2(x) & \phi_2(x) & \psi_2(x) \\ f_3(x) & \phi_3(x) & \psi_3(x) \end{vmatrix}+\begin{vmatrix} f_1(x) & \phi_1(x) & \psi_1(x) \\ f'_2(x) & \phi'_2(x) & \psi'_2(x) \\ f_3(x) & \phi_3(x) & \psi_3(x) \end{vmatrix}$$

$$+\begin{vmatrix} f_1(x) & \phi_1(x) & \psi_1(x) \\ f_2(x) & \phi_2(x) & \psi_2(x) \\ f'_3(x) & \phi'_3(x) & \psi'_3(x) \end{vmatrix}$$

Second Method :

Clearly $\Delta(x) = \Sigma f_1(x)\{\phi_2(x)\,\psi_3(x) - \phi_3(x)\,\psi_2(x)\}$.

$$\therefore \quad \Delta'(x) = \Sigma f'_1(x)\{\phi_2(x)\,\psi_3(x) - \phi_3(x)\,\psi_2(x)\}$$
$$+\Sigma f_1(x)\{\phi_2'(x)\,\psi_3(x) - \phi_3'(x)\,\psi_2(x)\}$$
$$+\Sigma f_1(x)\{\phi_2(x)\,\psi_3'(x) - \phi_3(x)\,\psi_2'(x)\}$$

$$\therefore \quad \Delta'(x) = \begin{vmatrix} f'_1(x) & \phi_1(x) & \psi_1(x) \\ f'_2(x) & \phi_2(x) & \psi_2(x) \\ f'_3(x) & \phi_3(x) & \psi_3(x) \end{vmatrix} + \begin{vmatrix} f_1(x) & \phi'_1(x) & \psi_1(x) \\ f_2(x) & \phi'_2(x) & \psi_2(x) \\ f_3(x) & \phi'_3(x) & \psi_3(x) \end{vmatrix}$$
$$\begin{vmatrix} f_1(x) & \phi_1(x) & \psi'_1(x) \\ f_2(x) & \phi_2(x) & \psi'_2(x) \\ f_3(x) & \phi_3(x) & \psi'_3(x) \end{vmatrix}$$

Thus the derivativo of a third ordor determinant $\Delta(x)$ is equal to the sum of the three determinants, each obtained by differentiating one column of $\Delta(x)$ leaving the other columns unaltered. Similarly, $\Delta'(x)$ is the sum of throe determinants each obtained by differentiating one row of $\Delta(x)$ leaving the other rows unaltered.

Tho similar result is true whatever be the order of the determinant.

Examples IV(B)

Find the differential coefficients of :

1. (i) $\{\phi(x)\}^n$. (ii) $(x^2+5)^7$. (iii) $\sqrt{(x^2+a^2)}$.

(iv) $1/(ax+b)$. (v) $(e^x)^3$. (vi) $\sqrt{\log x}$.

(vii) $\sin^n x$. (viii) $\tan^5 x$. (ix) $\sec^3 x$.

(x) $(\sin^{-1}x)^3$. (xi) $(\tan^{-1}x)^2$. (xii) $\sqrt{\phi(x)}$.

2. (i) $e^{\phi(x)}$. (ii) e^{ax}. (iii) e^{ax^2+bx+c}.

(iv) e^{x^4}. (v) $e^{\tan x}$. (vi) $e^{\sin^{-1}x}$.

(vii) $e^{\sqrt{(x+1)}} - e^{\sqrt{(x-1)}}$.

3. (i) $a^{\phi(x)}$. (ii) 7^{x^2+2x}. (iii) $10^{\log x^x}$.

4. (i) $\log \phi(x)$. (ii) $\log \sin x$. (iii) $\log \cos x$.

(iv) $\log (x+a)$. (v) $\log (ax+b)$. (vi) $\log \sqrt{x}$.

(vii) $\log (ax^2+bx+c)$. (viii) $\log (\log x)$.

(ix) $10^{\log \sin x}$. (x) $\log \tan^{-1}x$.

(xi) $\log (\sec x + \tan x)$. (xii) $\log_x a$.

(xiii) $\log_x \sin x$. (xiv) $\log_x (a+x)$.

(xv) $\log_{\sin x} x$ (xvi) $\log_{\sin x} (\sec x)$.

(xvii) $\log \tan (\frac{1}{4}\pi + \frac{1}{2}x)$. (xviii) $\log (x + \sqrt{x^2 \pm a^2})$.

(xix) $\log (\sqrt{x-a} + \sqrt{x-b})$. [*C. P. 1936*]

(xx) $\log_{10} (2x + \sqrt{4x^2+1})$.

(xxi) $\frac{1}{2} \log \{(1+x)/(1-x)\}$.

5. (i) $\sin \phi(x)$. (ii) $\cos \phi(x)$. (iii) $\tan \phi(x)$.

(iv) $\operatorname{cosec} \phi(x)$. (v) $\sec \phi(x)$. (vi) $\cot \phi(x)$.

(vii) $\sin ax$. (viii) $\cos (ax+b)$. (ix) $\cos^2 x$.

(x) $\tan mx$. (xi) $\operatorname{cosec}^3 x$.

(xii) $\sin 2x \cos x$. (xiii) $\cos 2x \cos 3x$.

(xiv) $\sin x^\circ$ (degrees). (xv) $e^{ax} \sin bx$.

(xvi) $e^{ax} \cos (bx+c)$. (xvii) $\tan 3x + \cot 4x$.

(xviii) $\sin x \sin 2x \sin 3x$. (xix) $a \tan^2 x + b \cot^2 x$.

(xx) $\sin^m x \cos^n x$. (xxi) $\sin^m x/\cos^n x$.

(xxii) $\cot x \coth x$. (xxiii) $\tanh x - \frac{1}{3} \tanh^3 x$.

(xxiv) $\log \tanh x$.

6. (i) $\sin^{-1}\phi(x)$. (ii) $\tan^{-1}\phi(x)$.

(iii) $\sec^{-1}\phi(x)$. (iv) $\sin^{-1}x^2$.

(v) $\tan^{-1}(\sqrt{x})$. (vi) $\tan^{-1}(x/a)$.

(vii) $\sin^{-1}(x/a)$. (viii) $\sec^{-1} x^3$.

(ix) $\cos^{-1}\sqrt{(ax+b)}$. (x) $\cot^{-1}(e^x)$.

(xi) $\sec^{-1}(\tan x)$. (xii) $\tan^{-1}(\sec x)$.

(xiii) $\tan^{-1}(1+x+x^2)$.

(xiv) $\cos^{-1}(8x^4-8x^2+1)$.

(xv) $\sin^{-1}(3x-4x^3)$.

(xvi) $\sec(\tan^{-1}x)$ [*C. P. 1940*]

(xvii) $\tan(\sin^{-1}x)$. (xviii) $\tan^{-1}(\tanh \frac{1}{2}x)$.

(xix) $\cot^{-1}(\operatorname{cosec} x+\cot x)$.

(xx) $\tan^{-1}(\sec x+\tan x)$.

(xxi) $\cot^{-1}(\sqrt{1+x^2}-x)$.

(xxii) $\cot^{-1}\dfrac{1+x}{1-x}$. (xxiii) $\cos^{-1}\dfrac{1-x^2}{1+x^2}$.

(xxiv) $\tan^{-1}\dfrac{a+bx}{b-ax}$. (xxv) $\sin^{-1}\dfrac{2x}{1+x^2}$.

(xxvi) $\sec^{-1}\dfrac{x^2+1}{x^2-1}$. (xxvii) $\tan^{-1}\dfrac{2x}{1-x^2}$.

(xxviii) $\tan^{-1}\dfrac{1}{\sqrt{(x^2-1)}}$. [*C. P. 1943*]

(xxix) $\tan^{-1}\dfrac{x}{\sqrt{(1-x^2)}}$. [*C. P. 1938*]

(xxx) $2\tan^{-1}\sqrt{\left(\dfrac{x-a}{b-x}\right)}$. (xxxi) $\tan^{-1}\dfrac{3x-x^3}{1-3x^2}$.

(xxxii) $\operatorname{sech}^{-1} x-\operatorname{cosech}^{-1}x$.

(xxxiii) $\tanh^{-1}(\tan \frac{1}{2}x)$.

(xxxiv) $\tanh^{-1}\{(x^2-1)/(x^2+1)\}$.

7. (i) $\cos\{\sqrt{(1+x^2)}\}$. (ii) $e^{\sqrt{\cot x}}$. [*C. P. 1943*]

(iii) $e^{\operatorname{cosec}^2\sqrt{x}}$. (iv) $e^{(\sin^{-1}x)^2}$.

(v) $3^{\sqrt{(1+x+x^2)}}$. (vi) $\log \tan \frac{1}{2}x$.

(vii) $\sqrt{(\log \sin x)}$. (viii) $(\log \sin x)^2$.

(ix) $\cos \{2 \sin^{-1} (\cos x)\}$. (x) $\sin^2 (\log x^2)$.

(xi) $\log \sec (ax+b)^3$. [*C. P. 1941*]

(xii) $\log \{2x+4+\sqrt{(4x^2+16x-12)}\}$.

(xiii) $\sqrt{(1+\log x, \log \sin x)}$. [*C. P. 1944*]

(xiv) $\tan \log \sin (e^{x^2})$.

(xv) $A(x+\sqrt{x^2-1})^n + B(x-\sqrt{x^2-1})^n$.

8. Find the differential coefficients of :

(i) x^x. (ii) $(1+x)^x$. (iii) $x^{\log x}$.

(iv) x^{1+x+x^2}. (v) a^{a^x}. (vi) e^{e^x}.

(vii) e^{x^2}. (viii) x^{e^x}. (ix) $(\sin x)^{\tan x}$.

(x) $x^{\cos^{-1}x}$. [*C. P. 1944*]

(xi) $(\sin x)^{\log x}$. [*C. P. 1943*]

(xii) x^{x^x}. [*C. P. 1937*]

(xiii) $(\sin x)^{\cos x} + (\cos x)^{\sin x}$.

(xiv) $(\tan x)^{\cot x} + (\cot x)^{\tan x}$.

9. Find the differential coefficients of :

(i) $(1-x)(1-2x)(1-3x)(1-4x)$.

(ii) $\sqrt[3]{x(x+1)(x+2)}$. (iii) $\sqrt{\left(\dfrac{1+x}{1-x}\right)}$.

(iv) $\left(\dfrac{a^2-x^2}{a^2+x^2}\right)^{\frac{1}{2}}$. (v) $\log \left\{e^x \left(\dfrac{x-1}{x+1}\right)^{\frac{3}{2}}\right\}$.

(vi) $x^3 \sqrt{\dfrac{x^2+4}{x^2+3}}$. [*C. P. 1941*]

(vii) $\dfrac{x^3\sqrt{x^2-12}}{\sqrt[3]{20-3x}}$ for $x=4$.

(viii) $\left(\dfrac{x}{1+\sqrt{1-x^2}}\right)^n$. [*C. P. 1936*]

(ix) $\dfrac{(4x+1)^{\frac{1}{2}}}{(2x+3)^{\frac{1}{3}}(5x-1)^{\frac{1}{5}}}$

(x) $\left(x^{\frac{b+c}{c-a}}\right)^{\frac{1}{a-b}} \times \left(x^{\frac{c+a}{a-b}}\right)^{\frac{1}{b-c}} \times \left(x^{\frac{a+b}{b-c}}\right)^{\frac{1}{c-a}}$.

10. Find $\dfrac{dy}{dx}$ in the following cases :

(i) $3x^4-x^2y+2y^3=0$. [*C. P. 1941*]

(ii) $x^4+x^2y^2+y^4=0$. [*C. P. 1939*]

(iii) $x^3+y^3+4x^2y-25=0$.

(iv) $x^3+y^3=3axy$. (v) $x^{\frac{2}{3}}+y^{\frac{2}{3}}=a^{\frac{2}{3}}$.

(vi) $ax^2+2hxy+by^2+2gx+2fy+c=0$.

(vii) $x=y\log(xy)$. (viii) $x^py^q=(x+y)^{p+q}$.

(ix) $y=x^y$. [*C. P. 1940*]

(x) $x^y=y^x$. [*C. P. 1945*]

(xi) $x^yy^x=1$. [*C. P. 1943*] (xii) $(\cos x)^y=(\sin y)^x$.

(xiii) $e^{xy}-4xy=2$.

(xiv) $\log(xy)=x^2+y^2$. [*C. P. 1943*]

11. Find $\dfrac{dy}{dx}$ when

(i) $x=a\cos\phi,\ y=b\sin\phi$.

(ii) $x=a\cos^3\theta,\ y=a\sin^3\theta$.

(iii) $x=at^2,\ y=2at$.

(iv) $x=\sin^2\theta$, $y=\tan\theta$. [*C. P. 1943*]

(v) $x=a\sec^2\theta$, $y=a\tan^3\theta$. [*C. P. 1942*]

(vi) $x=a(\cos t+\log\tan\frac{1}{2}t)$, $y=a\sin t$.

(vii) $x=a(\cos t+t\sin t)$, $y=a(\sin t-t\cos t)$.

(viii) $x=a(2\cos t+\cos 2t)$, $y=a(2\sin t-\sin 2t)$.

(ix) $x=2a\sin^2 t\cos 2t$, $y=2a\sin^2 t\sin 2t$.

(x) $x=3at/(1+t^3)$, $y=3at^2/(1+t^3)$. [*C. P. 1941*]

(xi) $\tan y=\dfrac{2t}{1-t^2}$, $\sin x=\dfrac{2t}{1+t^2}$. [*C. P. 1944*]

12. If $y=e^{\sin^{-1}x}$ and $z=e^{-\cos^{-1}x}$, then dy/dz is independent of x.

13. Differentiate the left-side functions with respect to the right side ones :

(i) x^5 w.r.t. x^2. (ii) $\sec x$ w.r.t. $\tan x$.

(iii) $\log_{10}x$ w.r.t. x^3. (iv) $\tan^{-1}x$ w.r.t. x^2.

(v) $\cos^{-1}\dfrac{1-x^2}{1+x^2}$ w.r.t. $\tan^{-1}\dfrac{2x}{1-x^2}$.

(vi) $\tan^{-1}\dfrac{\sqrt{(1+x^2)}-1}{x}$ w.r.t. $\tan^{-1}x$.

(vii) $x^{\sin^{-1}x}$ w.r.t. $\sin^{-1}x$. [*C. P 1938*]

14. Find the differential coefficients of :

(i) $\dfrac{1}{\sqrt{(1+x)}}+\dfrac{1}{\sqrt{(1-x)}}$. (ii) $\dfrac{1}{\sqrt{(x+a)}+\sqrt{(x+b)}}$.

(iii) $\sqrt{\left\{\dfrac{1+x+x^2}{1-x+x^2}\right\}}$. (iv) $\log\dfrac{x^2+x+1}{x^2-x+1}$.

(v) $\log\sqrt{\left\{\dfrac{1+\sin x}{1-\sin x}\right\}}$. (vi) $\log\sqrt{\left\{\dfrac{1-\cos x}{1+\cos x}\right\}}$.

(vii) $\tan^{-1}\sqrt{\left(\dfrac{1-x}{1+x}\right)}$.

(viii) $\tan^{-1}\dfrac{\cos x}{1+\sin x}$. [*C. P. 1942, '44*]

(ix) $\tan^{-1}\sqrt{\left(\dfrac{1-\cos x}{1+\cos x}\right)}$. (x) $\tan^{-1}\dfrac{\cos x-\sin x}{\cos x+\sin x}$.

(xi) $\sin^{-1}x+\sin^{-1}\sqrt{(1-x^2)}$.

(xii) $\sin^{-1}\{2ax\sqrt{(1-a^2x^2)}\}$.

(xiii) $\tan^{-1}\dfrac{\sqrt{(1+x^2)}-\sqrt{(1-x^2)}}{\sqrt{(1+x^2)}+\sqrt{(1-x^2)}}$.

(xiv) $\sin\left\{2\tan^{-1}\sqrt{\left(\dfrac{1-x}{1+x}\right)}\right\}$.

(xv) $\dfrac{\sqrt{(1+x)}+\sqrt{(1-x)}}{\sqrt{(1+x)}-\sqrt{(1-x)}}$.

(xvi) $\dfrac{\sqrt{(a^2+x^2)}+\sqrt{(a^2-x^2)}}{\sqrt{(a^2+x^2)}-\sqrt{(a^2-x^2)}}$.

(xvii) $\log\left(\dfrac{1+x}{1-x}\right)^{\frac{1}{4}}-\frac{1}{2}\tan^{-1}x$.

(xviii) $\log\sqrt{\left(\dfrac{a\cos x-b\sin x}{a\cos x+b\sin x}\right)}$.

(xix) $\log\dfrac{a+b\tan x}{a-b\tan x}$. [*C. P. 1942*]

(xx) $\tan^{-1}\left\{\sqrt{\left(\dfrac{a-b}{a+b}\right)}\tan\dfrac{x}{2}\right\}$.

(xxi) $\sin^{-1}\dfrac{a+b\cos x}{b+a\cos x}$. (xxii) $\cos^{-1}\dfrac{3+5\cos x}{5+3\cos x}$.

(xxiii) $\log\sqrt{\left\{\dfrac{\sqrt{1+x^2}+x}{\sqrt{1+x^2}-x}\right\}}$.

(xxiv) $\log\sqrt{\left\{\dfrac{\sqrt{1+x}+\sqrt{1-x}}{\sqrt{1+x}-\sqrt{1-x}}\right\}}$. (xxv) $x+\cfrac{1}{x+\cfrac{1}{x+\cfrac{1}{x}}}$

Find $\dfrac{dy}{dx}$ in the following cases (*Ex. 15-23*) :

15. $y = \frac{1}{3}x^3 \tan^{-1}x - \frac{1}{6}x^2 + \frac{1}{6}\log(1+x^2)$.

16. $y = \frac{1}{2}x\sqrt{(a^2 - x^2)} + \frac{1}{2}a^2 \sin^{-1}(x/a)$.

17. $y = \log(x + \sqrt{x^2 - a^2}) + \sec^{-1}(x/a)$.

18. $y = x\sqrt{(x^2 + a^2)} + a^2 \log(x + \sqrt{x^2 + a^2})$.

19. $x = \sqrt{(a^2 - y^2)} + \dfrac{a}{2}\log\dfrac{a - \sqrt{(a^2 - y^2)}}{a + \sqrt{(a^2 - y^2)}}$.

20. $y = \log\dfrac{1+x}{1-x} + \dfrac{1}{2}\log\dfrac{1+x+x^2}{1-x+x^2} + \sqrt{3}\tan^{-1}\dfrac{x\sqrt{3}}{1-x^2}$.

21. $y = \dfrac{1}{4\sqrt{2}}\log\dfrac{1+x\sqrt{2}+x^2}{1-x\sqrt{2}+x^2} + \dfrac{1}{2\sqrt{2}}\tan^{-1}\dfrac{x\sqrt{2}}{1-x^2}$.

22. $y = \frac{1}{4}x(x^2+1)^{\frac{3}{2}} - \frac{1}{8}x(x^2+1)^{\frac{1}{2}} - \frac{1}{8}\log(x + \sqrt{x^2+1})$.

23. $$y = \frac{1}{1 + x^{n-m} + x^{p-m}} + \frac{1}{1 + x^{m-n} + x^{p-n}} + \frac{1}{1 + x^{m-p} + x^{n-p}}.$$

24. If $f(x) = \left(\dfrac{a+x}{b+x}\right)^{a+b+2x}$, show that

$$f'(0) = \left(2\log\frac{a}{b} + \frac{b^2 - a^2}{ab}\right)\cdot\left(\frac{a}{b}\right)^{a+b}. \qquad [\textit{C. P. 1946}]$$

25. If $f(x) = \log\dfrac{\sqrt{(a+bx)} - \sqrt{(a-bx)}}{\sqrt{(a+bx)} + \sqrt{(a-bx)}}$,
find for what values of x, $1/f'(x) = 0$.

26. If $\sin x \sin\left(\dfrac{\pi}{n} + x\right)\sin\left(\dfrac{2\pi}{n} + x\right)$

$$\ldots\ldots\sin\left(\frac{n-1}{n}\pi + x\right) = \frac{\sin nx}{2^{n-1}},$$

show that

$$\cot x+\cot\left(\frac{\pi}{n}+x\right)+\cot\left(\frac{2\pi}{n}+x\right)$$
$$+\cdots\cdots+\cot\left(\frac{n-1}{n}\pi+x\right)=\pi\cot nx.$$

[*C. P. 1945*]

27. (i) From the relation (where n is odd)

$$2^{n-1}\cos\theta\cos\left(\theta+\frac{2\pi}{n}\right)\cos\left(\theta+\frac{4\pi}{n}\right)\cdots\cdots$$
$$\cdots\cos\left\{\theta+\frac{2(n-1)\pi}{n}\right\}=\cos n\theta,$$

deduce that

$$\tan\theta+\tan\left(\theta+\frac{2\pi}{n}\right)+\tan\left(\theta+\frac{4\pi}{n}\right)+\cdots\cdots$$
$$+\tan\left\{\theta+\frac{2(n-1)\pi}{n}\right\}=n\tan n\theta.$$

(ii) From the identity

$$\cos\frac{\theta}{2}\cos\frac{\theta}{2^2}\cos\frac{\theta}{2^3}\cdots\cdots\cos\frac{\theta}{2^n}=\frac{\sin\theta}{2^n\sin(\theta/2^n)},$$

show that

$$\frac{1}{2}\tan\frac{\theta}{2}+\frac{1}{2^2}\tan\frac{\theta}{2^2}+\cdots\cdots+\frac{1}{2^n}\tan\frac{\theta}{2^n}$$
$$=\frac{1}{2^n}\cot\frac{\theta}{2^n}-\cot\theta.$$

28. Find $f'(x)$ in the following cases and determine if it is continuous for $x=0$.

(i) $f(x)=0$ or $x^2\cos(1/x)$ according as x is or is not zero.

(ii) $f(x)=0$ or $x^3\cos(1/x)$ according as x is or is not zero.

29. If $(1+x)^n = c_0 + c_1 x + c_2 x^2 + \cdots\cdots + c_n x^n$, then

(i) $c_1 + 2c_2 + 3c_3 + \cdots\cdots + nc_n = n.2^{n-1}$.

(ii) $c_0 + 2c_1 + 3c_2 + \cdots\cdots + (n+1)c_n = (n+2)2^{n-1}$.

30. If $y = 1 + \dfrac{a_1}{x-a_1} + \dfrac{a_2 x}{(x-a_1)(x-a_2)}$

$$+\frac{a_3 x^2}{(x-a_1)(x-a_2)(x-a_3)},$$

show that

$$\frac{dy}{dx} = \frac{y}{x}\left\{\frac{a_1}{a_1-x} + \frac{a_2}{a_2-x} + \frac{a_3}{a_3-x}\right\}.$$

***31.** If $\Delta(x) = \begin{vmatrix} (x-a)^4 & (x-a)^3 & 1 \\ (x-b)^4 & (x-b)^3 & 1 \\ (x-c)^4 & (x-c)^3 & 1 \end{vmatrix}$, show that

$$\Delta'(x) = 3\begin{vmatrix} (x-a)^4 & (x-a)^2 & 1 \\ (x-b)^4 & (x-b)^2 & 1 \\ (x-c)^4 & (x-c)^2 & 1 \end{vmatrix}$$

***32.** If $\Delta(x) = \begin{vmatrix} \sin x & \cos x & \sin x \\ \cos x & -\sin x & \cos x \\ x & 1 & 1 \end{vmatrix}$

show that $\Delta'(x) = 1$.

***33.** If $f(x) = \begin{vmatrix} 1 & 1 & 1 & 1 \\ 1 & x & 1 & 1 \\ 1 & 1 & x & 1 \\ 1 & 1 & 1 & x \end{vmatrix}$

prove that $f'(x) = 3(x-1)^2$.

***34.** If the determinant of the 4th order

$$\begin{array}{cccc} x & a & a & a \\ a & x & a & a \\ a & a & x & a \\ a & a & a & x \end{array}$$

be denoted by Δ_4, show that $\Delta'_4 = 4\Delta_3$.

ANSWERS

1. (i) $n\{\phi(x)\}^{n-1}\phi'(x)$. (ii) $14x(x^2+5)^6$. (iii) $x/\sqrt{(x^2+a^2)}$.
(iv) $-a/(ax+b)^2$. (v) $3(e^x)^3$. (vi) $1/2x\sqrt{\log x}$.
(vii) $n\sin^{n-1}x\cos x$. (viii) $5\tan^4 x\sec^2 x$.
(ix) $3\sec^3 x\tan x$. (x) $\dfrac{3(\sin^{-1}x)^2}{\sqrt{1-x^2}}$. (xi) $\dfrac{2\tan^{-1}x}{1+x^2}$.
(xii) $\dfrac{\phi'(x)}{2\sqrt{\phi(x)}}$. **2.** (i) $e^{\phi(x)}.\phi'(x)$. (ii) ae^{ax}.
(iii) $(2ax+b)\,e^{ax^2+bx+c}$. (iv) $4x^3e^{x^4}$. (v) $\sec^2 x\,e^{\tan x}$.
(vi) $e^{\sin^{-1}x}\dfrac{1}{\sqrt{1-x^2}}$. (vii) $\dfrac{e^{\sqrt{x+1}}}{2\sqrt{x+1}}-\dfrac{e^{\sqrt{x-1}}}{2\sqrt{x-1}}$

3. (i) $(\log_e a.a^{\phi(x)}.\phi'(x)$. (ii) $2(x+1)\log 7.7^{x^2+2x}$.
(iii) $\log_e 10.10^{\log x^x}.(1+\log x)$. **4.** (i) $\phi'(x)/\phi(x)$ (ii) $\cot x$.
(iii) $-\tan x$. (iv) $1/(x+a)$. (v) $a/(ax+b)$. (vi) $1/2x$.
(vii) $(2ax+b)/(ax^2+bx+c)$. (viii) $1/x\log x$.
(ix) $10^{\log\sin x}\log_e 10.\cot x$. (x) $1/(1+x^2)\tan^{-1}x$. (xi) $\sec x$.
(xii) $-\log a/x(\log x)^2$. (xiii) $\dfrac{x\cot x.\log x-\log\sin x}{x(\log x)^2}$.
(xiv) $\dfrac{x\log x-(a+x)\log(a+x)}{x(a+x)(\log x)^2}$. (xv) $\dfrac{\log\sin x-x\cot x\log x}{x(\log\sin x)^2}$.
(xvi) $\dfrac{\tan x\log\sin x+\cot x\log\cos x}{(\log\sin x)^2}$. (xvii) $\sec x$.
(xviii) $\dfrac{1}{\sqrt{x^2\pm a^2}}$. (xix) $\dfrac{1}{2\sqrt{(x-a)(x-b)}}$. (xx) $\dfrac{2\log_{10}e}{\sqrt{4x^2+1}}$.

(xxi) $\frac{1}{1-x^2}$. 5. (i) $\cos \phi(x).\phi'(x)$. (ii) $-\sin \phi(x).\phi'(x)$.

(iii) $\sec^2\phi(x)\, \phi'(x)$. (iv) $-\operatorname{cosec} \phi(x) \cot \phi(x).\phi'(x)$.

(v) $\sec \phi(x) \tan \phi(x).\phi'(x)$. (vi) $-\operatorname{cosec}^2\phi(x).\phi'(x)$. (vii) $a \cos ax$.

(viii) $-a \sin(ax+b)$ (ix) $-\sin 2x$. (x) $m \sec^2 mx$.

(xi) $-3 \operatorname{cosec}^3 x \cot x$. (xii) $\frac{1}{2}(3 \cos 3x+\cos x)$.

(xiii) $-\frac{1}{2}(5 \sin 5x+\sin x)$ (xiv) $\frac{\pi}{180} \cos x^\circ$ (degrees).

(xv) $e^{ax}(a \sin bx+b \cos bx)$. (xvi) $e^{ax}\{a \cos (bx+c)-b \sin (bx+c)\}$.

(xvii) $3 \sec^2 3x-4 \operatorname{cosec}^2 4x$ (xviii) $\frac{1}{2}(\cos 2x+2 \cos 4x-3 \cos 6x)$.

(xix) $2(a \tan x \sec^2 x-b \cot x \operatorname{cosec}^2 x)$.

(xx) $\sin^{m-1}x \cos^{n-1}x\,(m \cos^2 x-n \sin^2 x)$.

(xxi) $\frac{\sin^{m-1}x}{\cos^{n+1}x}(m \cos^2 x+n \sin^2 x)$.

(xxii) $-\operatorname{cosec}^2 x \coth x-\cot x \operatorname{cosech}^2 x$.

(xxiii) $\operatorname{sech}^4 x$. (xxiv) $2 \operatorname{cosech} 2x$.

6. (i) $\frac{\phi'(x)}{\sqrt{1-\{\phi(x)\}^2}}$. (ii) $\frac{\phi'(x)}{1+\{\phi(x)\}^2}$. (iii) $\frac{\phi'(x)}{\phi(x).\sqrt{\{\phi(x)\}^2-1}}$

(iv) $\frac{2x}{\sqrt{1-x^4}}$. (v) $\frac{1}{2(1+x)\sqrt{x}}$. (vi) $\frac{a}{a^2+x^2}$.

(vii) $\frac{1}{\sqrt{a^2-x^2}}$. (viii) $\frac{3}{x\sqrt{x^6-1}}$.

(ix) $-\frac{1}{\sqrt{1-(ax+b)}}\cdot\frac{a}{2\sqrt{ax+b}}$. (x) $-\frac{e^x}{1+e^{2x}}$.

(xi) $\frac{1}{\sin x\sqrt{\sin^2 x-\cos^2 x}}$. (xii) $\frac{\sin x}{1+\cos^2 x}$.

(xiii) $\frac{1+2x}{1+(1+x+x^2)^2}$. (xiv) $-\frac{4}{\sqrt{1-x^2}}$. (xv) $\frac{3}{\sqrt{1-x^2}}$.

(xvi) $\frac{x}{\sqrt{1+x^2}}$. (xvii) $(1-x^2)^{-\frac{3}{2}}$. (xviii) $\frac{1}{2}\operatorname{sech} x$.

(xix) $\frac{1}{2}$. (xx) $\frac{1}{2}$. (xxi) $\frac{1}{2}\cdot\frac{1}{1+x^2}$. (xxii) $-\frac{1}{1+x^2}$.

(xxiii) $\dfrac{2}{1+x^2}$. (xxiv) $\dfrac{1}{1+x^2}$. (xxv) $\dfrac{2}{1+x^2}$.

(xxvi) $-\dfrac{2}{1+x^2}$. (xxvii) $\dfrac{2}{1+x^2}$. (xxviii) $-\dfrac{1}{x\sqrt{x^2-1}}$.

(xxix) $\dfrac{1}{\sqrt{1-x^2}}$. (xxx) $\dfrac{1}{\sqrt{(x-a)(b-x)}}$.

(xxxi) $\dfrac{3}{1+x^2}$. (xxxii) $-\dfrac{1}{x}\left[\dfrac{1}{\sqrt{1-x^2}}-\dfrac{1}{\sqrt{1+x^2}}\right]$.

(xxxiii) $\frac{1}{2}\sec x$. (xxxiv) $1/x$.

7. (i) $\dfrac{-x\sin(\sqrt{1+x^2})}{\sqrt{1+x^2}}$. (ii) $\dfrac{-e^{\sqrt{\cot x}}}{2\sin^2 x\,.\sqrt{\cot x}}$.

(iii) $-e^{\operatorname{cosec}^2\sqrt{x}}\dfrac{\cos\sqrt{x}}{\sqrt{x}\sin^3\sqrt{x}}$. (iv) $\dfrac{2\sin^{-1}x}{\sqrt{1-x^2}}e^{(\sin^{-1}x)^2}$.

(v) $\dfrac{(1+2x)\log 3}{2\sqrt{1+x+x^2}}.3^{\sqrt{(1+x+x^2)}}$. (vi) $\operatorname{cosec} x$.

(vii) $\frac{1}{2}(\log\sin x)^{-\frac{1}{2}}\cot x$. (viii) $2\cot x\log\sin x$

(ix) $2\sin 2x$. (x) $2x^{-1}\sin(4\log x)$.

(xi) $3a(ax+b)^2\tan(ax+b)^3$.

(xii) $\dfrac{1}{\sqrt{x^2+4x-3}}$. (xiii) $\dfrac{\log\sin x+x\cot x\log x}{2x\sqrt{(1+\log x\log\sin x)}}$.

(xiv) $\sec^2(\log\sin e^{x^2})\cot(e^{x^2})\,e^{x^2}.2x$.

(xv) $\dfrac{n}{\sqrt{x^2-1}}\{A(x+\sqrt{x^2-1})^n-B(x-\sqrt{x^2-1})^n\}$.

8. (i) $x^x(\log x+1)$. (ii) $(1+x)^x\left\{\log(1+x)+\dfrac{x}{1+x}\right\}$.

(iii) $2\log x.x^{(\log x-1)}$. (iv) $x^{1+x+x^2}\{(2x+1)\log x+x^{-1}+1+x\}$.

(v) $a^{a^x}.a^x(\log a)^2$. (vi) $e^{e^x}.e^x$. (vii) $e^{x^x}.x^x(\log x+1)$.

(viii) $x^{e^x}.e^x(x^{-1}+\log x)$. (ix) $(\sin x)^{\tan x}\{\sec^2 x\log(\sin x)+1\}$.

(x) $x^{\cos^{-1}x}\left\{-\dfrac{\log x}{\sqrt{(1-x^2)}}+\dfrac{\cos^{-1}x}{x}\right\}$.

(xi) $(\sin x)^{\log x}\{x^{-1}\log\sin x+\log x\cot x\}$.

(xii) $x^{x^x} . x^x \{\log x\,(\log x+1)+1/x\}$.

(xiii) $(\sin x)^{\cos x}\{\cos x \cot x-\sin x \log \sin x\}$
$+(\cos x)^{\sin x}\{\cos x \log \cos x-\sin x \tan x\}$.

(xiv) $(\tan x)^{\cot x}\{\operatorname{cosec}^2 x\,(1-\log \tan x)\}$
$+(\cot x)^{\tan x}\{\sec^2 x\,(\log \cot x-1)\}$.

9. (i) $96x^3-150x^2+7x-10$. (ii) $\dfrac{3x^2+6x+2}{3\{x(x+1)(x+2)\}^{\frac{2}{3}}}$.

(iii) $\dfrac{1}{(1-x)\sqrt{1-x^2}}$. (iv) $\dfrac{-2a^2x}{(a^2+x^2)\sqrt{a^4-x^4}}$.

(v) $\dfrac{x^2+2}{x^4-1}$. (vi) $\dfrac{x^2(3x^4+20x^2+36)}{(x^2+4)^{\frac{1}{2}}(x^2+3)^{\frac{3}{2}}}$. (vii) 120.

(viii) $\dfrac{ny}{x\sqrt{(1-x^2)}}$. (ix) $\dfrac{-(5+2x+18x^2)}{(4x+1)^{\frac{3}{4}}(2x+3)^{\frac{3}{2}}(5x-1)^{\frac{6}{5}}}$. (x) 0.

10 (i) $\dfrac{2x(6x^2-y)}{x^2-6y^2}$. (ii) $-\dfrac{x(2x^2+y^2)}{y(x^2+2y^2)}$. (iii) $-\dfrac{3x^2+8xy}{3y^2+4x^2}$.

(iv) $\dfrac{x^2-ay}{ax-y^2}$. (v) $-\left(\dfrac{y}{x}\right)^{\frac{1}{3}}$. (vi) $-\dfrac{ax+hy+g}{hx+by+f}$.

(vii) $\dfrac{y(x-y)}{x(x+y)}$. (viii) $\dfrac{y}{x}$. (ix) $\dfrac{y^2}{x(1-y\log x)}$.

(x) $\dfrac{y\,(x\log y-y)}{x\,(y\log x-x)}$. (xi) $-\dfrac{y^2(1-\log x)}{x^2(1-\log y)}$.

(xii) $\dfrac{y\tan x+\log \sin y}{\log \cos x-x\cot y}$. (xiii) $-\dfrac{y}{x}$. (xiv) $\dfrac{y(2x^2-1)}{x(1-2y^2)}$.

11. (i) $-(b/a)\cot\phi$. (ii) $-\tan\theta$. (iii) $1/t$.

(iv) $\frac{1}{2}\sec^3\theta \operatorname{cosec}\theta$. (v) $\frac{3}{2}\tan\theta$. (vi) $\tan t$

(vii) $\tan t$. (viii) $-\tan\frac{1}{2}t$. (ix) $\tan 3t$.

(x) $t(2-t^3)(1-2t^3)$. (xi) 1

13. (i) $\frac{4}{3}x^3$. (ii) $\sin x$. (iii) $\frac{1}{3}x^{-3}\log_{10} e$.

(iv) $1/2x\,(1+x^2)$. (v) 1. (vi) $\frac{1}{2}$.

(vii) $x^{\sin^{-1}x}\left\{\log x+\sin^{-1}x\cdot\dfrac{\sqrt{1-x^2}}{x}\right\}$.

14. (i) $\frac{1}{2}[(1-x)^{\frac{3}{2}}-(1+x)^{\frac{3}{2}}]$. (ii) $\dfrac{1}{2(a-b)}\left[\dfrac{1}{\sqrt{x+a}}-\dfrac{1}{\sqrt{x+b}}\right]$.

(iii) $\dfrac{1-x^2}{(1+x+x^2)^{\frac{1}{2}}(1-x+x^2)^{\frac{3}{2}}}$. (iv) $\dfrac{2(1-x^2)}{1+x^2+x^4}$. (v) $\sec x$.

(vi) $\operatorname{cosec} x$ (vii) $\dfrac{-1}{2\sqrt{1-x^2}}$. (viii) $-\frac{1}{2}$. (ix) $\frac{1}{2}$.

(x) -1. (xi) 0. (xii) $\dfrac{2a}{\sqrt{1-a^2x^2}}$. (xiii) $\dfrac{x}{\sqrt{(1-x^4)}}$.

(xiv) $\dfrac{-x}{\sqrt{(1-x^2)}}$. (xv) $-\dfrac{1}{x^2\sqrt{(1-x^2)}}-\dfrac{1}{x^2}$.

(xvi) $-\dfrac{2a^2}{x^3}\left\{1+\dfrac{a^2}{\sqrt{a^4-x^4}}\right\}$. (xvii) $\dfrac{x^2}{1-x^4}$.

(xviii) $\dfrac{-ab}{a^2\cos^2 x-b^2\sin^2 x}$. (xix) $\dfrac{2ab}{a^2\cos^2 x-b^2\sin^2 x}$.

(xx) $\dfrac{\sqrt{(a^2-b^2)}}{2(a+b\cos x)}$. (xxi) $\dfrac{-\sqrt{(b^2-a^2)}}{b+a\cos x}$. (xxii) $\dfrac{4}{5+3\cos x}$.

(xxiii) $\dfrac{1}{\sqrt{1+x^2}}$. (xxiv) $\dfrac{-1}{2x\sqrt{1-x^2}}$. (xxv) $1-\dfrac{x^4+x^2+2}{(x^3+2x)^2}$.

15. $x^2\tan^{-1}x$. **16** $\sqrt{(a^2-x^2)}$ **17** $\dfrac{1}{x}\sqrt{\dfrac{x+a}{x-a}}$.

18. $2\sqrt{(a^2+x^2)}$. **19.** $\dfrac{y}{\sqrt{(a^2-y^2)}}$. **20.** $\dfrac{6}{1-x^6}$.

21. $\dfrac{1}{1+x^4}$. **22.** $x^2\sqrt{(x^2+1)}$ **23.** 0. **25.** $0, \pm(a/b)$.

28. (i) $f'(x)=0$ or $\sin(1/x)+2x\cos(1/x)$ according as x is or is not zero ; $f'(x)$ is discontinuous for $x=0$.

(ii) $f'(x)=0$ or $3x^2\cos(1/x)+x\sin(1/x)$ according as x is or is not zero ; $f'(x)$ is continuous for $x=0$.

4·14. Significance of derivative and its sign.

A very important aspect of a derivative, following from its definition, is as a **rate-measurer.** This will be clear from the following examples.

Let s denote the length of the path covered by a moving particle in any time t. Clearly, as the particle moves continuously, s has a definite value for every value of t, and accordingly by definition, s is a function of t. If $s+\Delta s$ be the value of s corresponding to the value $t+\Delta t$ of t, then as Δs denotes the distance moved over by the particle in time Δt, the ratio $\frac{\Delta s}{\Delta t}$ in the limit, when Δt becomes infinitely small, represents the ratio at which the particle is describing its path per unit of time at the moment. But, on the other hand, $\underset{\Delta t \to 0}{Lt} \frac{\Delta s}{\Delta t}$ is by definition (since Δs and Δt denote corresponding changes in s and t), the derivative of s with respect to t. Thus the derivative $\frac{ds}{dt}$ respresents the rate of change of s with respect to t, *i.e.*, the speed of the moving particle.

More generally, if y be a function of the variable x, changing continuously with x, then Δy being the change in y corresponding to a change Δx of x, the derivative $\frac{dy}{dx} = \underset{\Delta x \to 0}{Lt} \frac{\Delta y}{\Delta x}$ represents the *rate of change of y with respect to x.*

Thus v being the velocity of a moving particle at time t, $\frac{dv}{dt}$ represents the time-rate of change of velocity, *i.e.*, its acceleration; again, if V be the volume of a quantity of gas enclosed in a flexible vessel at a constant temperature, when its pressure is p which we can change at pleasure, $\frac{dV}{dp}$ represents the rate of change of volume with pressure; and so on.

Next we may note, that y changing with x, *if y increases when x is increased and diminishes when x is diminished,* the corresponding changes Δy and Δx are of the same sign, and accordingly the ratio $\dfrac{\Delta y}{\Delta x}$ is positive. Hence, $\dfrac{dy}{dx}$ (when it exists and is $\neq 0$) *is positive.* Similarly, *if y decreases when x is increased, or increases when x is diminished,* $\dfrac{dy}{dx}$ *is negative.*

Conversely, a positive sign of $\dfrac{dy}{dx}$ at a point c indicates that in the neighbourhood of the point, y increases or decreases with x, *i.e.*, both y and x increase or decrease together. On the other hand, a negative sign of $\dfrac{dy}{dx}$ means that y decreases when x increases and vice versa near the point.

A formal proof of the above result is given below.

A theorem on the sign of f′(x).

If $f'(a) > 0$, prove that $f(x) < f(a)$ for all values of $x < a$ but sufficiently near to a, and $f(x) > f(a)$ for all values of $x > a$ but sufficiently near to a.

Since $\quad f'(a) > 0$

$$\therefore \quad \underset{h\to 0+0}{Lt} \frac{f(a+h)-f(a)}{h} > 0, \text{ and}$$

$$\underset{h\to 0+0}{Lt} \frac{f(a-h)-f(a)}{-h} > 0.$$

$\therefore$ for all sufficiently small values of h, we have $f(a-h) < f(a) < f(a+h)$.

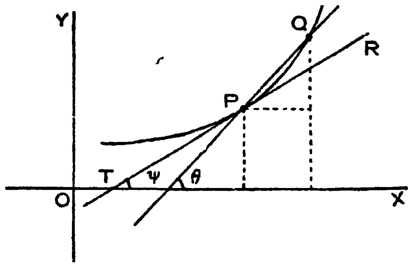
In other words, there exists some *neighbourhood* $(a-\delta, a+\delta)$ of a in which

$$f(x) > f(a) \text{ for } x > a, f(x) < f(a) \text{ for } x < a,$$

i.e., $f(x) > f(a)$ for $x > a$ but sufficiently near to a

$f(x) < f(a)$ for $x < a$ but sufficiently near to a.

When the function $y = f(x)$ is represented graphically, a **geometrical interpretation** of the derivative $\frac{dy}{dx}$ corresponding to any value of x may be given as follows :



Let P be a point (x, y) on the curve, and Q a neighbouring point $(x + \Delta x, y + \Delta y)$ which may be taken on either side of P, so that Δx may have any sign. The equation to the line PQ is (X, Y denoting current co-ordinates),

$$Y - y = \frac{y + \Delta y - y}{x + \Delta x - x}(X - x) = \frac{\Delta y}{\Delta x}(X - x).$$

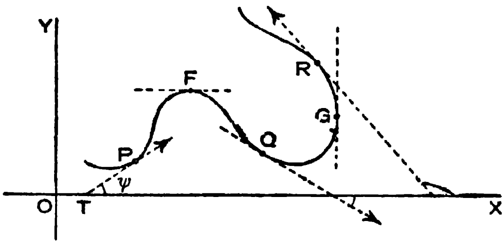
If θ be the inclination of this line PQ to the x-axis, the slope of the line, *i e*, $\tan\theta = \frac{\Delta y}{\Delta x}$ (i)

Now let Q approach P along the curve indefinitely closely, so that $\Delta x \to 0$. If the straight line PQ tends to a definite limiting position TPR as Q approaches P from either side, then TPR is called the tangent to the curve at P. In this case, if ψ be the inclination of TPR to the x-axis, then as PQ tends to TPR, $\theta \to \psi$. Also, as $\Delta x \to 0$, $\underset{\Delta x \to 0}{Lt} \frac{\Delta y}{\Delta x} = \frac{dy}{dx}$ from definition. Thus (i) leads to

$$\tan \psi = \frac{dy}{dx}.$$

Hence, *the derivative* $\frac{dy}{dx}$ *for any value of* x, *when it exists, is the trigonometrical tangent of the inclination* (otherwise known as *slope or gradient*) *of the tangent line at the corresponding point* P *on the curve* $y = f(x)$.

Also, if $\frac{dy}{dx}$ $(= \tan \psi)$ be positive, ψ is acute (as at P) in the figure below, and at that point y increases with x. If $\frac{dy}{dx}$ *i.e.*, $\tan \psi$ be negative, ψ is negative (as at Q), or is obtuse (as at R), and y diminishes when x increases, or vice versa.



At a point where $\frac{dy}{dx} = 0$, the tangent line is parallel to the x-axis (as at F), and at a point when $\frac{dy}{dx} \to \infty$, *i.e.*, $\frac{dx}{dy} \to 0$, the tangent line is parallel to the y-axis (as at G).

4·15. Differentials.

If $f'(x)$ is the derivative of $f(x)$, and Δx is an increment of x, then the *differential* of $f(x)$, denoted by the symbol $df(x)$, is *defined* by the relation

$$df(x)=f'(x)\,\Delta x. \qquad \dots \quad \text{(i)}$$

If $f(x)=x$, then $f'(x)=1$, and (i) reduces to $dx=\Delta x$. Thus, when x is the independent variable, the differential of $x (=dx)$ is identical with Δx. Hence, if $y=f(x)$, the relation (i) becomes

$$\mathbf{dy=f'(x)\,dx} \qquad \dots \quad \text{(ii)}$$

i.e., the differential of a function is equal to its derivative multiplied by the differential of the independent variable.

Thus, if $y=\tan x$, $dy=\sec^2 x\,dx$.

From the definition of the differential of a function, the following formulæ for finding differentials are obvious :

$$d(c)=0, \text{ where } c \text{ is a constant ;}$$

$$d(u+v-w)=du+dv-dw\ ;$$

$$d(uv)=u\,dv+v\,du\ ;\ d\left(\frac{u}{v}\right)=\frac{v\,du-u\,dv}{v^2}.$$

Differentials are especially useful in applications of integral calculus.

Note 1. The students should note carefully that although for the independent variable x, increment Δx and differential dx are equal, this is generally not the case with the dependent variable y, *i.e.*, $\Delta y \neq dy$ generally.

Note 2. The relation (ii) can be written as $dy/dx=f'(x)$; thus the quotient of the differentials of y and x is equal to the derivative of y with respect to x.

Probably on account of the position that $f'(x)$ occupies in equation (ii) above, $f'(x)$ is called the *differential coefficient.*

4·16. Approximate Calculations and Small Errors.

If $y=f(x)$, since $\underset{\Delta x \to 0}{Lt} \frac{\Delta y}{\Delta x}=f'(x)$, Δy is *approximately* $=f'(x)\,\Delta x$ for small values of Δx. Thus dy and Δy may be taken as approximately equal, when Δx $(=dx)$ is small. Hence, when only an *approximate value* of the change of a function is desired, it is usually convenient to calculate the value of the corresponding differential and use this value.

Small errors arising in the value of a function due to an assumed small error in the independent variable may also be calculated on the same principle.

As an illustration let us consider the following cases :

The radius of a sphere is found by measurement to be 7 inches ; if an error of ·01 inch is made in measuring the radius, find the error made in calculating the surface-area of sphere.

If S be the surface-area of the sphere of radius r,

$$S=4\pi r^2.$$

$$\therefore \quad dS=8\pi r\ dr.$$

Here $r=7$ and $dr=·01$.

$\therefore$ approximate error in the calculation of the surface-area

$$=dS=8\times\tfrac{22}{7}\times 7\times ·01=1·76 \text{ sq. in.}$$

Note 1. The *actual error* is $4\pi\{(7·01)^2-7^2\}$ which is very nearly equal to 1·76.

Note 2. If dx is the error in x, then the ratios (i) $\frac{dx}{x}$ and (ii) $100\cdot\frac{dx}{x}$ are called respectively the *relative error* (*i.e.*, error per unit x) and the *percentage error*. They may be easily obtained by logarithmic differentiation.

4·17. Illustrative Examples.

Ex. 1. *If the area of a circle increases at a uniform rate, show that the rate of increase of the perimeter varies inversely as the radius.*
[*C. P. 1930*]

At any time t, let A be the area, P the perimeter and r the radius of the circle Then $A=\pi r^2$; $P=2\pi r$. $\therefore$ $P^2=4\pi A$.

$\therefore$ differentiating both sides with respect to t, we have,

$$2P\frac{dP}{dt}=4\pi\frac{dA}{dt}, \text{ i.e., } \frac{dP}{dt}=\frac{2\pi}{P}\cdot\frac{dA}{dt}=\frac{1}{r}\cdot\frac{dA}{dt}.$$

Since $\frac{dA}{dt}$ is constant, $\therefore$ $\frac{dP}{dt}\propto\frac{1}{r}$.

Ex. 2. *A ladder AB, 25 ft. long, leans against a vertical wall. If the lower end A, which is at a distance of 7 feet from the bottom of the wall, is being moved away on the ground from the wall at the rate of 2 ft per second, find how fast is the top B descending on the wall.*

Let the distance of A and B from O, the bottom of the wall, at time t be x and y. Then the velocities of A and B are $\frac{dx}{dt}$ and $\frac{dy}{dt}$;

hence, as given here, $\frac{dx}{dt}=2$.

Now, $x^2+y^2=25^2$.

Differentiating with respect to t, $2x\frac{dx}{dt}+2y\frac{dy}{dt}=0$.

$$\frac{dy}{dt}=-\frac{x}{y}\frac{dx}{dt}. \qquad \cdots \quad (1)$$

When $x=7$, $y^2=25^2-7^2=576=24^2$. $\therefore$ $y=24$.

Hence when $x=7$, $y=24$, $\frac{dx}{dt}=2$;

$\therefore$ from (1), $\frac{dy}{dt}=-\frac{7}{24}\times 2=-\frac{7}{12}$ ft. per sec. $=-7$ inches per sec.

$\therefore$ the end B is moving at the rate of 7 inches per second *towards O* ($\because$ dy/dt is negative) *i.e.*, B is descending at that rate.

Ex. 3. *The adiabatic law for the expansion of air is $PV^{1\cdot 4}=k$, where k is a constant. If at a given time the volume is observed to* *be 20 cu. ft. and the pressure is 50 lbs. per square inch, at what rate is the pressure changing if the volume is decreasing at the rate of 2 cu. ft. per sec.?*

$$PV^{1\cdot 4}=k.$$

Taking logarithm of both sides and differentiating with respect to t,

$$\frac{1}{P}\frac{dP}{dt}+1\cdot 4\,\frac{1}{V}\frac{dV}{dt}=0.$$

When $V=20$, $P=50\times 144$ lbs. per sq. ft., and $\dfrac{dV}{dt}=-2$ then ;

$\therefore$ $$\frac{1}{50\times 144}\times\frac{dP}{dt}+1\cdot 4\times\frac{1}{20}(-2)=0.$$

$\therefore$ $$\frac{dP}{dt}=\cdot 14\times 50\times 144 \text{ lbs. sq. ft.}$$

$$=\cdot 14\times 50, \text{ i.e., } 7 \text{ lbs. per sq inch per sec.}$$

Ex. 4. *If $y=2x-\tan^{-1}x-\log(x+\sqrt{1+x^2})$, show that y continually increases as x changes from zero to positive infinity.*

[*C. P. 1942*]

Here $$\frac{dy}{dx}=2-\frac{1}{1+x^2}-\frac{1}{x+\sqrt{1+x^2}}\cdot\left\{1+\frac{2x}{2\sqrt{1+x^2}}\right\}$$

$$=2-\frac{1}{1+x^2}-\frac{1}{\sqrt{1+x^2}}.$$

Since $1+x^2$ and $\sqrt{1+x^2}$ are each greater than 1,

$\therefore$ $\dfrac{1}{1+x^2}$ and $\dfrac{1}{\sqrt{1+x^2}}$ are each less than 1.

$\therefore$ $\dfrac{dy}{dx}$ is positive ; also $y=0$, when $x=0$.

$\therefore$ for positive values of x, y must be positive and continually increases as x increases from 0 to ∞.

Ex. 5. *Find approximately the value of tan 46°, given 1° = ·01745 radians.*

Let $y=\tan x$. $\therefore$ $dy=\sec^2 x\,dx$.

Thus taking $x=45°$ $(=\frac{1}{4}\pi)$, $dx=1°=\cdot 01745$, we have

$$dy=2\times\cdot 01745=\cdot 03490.$$

Hence, for an increment of 1° in the angle, the increment in the value of tan 45° is $\cdot 03490$.

$\therefore$ $\tan 46^\circ = \tan 45^\circ + \cdot 03490 = 1\cdot 03490$ approximately.

Ex. 6. *If in a triangle the side c and angle C remain constant, while the remaining elements are changed slightly, show that*

$$\frac{da}{\cos A} + \frac{db}{\cos B} = 0.$$

In any triangle, $\dfrac{a}{\sin A} = \dfrac{b}{\sin B} = \dfrac{c}{\sin C}$.

Since c and C are constant, $\therefore$ $c/\sin C = \text{constant} = K$ suppose.

$$\therefore \quad \frac{a}{\sin A} = \frac{b}{\sin B} = K.$$

$\therefore$ $a = K \sin A$, and $\therefore$ $da = K \cos A \, dA$.

Also $b = K \sin B$, $\therefore$ $db = K \cos B \, dB$.

$$\therefore \quad \frac{da}{\cos A} + \frac{db}{\cos B} = K.(dA + dB) = K.d(A+B)$$

$$= K.d\,(\pi - C)$$

$$= K \times 0 = 0. \qquad (\because C \text{ is constant})$$

Examples IV(C)

1. A point moves on the parabola $3y = x^2$ in such a way that when $x = 3$, the abscissa is increasing at the rate of 3 ft. per second. At what rate is the ordinate increasing at that point ?

2. A toy spherical balloon being inflated, the radius is increasing at the rate of $\frac{1}{11}$ inch per second. At what rate would the volume be increasing at the instant when $r = 3\frac{1}{2}$ inches ?

3. A circular plate of metal expands by heat so that its radius increases at the rate of $\cdot 25$ inches per second. Find the rate at which the surface-area is increasing when the radius is 7 inches.

4. The candle-power of an incandescent lamp and its voltage V are connected by the equation $C=\dfrac{5V^6}{10^{11}}$.

Find the rate at which the candle-power increases with the voltage when $V=200$.

5. If Q units be the heat required to raise the temperature of 1 gramme of water from $0°C$ to $t°C$, then it is known that $Q=t+10^{-5}.2t^2+10^{-7}.3t^3$. Find the specific heat at $50°C$, the specific heat being the rate of increase of heat per unit degree rise of temperature.

6. A man 5 ft. tall walks away from a lamp-post $12\frac{1}{2}$ ft. high at the rate of 3 miles per hour.

(i) How fast is the farther end of his shadow moving on the pavement ?

(ii) How fast is his shadow lengthening ?

7. If a particle moves according to the law $x \propto t^2$, when x is the distance (measured from a fixed point) travelled in time t, show that the velocity will be proportional to time and the rate of change of velocity will be constant.

8. Water is poured into an inverted conical vessel of which the radius of the base is 6 ft. and height 12 ft., at the rate of $5\frac{1}{2}$ cu. inches per minute. At what rate is the water-level rising at the instant when the depth is $3\frac{1}{2}$ inches ?

9. If the side of an equilateral triangle increases at the rate of $\sqrt{3}$ ft. per second and its area at the rate of 12 sq. ft. per second, find the side of the triangle.

10. If in the rectilinear motion of a particle $s=ut+\frac{1}{2}ft^2$, when u and f are constant, prove that the velocity at time t is $u+ft$ and acceleration is f.

11. A man is walking at the rate of 5 miles per hour towards the foot of a building 40 ft. high. At what rate is he approaching the top when he is 30 ft. from the foot of the building ?

***12.** A circular ink-blot grows at the rate of 2 sq. inches per second. Find the rate at which the radius is increasing after $2\frac{6}{11}$ seconds.

***13.** The volume of a right circular cone remains constant. If the radius of the base is increasing at the rate of 3 inches per second, how fast is the altitude changing when the altitude is 8 inches and radius 6 inches ?

***14.** Sand is being poured on the ground and forms a pile which has always the shape of a right circular cone whose height is equal to the radius of the base. If the sand is falling at the rate of 7·7 cu. ft. per sec., how fast is the height of the pile increasing when the height is 3·5 ft. ?

***15.** The marginal cost of a commodity being the rate of change in cost for change in the output, if $f(x) \equiv ax \cdot \dfrac{x+b}{x+c} + d$ $(b > c)$ be the total cost of an output x, show that the marginal cost falls continuously as the output increases.

16. (i) An aeroplane is flying horizontally at a height of $\frac{2}{3}$ mile with a velocity of 15 miles an hour. Find the rate at which it is receding from a fixed point on the ground which it passed over 2 minutes ago.

(ii) A kite is 300 ft. high and there are 500 ft. of cord out. If the wind moves the kite horizontally at the rate of 5 miles per hour directly away from the person who is flying it, how fast is the cord being paid ?

17. If $\phi(x) = (x-1)\,e^x + 1$, show that $\phi(x)$ is positive for all positive values of x. [*C. P. 1943*]

18. If $f(x) = \cos x + \cos^2 x + x \sin x$, then $f(x)$ continually diminishes as x increases from 0 to $\frac{1}{2}\pi$.

19. Show that for $0 < \theta < \frac{1}{2}\pi$,

(i) $\dfrac{\sin\theta}{\theta}$ continually diminishes as θ continually increases.

(ii) $\dfrac{4 \sin \theta}{2+\cos \theta} - \theta$ increases with θ.

***20.** (i) Prove that if $0 < x < \frac{1}{2}\pi$,

(*a*) $1 - \dfrac{1}{2\,!}x^2 < \cos x < 1 - \dfrac{1}{2\,!}x^2 + \dfrac{1}{4\,!}x^4$.

(*b*) $x - \dfrac{1}{3\,!}x^3 < \sin x < x - \dfrac{1}{3\,!}x^3 + \dfrac{1}{5\,!}x^5$.

(ii) Show that if $x > 0$,

(*a*) $x > \log(1+x) > x - \frac{1}{2}x^2$.

(*b*) $\frac{1}{2}x^2 + 2x + 3 > (3-x)\,e^x$.

21. Given $y = \dfrac{a \sin x + b \cos x}{c \sin x + d \cos x}$, prove that

(i) if $a=1$, $b=2$, $c=3$, $d=4$, then y decreases for all values of x ;

and (ii) if $a=2$, $b=1$, $c=3$, $d=4$, then y increases for all values of x.

22. Find the range of values of x for which each of the functions,

(i) $x^3 - 3x^2 - 24x + 30$, (ii) $x^3 - 9x^2 + 24x - 16$,
increases with x.

(iii) $2x^3 - 9x^2 + 12x - 3$, (iv) $x^4 - 4x^3 + 4x^2 + 40$,
decreases as x increases.

23. Show that the function $x^3 - 3x^2 + 6x - 8$ increases with x.

24. Find the approximate values of the following by the method of differentials :

(i) $\log_e 10{\cdot}1$, given $\log_e 10 = 2{\cdot}303$.

(ii) $\log_{10} 10{\cdot}1$, given $\log_{10} e = {\cdot}4343$.

(iii) $\sqrt{6{\cdot}33}$, given $\sqrt{6{\cdot}25} = 2{\cdot}5$.

(iv) $\sec^2 46°$, given $1° = \cdot 0175$ radian.

(v) $\sin 62°$, given $\sin 60° = \cdot 86603$.

25. What is the approximate change in $\sin\theta$ per minute change in θ when $\theta = 60°$? (given $1' = \cdot 00029$ radian.)

***26.** Find approximately the values of :

(i) $x^3 + 4x^2 + 2x + 2$, when $x = 2\cdot 00012$.

(ii) $x^4 + 4x^2 + 1$, when $x = 1\cdot 997$.

***27.** Find approximately the difference in areas of two circles of radii $7''$ and $7\cdot 01''$.

***28.** What error in the common logarithm of a number will be produced by an error of 1% in the number ?

$$[\ \log_{10} e = \cdot 4343\ ]$$

29. Find the relative error (*i.e.*, error per unit area) in calculating the area of a triangle two of whose sides are 5 and 6 inches, when the included angle is taken as 45° instead of 45° 2′.

30. Show that the relative error in computing the volume of a sphere, due to an error in measuring the radius, is approximately equal to three times the relative error in the radius.

31. The angle of elevation of the top of a tower as observed from a distance of 100 ft. from the foot of the tower is found to be 60° ; if the angle of elevation was really 60° 1′, obtain approximately the error in the calculated height. [$1' = \cdot 00029$ radian.]

32. The pressure p and the volume v of a gas are connected by the relation $pv^{1\cdot 4} = k$, where k is a constant. If there be an increase of ·7 per cent. in the pressure, show that there is a decrease of ·5 per cent. in the volume.

33. An electric current C as measured by a galvanometer is given by the relation $C \propto \tan\theta$. Find the percentage error in the current corresponding to an error of ·7 per cent. in the measurement of θ, when $\theta = 45°$.

***34.** The time T of a complete oscillation of a simple pendulum of length l is given by the relation $T=2\pi\sqrt{(l/g)}$, where g is a constant. Find approximately the percentage error in the calculated value of T corresponding to an error of 1 per cent. in the value of l.

***35.** (i) In a triangle if the sides and angles receive small variations, but a and B are constants, show that

$$\tan A\, db = b\, dC.$$

(ii) In a triangle if the sides a, b be constant and the base angles A and B vary, show that

$$\frac{dA}{\sqrt{a^2-b^2\sin^2 A}}=\frac{dB}{\sqrt{b^2-a^2\sin^2 B}}.$$

***36.** If a triangle ABC inscribed in a fixed circle be slightly varied in such a way as to have its vertices always on the circle, then

$$\frac{da}{\cos A}+\frac{db}{\cos B}+\frac{dc}{\cos C}=0.$$

ANSWERS

1. 6 ft. per sec. **2.** 14 cu. inches per sec.

3. 11 sq. in. per second. **4.** 96. **5.** 1·00425.

6. (i) 5 miles per hour. (ii) 2 miles per hour or 2·93 ft./sec.

8. $\frac{4}{7}$ ins. per minute. **9.** 8 ft. **11.** 3 miles per hour.

12. ·25 inches per second. **13.** Decreasing 8 ins. per sec.

14. ·2 ft. per sec.

16. (i) 9 miles per hour. (ii) 4 miles per hour.

22. (i) $x>4$ or <-2. (ii) $x>4$ or <2. (iii) $1<x<2$.

(iv) $x<0$ and $1<x<2$. **24.** (i) 2·313. (ii) 1·0043.

(iii) 2·516. (iv) 2·07. (v) ·8835. **25.** ·00015.

26. (i) 30·0036. (ii) 32·856. **27.** ·44 sq. inches.

28. ·0043. **29.** ·00058. **31.** ·116 ft. **33.** 1·1. **34.** ·5.

CHAPTER V

SUCCESSIVE DIFFERENTIATION

5·1. Definitions and Notations.

We have seen that the derivative of a function of x say $f(x)$, is in general a function of x. This new function (*i.e.*, the derivative) may have a derivative, which is called the *second derivative* (or *second differential coefficient*) of $f(x)$, the original derivative being called the *first derivative* (or *first differential coefficient*). Similarly, the derivative of the second derivative is called the *third derivative*; and so on for the *nth derivative*.

Thus, if $y=x^3$, $\dfrac{dy}{dx}=3x^2$.

Again, $\dfrac{d}{dx}\left(\dfrac{dy}{dx}\right)=\dfrac{d}{dx}(3x^2)=6x$.

Now, $\dfrac{d}{dx}\left(\dfrac{dy}{dx}\right)$ is denoted by $\dfrac{d^2y}{dx^2}$.

$\therefore$ $\dfrac{d^2y}{dx^2}$ (*i.e*, the second derivative of y with respect to x) in this case is $6x$.

Again, $\dfrac{d}{dx}\left(\dfrac{d^2y}{dx^2}\right)=\dfrac{d}{dx}(6x)=6$.

Now, $\dfrac{d}{dx}\left(\dfrac{d^2y}{dx^2}\right)$ is denoted by $\dfrac{d^3y}{dx^3}$.

$\therefore$ $\dfrac{d^3y}{dx^3}$ (*i.e.*, the third derivative of y with respect to x) is 6 here.

Similarly, the nth derivative of y with respect to x is generally denoted by $\dfrac{d^n y}{dx^n}$.

If $y=f(x)$, the successive derivatives are also denoted by

$$
\begin{array}{llll}
 & y_1, & y_2, & y_3, \ldots\ldots\ldots\ldots y_n \\
\text{or,} & y', & y'', & y''', \ldots\ldots\ldots\ldots y^{(n)} \\
\text{or,} & \dot{y}, & \ddot{y}, & \dddot{y}, \ldots\ldots\ldots\ldots \\
\text{or,} & f'(x), & f''(x), & f'''(x), \ldots\ldots\ldots\ldots f^{(n)}(x) \\
\text{or,} & Df(x), & D^2 f(x), & D^3 f(x) \ldots\ldots\ldots\ldots D^n f(x)
\end{array}
$$

D standing for the symbol $\dfrac{d}{dx}$.

5·2. The nth derivatives* of some special functions.

(i) $\mathbf{y = x^n}$, *where n is a positive integer.*

$y_1 = nx^{n-1}$; $y_2 = n(n-1)\,x^{n-2}$; $y_3 = n(n-1)(n-2)\,x^{n-3}$; and proceeding in a similar manner,

$$y_r = n(n-1)(n-2)\ldots\{n-(r-1)\}x^{n-r}\ (r < n)$$

$$\therefore \quad y_n = n(n-1)(n-2)\ldots 3.2.1 = n\,!$$

$$\text{i.e., } \mathbf{D^n(x^n) = n\,!}$$

Cor. Since $y_n = n\,!$, which is a constant, $y_{n+1}, y_{n+2}\ldots\ldots\ldots$ etc. are all zeroes in this case.

(ii) $\mathbf{y = (ax+b)^m}$, where *m is any number.*

$y_1 = ma\,(ax+b)^{m-1}$; $y_2 = m(m-1)a^2\,(ax+b)^{m-2}$;

$y_3 = m(m-1)(m-2)a^3\,(ax+b)^{m-3}$; and proceeding similarly,

$$y_n = m(m-1)(m-2)\ldots(m-n+1)\,a^n\,(ax+b)^{m-n}.$$

$$\therefore\ \mathbf{D^n(ax+b)^m = m(m-1)(m-2)\ldots(m-n+1)a^n(ax+b)^{m-n}}.$$

---

* Strictly speaking, in these cases, the nth derivatives are to be established generally by the method of Induction.

If m be a positive integer greater than n,

since $m(m-1)(m-2)\ldots(m-n+1)=\dfrac{m\,!}{(m-n)\,!}$.

$$\therefore \quad \mathbf{D^n(ax+b)^m=\frac{m\,!}{(m-n)\,!}\,a^n\,(ax+b)^{m-n}}, = {}^mP_n\, a^r (ax+b)^n$$

m being a positive integer greater than n.

Note. *If m be a positive integer less than n, $D^n\ (ax+b)^m=0$.*
When $m=n$, $D^n\ (ax+b)^n=a^n.\ n\,!$

(iii) $\mathbf{y=e^{ax}}$.

$$\therefore \quad y_1=ae^{ax}\ ;\ y_2=a^2e^{ax}\ ;\ y_3=a^3e^{ax}\ ;\ \ldots y_n=a^ne^{ax}.$$

$$\therefore \quad \mathbf{D^n\ (e^{ax})=a^ne^{ax}}.$$

Cor. (i) $\mathbf{D^n\ (e^x)=e^x}$.

Cor. (ii) $y=a^x=e^{x\log_e a}$. $\quad\therefore\quad \mathbf{D^n\,(a^x)=(\log_e a)^n a^x}$.

(iv) $\mathbf{y=\dfrac{1}{x+a}}$.

$$\therefore \quad y_1=-1.\,(x+a)^{-2}\ ;\ y_2=(-1)(-2)(x+a)^{-3}$$
$$=(-1)^2.2\,!\,(x+a)^{-3}.$$

Similarly, $y_3=(-1)^3\ 3\,!\,(x+a)^{-4}$ etc.

$$\therefore \quad \mathbf{D^n\left(\frac{1}{x+a}\right)=\frac{(-1)^n\,n\,!}{(x+a)^{n+1}}}.$$

Cor. Proceeding as above, $\mathbf{D^n\,\dfrac{1}{(ax+b)^m}=\dfrac{(-1)^n a^n\ (m+n-1)\,!}{(m-1)\,!\ (ax+b)^{m+n}}}$.

(v) $\mathbf{y=\log\,(x+a)}$.

$\therefore \quad y_1=\dfrac{1}{x+a}$. Hence as in (iv) above,

$$\mathbf{D^n\,\{\log\,(x+a)\}=\frac{(-1)^{n-1}(n-1)\,!}{(x+a)^n}}.$$

Cor. $\mathbf{D^n\,\{\log\,(ax+b)\}=\dfrac{(-1)^{n-1}\,(n-1)\,!\,a^n}{(ax+b)^n}}$.

(vi) $\mathbf{y = \sin(ax+b)}$.

$$y_1 = a\cos(ax+b) = a\sin(\tfrac{1}{2}\pi + ax + b).$$

$$y_2 = a^2\cos(\tfrac{1}{2}\pi + ax + b) = a^2\sin(2.\tfrac{1}{2}\pi + ax + b).$$

$$y_3 = a^3\cos(2.\tfrac{1}{2}\pi + ax + b) = a^3\sin(3.\tfrac{1}{2}\pi + ax + b) \text{ etc.}$$

$$\therefore\quad \mathbf{D^n\{\sin(ax+b)\} = a^n \sin\left(\frac{n\pi}{2} + ax + b\right)}.$$

Similarly, $\mathbf{D^n\{\cos(ax+b)\} = a^n \cos\left(\frac{n\pi}{2} + ax + b\right)}$.

As particular cases when $b=0$,

$$\mathbf{D^n(\sin ax) = a^n \sin\left(\frac{n\pi}{2} + ax\right)};$$

$$\mathbf{D^n(\cos ax) = a^n \cos\left(\frac{n\pi}{2} + ax\right)}.$$

5·3. The nth derivatives of rational algebraic functions.

The nth derivative of a fraction whose numerator and denominator are both rational integral algebraic functions may be conveniently obtained by resolving the fraction into *partial fractions*. This is shown in Ex. 4, Art. 5·4. The rules for decomposing a fraction into partial fractions are given in the Appendix.

Even when the denominator of a given algebraic fraction cannot be broken up into *real* linear factors, the above method of decomposition can be used by resolving the denominator into *imaginary* linear factors. In this case *De Moivre's theorem* is conveniently applied to put the final result in the real form. This is illustrated in Ex. 5, Art. 5·4.

5·4. Illustrative Examples.

Ex. 1. *If $y = \sin^3 x$, find y_n.*

$$\sin 3x = 3 \sin x - 4 \sin^3 x.$$

$\therefore \quad y = \sin^3 x = \frac{1}{4} [3 \sin x - \sin 3x].$

$\therefore \quad y_n = \frac{1}{4} [3 \sin (x + \frac{1}{2}n\pi) - 3^n \sin (3x + \frac{1}{2}n\pi)].$

Ex. 2. *If $y = \sin 3x \cos 2x$, find y_n.*

$$y = \tfrac{1}{2}.2 \sin 3x \cos 2x = \tfrac{1}{2} (\sin 5x + \sin x).$$

$\therefore \quad y_n = \frac{1}{2} [5^n \sin (\frac{1}{2}n\pi + 5x) + \sin (\frac{1}{2}n\pi + x)].$

Ex. 3. *If $y = e^{ax} \sin bx$, find y_n.*

$$y_1 = e^{ax}.a.\sin bx + e^{ax}.\cos bx\ b$$
$$= e^{ax} (a \sin bx + b \cos bx).$$

Let $a = r \cos \phi$, $b = r \sin \phi$, so that

$$r = (a^2 + b^2)^{\frac{1}{2}},\ \phi = \tan^{-1} (b/a).$$

$\therefore \quad y_1 = re^{ax} \sin (bx + \phi).$

Similarly, $y_2 = re^{ax} [a \sin (bx + \phi) + b \cos (bx + \phi)]$

$$= r^2 e^{ax} \sin (bx + 2\phi), \text{ as before.}$$

In a similar way, $y_3 = r^3 e^{ax} \sin (bx + 3\phi)$, etc., and generally,

$$y_n = r^n e^{ax} \sin (bx + n\phi),$$

i.e., $\mathbf{D^n (e^{ax} \sin bx) = (a^2 + b^2)^{\frac{1}{2}n} e^{ax} \sin (bx + n \tan^{-1} b/a)}.$

Note. Similarly,

$$\mathbf{D^n (e^{ax} \cos bx) = (a^2 + b^2)^{\frac{1}{2}n} e^{ax} \cos (bx + n \tan^{-1} b/a)}.$$

Again, if $y = e^{ax} \sin (bx + c)$,

$$y_n = (a^2 + b^2)^{\frac{n}{2}} e^{ax} \sin (bx + c + n \tan^{-1} b/a)$$

and if $y = e^{ax} \cos (bx + c)$,

$$y_n = (a^2 + b^2)^{\frac{n}{2}} e^{ax} \cos (bx + c + n \tan^{-1} b/a).$$

Ex. 4. *If* $y=\dfrac{x^2+x-1}{x^3+x^2-6x}$, *find* y_n.

$$x^3+x^2-6x=x(x^2+x-6)=x(x+3)(x-2).$$

Let $\dfrac{x^2+x-1}{x^3+x^2-6x}=\dfrac{A}{x}+\dfrac{B}{x+3}+\dfrac{C}{x-2}$.

Multiplying both sides by $x(x+3)(x-2)$, we get

$$x^2+x-1=A(x+3)(x-2)+Bx\,(x-2)+Cx\,(x+3).$$

Putting $x=0,\ -3,\ 2$ successively on both sides, we get

$$A=\tfrac{1}{6},\ B=\tfrac{1}{3},\ C=\tfrac{1}{2}.$$

$$\therefore\quad y=\frac{1}{6}\cdot\frac{1}{x}+\frac{1}{3}\cdot\frac{1}{x+3}+\frac{1}{2}\cdot\frac{1}{x-2}.$$

$$\therefore\quad y_n=(-1)^n n!\left[\frac{1}{6}\cdot\frac{1}{x^{n+1}}+\frac{1}{3}\cdot\frac{1}{(x+3)^{n+1}}+\frac{1}{2}\cdot\frac{1}{(x-2)^{n+1}}\right].$$

Ex. 5. *If* $y=\dfrac{1}{x^2+a^2}$, *find* y_n.

$$y=\frac{1}{(x+ia)(x-ia)}=\frac{1}{2ia}\left[\frac{1}{x-ia}-\frac{1}{x+ia}\right].$$

$$\therefore\quad y_n=\frac{(-1)^n n!}{2ia}\left[\frac{1}{(x-ia)^{n+1}}-\frac{1}{(x+ia)^{n+1}}\right]$$

$$=\frac{(-1)^n n!}{2ia}\left[(x-ia)^{-(n+1)}-(x+ia)^{-(n+1)}\right].$$

Put $x=r\cos\theta,\ a=r\sin\theta$,

so that $r=(x^2+a^2)^{\frac{1}{2}},\ \theta=\tan^{-1}a/x$.

Now $(x-ia)^{-(n+1)}=r^{-(n+1)}(\cos\theta-i\sin\theta)^{-(n+1)}$

$$=r^{-(n+1)}\{\cos(n+1)\theta+i\sin(n+1)\theta\}.$$

$(x+ia)^{-(n+1)}=r^{-(n+1)}(\cos\theta+i\sin\theta)^{-(n+1)}$

$$=r^{-(n+1)}\{\cos(n+1)\theta-i\sin(n+1)\theta\}.$$

$$\therefore\quad y_n=\frac{(-1)^n n!}{2ia}\cdot r^{-(n+1)}.\,2i\sin(n+1)\theta.$$

Since $r=a/\sin\theta$, $r^{-(n+1)}=a^{-(n+1)}/\sin^{-(n+1)}\theta=\sin^{n+1}\theta/a^{n+1}$.

$$\therefore\quad \mathbf{D^n\left(\frac{1}{x^2+a^2}\right)=\frac{(-1)^n.n!}{a^{n+2}}\sin^{n+1}\theta\sin(n+1)\theta,}$$

where $\theta=\tan^{-1}\dfrac{a}{x}=\cot^{-1}\dfrac{x}{a}$.

Note. *If* $y=\dfrac{1}{(x+b)^2+a^2}$,

$$y_n=\frac{(-1)^n n!}{a^{n+2}}\sin^{n+1}\theta\sin(n+1)\theta,$$

where $\theta=\tan^{-1}\{a/(b+x)\}=\cot^{-1}\{(b+x)/a\}$.

Cor. *If* $y=\tan^{-1}x$, $y_1=\dfrac{1}{1+x^2}$; hence,

$$\mathbf{D^n(\tan^{-1}x)=(-1)^{n-1}(n-1)!\sin^n\theta\sin n\theta,}$$

where $\boldsymbol{\theta=\tan^{-1}(1/x)=\cot^{-1}x}$.

Examples V(A)

1. Find y_n in the following cases :

(i) $y=(a-bx)^m$. (ii) $y=1/(ax+b)^m$. (iii) $y=1/(a-x)$.

(iv) $y=\log(ax+b)^p$. (v) $y=\log\{(a-x)/(a+x)\}$.

(vi) $y=\sqrt{x}$. (vii) $y=1/\sqrt{x}$. (viii) $y=(2-3x)^n$.

(ix) $y=\log(ax+x^2)$. (x) $y=10^{3-2x}$. (xi) $y=x/(a+bx)$.

(xii) $y=(a-x)/(a+x)$. (xiii) $y=x^n/(x-1)$. (xiv) $y=\sin^2 x$.

(xv) $y=\cos 2x\cos x$. (xvi) $y=\cos^3 x$.

(xvii) $y=\sin^2 x\cos^2 x$. (xviii) $y=\sin x\sin 2x\sin 3x$.

(xix) $y=e^x\cos x$. (xx) $y=e^x\sin x\sin 2x$.

(xxi) $y=e^{3x}\sin 4x$. (xxii) $y=e^x\sin^2 x$.

2. Find y_3, if

(i) $y=x^2\log x$. (ii) $y=e^{\sin x}$.

(iii) $y=e^{1/x}$. (iv) $y=\sin^{-1}x$.

3. Find the nth derivatives of the following functions :

(i) $\dfrac{1}{x^2-a^2}$. (ii) $\dfrac{1}{x^2+16}$. (iii) $\tan^{-1}\dfrac{x}{a}$.

(iv) $\dfrac{x}{x^2+a^2}$. (v) $\dfrac{1}{x^4-a^4}$. (vi) $\dfrac{1}{x^2+x+1}$.

(vii) $\dfrac{1}{(x^2+a^2)(x^2+b^2)}$. (viii) $\dfrac{1}{4x^2+4x+5}$.

(ix) $\dfrac{x^4}{(x-1)(x-2)}$. (x) $\dfrac{x^2+1}{(x-1)(x-2)(x-3)}$.

(xi) $\dfrac{x^2}{(x+1)^2(x+2)}$. (xii) $\dfrac{x^2}{(x-a)(x-b)}$.

(xiii) $\tan^{-1}\dfrac{1+x}{1-x}$. (xiv) $\sin^{-1}\dfrac{2x}{1+x^2}$.

(xv) $\tan^{-1}\dfrac{\sqrt{1+x^2}-1}{x}$. (xvi) $\cot^{-1}\dfrac{x}{a}$.

4. If $y=x^{2n}$, where n is a positive integer, show that
$$y_n=2^n\{1.3.5\ldots(2n-1)\}x^n.$$

5. If $u=\sin ax+\cos ax$, show that
$$u_n=a^n\{1+(-1)^n\sin 2ax\}^{\frac{1}{2}}.$$

6. If $ax^2+2hxy+by^2=1$, show that
$$\frac{d^2y}{dx^2}=\frac{h^2-ab}{(hx+by)^3}.$$

7. Find y_2, if

(i) $\sin x+\cos y=1$. (ii) $y=\tan(x+y)$.

(iii) $x^3+y^3-3axy=0$.

8. Find $\dfrac{d^2y}{dx^2}$ in the following cases :

(i) If $x=a\cos\theta$, $y=b\sin\theta$.

(ii) If $x=a(\theta+\sin\theta)$, $y=a(1-\cos\theta)$.

9. If $x=f(t)$, $y=\phi(t)$, then prove that
$$\frac{d^2y}{dx^2}=\frac{x_1y_2-y_1x_2}{x_1^{\,3}},$$
where suffixes denote differentiation with respect to t.

10. If $x \sin\theta + y\cos\theta = a$ and $x\cos\theta - y\sin\theta = b$, prove that

$$\frac{d^p x}{d\theta^p}\cdot\frac{d^q y}{d\theta^q} - \frac{d^q x}{d\theta^q}\cdot\frac{d^p y}{d\theta^p} \text{ is constant.}$$

11. Show that $\dfrac{d^n}{dx^n}\left[\dfrac{1}{x^2+1}\right] = \dfrac{(-1)^n n!}{(x^2+1)^{n+1}}$

$$\left[(n+1)x^n - \binom{n+1}{3}x^{n-2} + \binom{n+1}{5}x^{n-4} - \cdots\right].$$

12. If $y = \sin mx$, show that

$$\begin{vmatrix} y & y_1 & y_2 \\ y_3 & y_4 & y_5 \\ y_6 & y_7 & y_8 \end{vmatrix} = 0$$

where the suffixes of y denote the order of differentiations of y with respect to x.

ANSWERS

1. (i) $(-1)^n m(m-1)(m-2)\cdots(m-n+1)\, b^n (a-bx)^{m-n}$.

(ii) $\dfrac{(-1)^n m(m+1)(m+2)\cdots(m+n-1)\, a^n}{(ax+b)^{m+n}}$

(iii) $\dfrac{n!}{(a-x)^{n+1}}$.

(iv) $\dfrac{(-1)^{n-1} p.a^n (n-1)!}{(ax+b)^n}$.

(v) $(n-1)!\left\{\dfrac{-1}{(a-x)^n} + \dfrac{(-1)^n}{(a+x)^n}\right\}$.

(vi) $(-1)^{n-1}.\dfrac{1.3.5\cdots(2n-3)}{2^n x^{n-\frac{1}{2}}}$.

(vii) $(-1)^n.\dfrac{1.3.5\cdots(2n-1)}{2^n x^{n+\frac{1}{2}}}$.

(viii) $(-1)^n.3^n.n!$

(ix) $(-1)^{n-1}(n-1)!\left\{\dfrac{1}{x^n} + \dfrac{1}{(x+a)^n}\right\}$.

(x) $10^{3-2x}.(-2)^n.(\log_e 10)^n$.

(xi) $\dfrac{(-1)^{n+1} ab^{n-1}.n!}{(a+bx)^{n+1}}$.

(xii) $\dfrac{2a(-1)^n.n!}{(a+x)^{n+1}}$.

(xiii) $\dfrac{(-1)^n.n!}{(x-1)^{n+1}}$.

(xiv) $-2^{n-1}\cos(\frac{1}{2}n\pi + 2x)$.

(xv) $\frac{1}{2}\{3^n \cos(\frac{1}{2}n\pi+3x)+\cos(\frac{1}{2}n\pi+x)\}$.

(xvi) $\frac{1}{4}\{3\cos(\frac{1}{2}n\pi+x)+3^n\cos(\frac{1}{2}n\pi+3x)\}$.

(xvii) $-2^{2n-3}\cos(\frac{1}{2}n\pi+4x)$.

(xviii) $\frac{1}{4}\{4^n\sin(\frac{1}{2}n\pi+4x)+2^n\sin(\frac{1}{2}n\pi+2x)-6^n\sin(\frac{1}{2}n\pi+6x)\}$.

(xix) $2^{\frac{1}{2}n}\, e^x \cos(x+\frac{1}{4}n\pi)$.

(xx) $\frac{1}{2}e^x\{2^{\frac{1}{2}n}\cos(x+\frac{1}{4}n\pi)-10^{\frac{1}{2}n}\cos(3x+n\tan^{-1}3)\}$.

(xxi) $5^n e^{3x}\sin(4x+n\tan^{-1}\frac{4}{3})$.

(xxii) $\frac{1}{2}e^x\{1-5^{\frac{1}{2}n}\cos(2x+n\tan^{-1}2)\}$.

2. (i) $2/x$. (ii) $-e^{\sin^2 x}\cos x.\sin x.(\sin x+3)$.

(iii) $-\dfrac{(1+6x+6x^2)e^{1/x}}{x^6}$. (iv) $\dfrac{1+2x^2}{(1-x^2)^{5/2}}$.

3. (i) $\dfrac{(-1)^n n!}{2a}\left\{\dfrac{1}{(x-a)^{n+1}}-\dfrac{1}{(x+a)^{n+1}}\right\}$.

(ii) $\dfrac{(-1)^n n!\sin^{n+1}\theta\sin(n+1)\theta}{4^{n+2}}$, where $\theta=\tan^{-1}\dfrac{4}{x}$.

(iii) $\dfrac{(-1)^{n-1}(n-1)!\sin^n\theta\sin n\theta}{a^n}$, where $\theta=\tan^{-1}\dfrac{a}{x}$.

(iv) $\dfrac{(-1)^n n!}{a^{n+1}}\sin^{n+1}\theta\cos(n+1)\theta$, where $\theta=\tan^{-1}\dfrac{a}{x}$.

(v) $\dfrac{(-1)^n n!}{4a^3}\left[\dfrac{1}{(x-a)^{n+1}}-\dfrac{1}{(x+a)^{n+1}}-\dfrac{2}{a^{n+1}}\sin^{n+1}\theta\sin(n+1)\theta\right]$,

where $x=a\cot\theta$.

(vi) $\dfrac{(-1)^n.2^{n+2}n!}{(\sqrt{3})^{n+2}}\sin^{n+1}\theta\sin(n+1)\theta$,

where $\theta=\tan^{-1}\{\sqrt{3}/(2x+1)\}$.

(vii) $\dfrac{(-1)^n n!}{a^2-b^2}\left[\dfrac{\sin^{n+1}\theta\sin(n+1)\theta}{b^{n+2}}-\dfrac{\sin^{n+1}\phi\sin(n+1)\phi}{a^{n+2}}\right]$,

where $x=b\cot\theta=a\cot\phi$.

(viii) $(-1)^n n!\,\frac{1}{2}\sin^{n+1}\theta\sin(n+1)\theta$, where $\cot\theta=x+\frac{1}{2}$.

(ix) $(-1)^n n!\left\{\dfrac{16}{(x-2)^{n+1}}-\dfrac{1}{(x-1)^{n+1}}\right\}$, when $n>2$.

(x) $(-1)^n n!\left\{\dfrac{1}{(x-1)^{n+1}}-\dfrac{5}{(x-2)^{n+1}}+\dfrac{5}{(x-3)^{n+1}}\right\}$.

(xi) $(-1)^n n!\left\{\frac{n+1}{(x+1)^{n+2}}-\frac{3}{(x+1)^{n+1}}+\frac{4}{(x+2)^{n+1}}\right\}$.

(xii) $\frac{(-1)^n n!}{(a-b)}\left\{\frac{a^2}{(x-a)^{n+1}}-\frac{b^2}{(x-b)^{n+1}}\right\}$.

(xiii) $(-1)^{n-1}(n-1)!\sin^n\theta\sin n\theta$, where $\cot\theta=x$.

(xiv) $2(-1)^{n-1}(n-1)!\sin^n\theta\sin n\theta$, where $\cot\theta=x$.

(xv) $\frac{1}{2}(-1)^{n-1}(n-1)!\sin^n\theta\sin n\theta$, where $\cot\theta=x$.

(xvi) $\frac{(-1)^n(n-1)!\sin^n\theta\sin n\theta}{a^n}$, where $\theta=\cot^{-1}\frac{x}{a}$.

7. (i) $-\frac{\sin^2 x+\cos y}{\sin^3 y}$. (ii) $-\frac{2(1+y^2)}{y^5}$. (iii) $-\frac{2a^3xy}{(y^2-ax)^3}$.

8. (i) $-\frac{b}{a^2}\operatorname{cosec}^3\theta$. (ii) $\frac{1}{4a}\sec^4\frac{\theta}{2}$.

5·5. Leibnitz's Theorem*. (*nth derivative of the product of two functions*)

If u and v are two functions of x, then the nth derivative of their product *i.e.*,

$$(\mathbf{uv})_n=\mathbf{u}_n\mathbf{v}+{}^n\mathbf{c}_1\mathbf{u}_{n-1}\mathbf{v}_1+{}^n\mathbf{c}_2\mathbf{u}_{n-2}\mathbf{v}_2+\cdots\cdots+{}^n\mathbf{c}_r\,\mathbf{u}_{n-r}\mathbf{v}_r+\cdots\cdots+\mathbf{uv}_n,$$

where the suffixes in u and v denote the order of differentiations of u and v with respect to x.

Let $y=uv$.

By actual differentiation, we have

$$y_1=u_1v+uv_1$$

$$y_2=u_2v+2u_1v_1+uv_2=u_2v+{}^2c_1u_1v_1+uv_2$$

$$y_3=u_3v+3u_2v_1+3u_1v_2+uv_3$$

$$=u_3v+{}^3c_1u_2v_1+{}^3c_2u_1v_2+uv_3.$$

*Leibnitz (1646-1716), was a German mathematician, who invented Calculus in Germany, as Newton did in England.

The theorem is thus seen to be true when $n = 2$ and 3.

Let us *assume* therefore that

$$y_n = u_n v + {}^n c_1 u_{n-1} v_1 + {}^n c_2 u_{n-2} v_2 + \cdots\cdots + {}^n c_r u_{n-r} v_r + \cdots\cdots + u v_n,$$

where n has any particular value.

$\therefore$ differentiating,

$$y_{n+1} = u_{n+1} v + ({}^n c_1 + 1) u_n v_1 + ({}^n c_2 + {}^n c_1) u_{n-1} v_2 + \cdots + ({}^n c_r + {}^n c_{r-1}) u_{n-r+1} v_r + \cdots\cdots + u v_{n+1}.$$

Since, ${}^n c_r + {}^n c_{r-1} = {}^{n+1} c_r$ and ${}^n c_1 + 1 = {}^{n+1} c_1$, $(1 = {}^n c_0)$

$$\therefore \quad y_{n+1} = u_{n+1} v + {}^{n+1} c_1 u_n v_1 + {}^{n+1} c_2 u_{n-1} v_2 + \cdots\cdots + {}^{n+1} c_r u_{n-r+1} v_r + \cdots\cdots + u v_{n+1}.$$

Thus, if the theorem holds for n differentiations, it also holds for $n+1$. But it was proved to hold for 2 and 3 differentiations; hence it holds for four, and so on, and thus the theorem is true for every positive integral value of n.

5·6. Important results of symbolic operation.

If $F(D)$ be any rational integral algebraic function of D or $\dfrac{d}{dx}$ (the symbolic operator), *i.e.*, if

$$F(D) = A_n D^n + A_{n-1} D^{n-1} + \cdots\cdots + A_1 D + A,$$

$$= \Sigma A_r D^r, \text{ where } A_r \text{ is independent of } D, \text{ then}$$

(i) $\mathbf{F(D)\, e^{ax} = F(a)\, e^{ax}}$.

(ii) $\mathbf{F(D)\, e^{ax}\, V = e^{ax}\, F(D+a) V}$, **V being a function of x.**

(iii) $$\mathbf{F(D^2) \begin{Bmatrix} \sin (ax+b) \\ \cos (ax+b) \end{Bmatrix} = F(-a^2) \begin{Bmatrix} \sin (ax+b) \\ \cos (ax+b). \end{Bmatrix}}$$

Proof :

(i) Since $D^r e^{ax} = a^r e^{ax}$,

$\therefore \quad F(D)e^{ax} = \Sigma A_r D^r (e^{ax}) = \Sigma A_r a^r e^{ax} = (\Sigma A_r a^r) e^{ax}$
$= F(a) e^{ax}$.

(ii) Let $y = e^{ax} V$; since $D^r e^{ax} = a^r e^{ax}$,

$\therefore$ by Leibnitz's Theorem, we have

$$y_n = e^{ax} (a^n V + {}^n c_1 a^{n-1} DV + {}^n c_2 a^{n-2} D^2 V + \cdots\cdots + D^n V)$$

which by analogy with the Binomial theorem may be written as

$$D^n (e^{ax} V) = e^{ax} (D+a)^n V.$$

$$\begin{aligned} \therefore \quad F(D)e^{ax} V &= (\Sigma A_r D^r) e^{ax} V \\ &= \Sigma A_r D^r e^{ax} V \\ &= e^{ax} \Sigma A_r (D+a)^r V \\ &= e^{ax} F(D+a) V. \end{aligned}$$

(iii) We have $D \sin(ax+b) = a \cos(ax+b)$, and so
$D^2 \sin(ax+b) = (-a^2) \sin(ax+b)$;
$\therefore \quad D^{2r} \sin(ax+b) = (-a^2)^r \sin(ax+b)$.

Hence, as in (i) and (ii), it follows that

$$F(D^2) \sin(ax+b) = F(-a^2) \sin(ax+b).$$

Similarly, $F(D^2) \cos(ax+b) = F(-a^2) \cos(ax+b)$.

5·7. Illustrative Examples.

Ex. 1. *If $y = e^{ax} x^3$, find y_n.*

Let $u = e^{ax}$, $v = x^3$. Now, $u_n = a^n e^{ax}$.

$\therefore$ by Leibnitz's Theorem,

$$y_n = a^n e^{ax} x^3 + n.a^{n-1} e^{ax}.3x^2 + \frac{n(n-1)}{2!} \cdot a^{n-2} e^{ax}.6x$$
$$+ \frac{n(n-1)(n-2)}{3!} a^{n-3} e^{ax} 6$$
$$= e^{ax} a^{n-3} \{a^3 x^3 + 3na^2 x^2 + 3n(n-1) ax + n(n-1)(n-2)\}.$$

Ex. 2. *If $y = a \cos(\log x) + b \sin(\log x)$, show that*

$$x^2 y_2 + x y_1 + y = 0.$$

Differentiating,

$$y_1 = -a \sin(\log x) \cdot \frac{1}{x} + b \cos(\log x) \cdot \frac{1}{x};$$

$$\therefore \quad x y_1 = -a \sin(\log x) + b \cos(\log x).$$

Differentiating again,

$$x y_2 + y_1 = -a \cos(\log x) \cdot \frac{1}{x} - b \sin(\log x) \cdot \frac{1}{x}.$$

$$\therefore \quad x^2 y_2 + x y_1 = -(a \cos \log x + b \sin \log x) = -y.$$

$$\therefore \quad x^2 y_2 + x y_1 + y = 0.$$

Note. This is called the *differential equation* formed from the above equation.

Ex. 3. *Differentiate n times the equation*

$$(1+x^2)\, y_2 + (2x-1)\, y_1 = 0.$$

By Leibnitz's Theorem,

$$\frac{d^n}{dx^n}\{y_2\,(1+x^2)\} = y_{n+2}\,(1+x^2) + n . y_{n+1}\, 2x + \frac{n(n-1)}{2\,!}\, y_n . 2$$

$$\frac{d^n}{dx^n}\{y_1\,(2x-1)\} = y_{n+1}\,(2x-1) + n\, y_n . 2$$

$\therefore$ adding,

$$(1+x^2)\, y_{n+2} + \{2\,(n+1)\,x - 1\}\, y_{n+1} + n(n+1)\, y_n = 0.$$

Ex. 4. *Find the value of y_n for $x=0$, when $y = e^{a \sin^{-1} x}$.*

From the value of y, when $x=0$, $y=1$.

$$\text{Here} \quad y_1 = e^{a \sin^{-1} x} . a / \sqrt{(1-x^2)} \qquad \cdots \qquad \cdots \quad (1)$$

$$= ay / \sqrt{(1-x^2)}.$$

$$\therefore \quad y_1^2 (1-x^2) = a^2 y^2.$$

Differentiating, $2 y_1 y_2\,(1-x^2) + y_1^2\,(-2x) = 2a^2 y y_1$,

$$\text{or,} \quad (1-x^2)\, y_2 - x y_1 - a^2 y = 0. \qquad \cdots \qquad \cdots \quad (2)$$

Differentiating this n times by Leibnitz's Theorem as in Ex. 3, we easily get $\quad (1-x^2)\, y_{n+2} - (2n+1)\, x y_{n+1} - (n^2 + a^2)\, y_n = 0.$

Putting $x=0$, $(y_{n+2})_0=(n^2+a^2)\ (y_n)_0$. $\quad\cdots\quad$ (3)

Replacing n by $n-2$, we get similarly

$$(y_n)_0=\{(n-2)^2+a^2\}\ (y_{n-2})_0$$
$$=\{(n-2)^2+a^2\}\{(n-4)^2+a^2\}(y_{n-4})_0=\text{etc.}$$

Also from (1) and (2), $(y_1)_0=a$, $(y_2)_0=a^2$.

Thus, $\quad (y_n)_0=\{(n-2)^2+a^2\}\{(n-4)^2+a^2\}\cdots\cdots$

$$(4^2+a^2)\ (2^2+a^2)\ a^2, \text{ if } n \text{ is even}$$

and, $\quad =\{(n-2)^2+a^2\}\{(n-4)^2+a^2\}\cdots\cdots$

$$(3^2+a^2)(1^2+a^2)a, \text{ if } n \text{ is odd.}$$

Note. The value of y_n for $x=0$ is shortly denoted by $(y_n)_0$.

Examples V(B)

1. Find y_n in the following cases :

(i) $y=x^2e^{ax}$. (ii) $y=x^3 \sin x$.

(iii) $y=x^3 \log x$. (iv) $y=x^2 \tan^{-1}x$.

(v) $y=e^{ax} \cos bx$. (vi) $y=\log (ax+x^2)$.

(vii) $y=x^n(1-x)^n$. (viii) $y=x^n/(1+x)$.

2. If $y=A \sin mx+B \cos mx$, then $y_2+m^2y=0$.

3. If $y=Ae^{mx}+Be^{-mx}$, then $y_2-m^2y=0$.

4. If $y=e^{ax} \sin bx$, then $y_2-2ay_1+(a^2+b^2)\,y=0$.

5. If $y=\log (x+\sqrt{a^2+x^2})$, then $(a^2+x^2)\,y_2+xy_1=0$.

6. If $y=(x+\sqrt{1+x^2})^m$, then

$$(1+x^2)\,y_2+xy_1-m^2y=0.$$

7. If $y=\tan^{-1}x$, then

(i) $(1+x^2)\,y_1=1$

and (ii) $(1+x^2)\,y_{n+1}+2nxy_n+n(n-1)\,y_{n-1}=0$.

Find also the value of $(y_n)_0$.

***25.** Show that $\dfrac{d^n}{dx^n}(e^{-x}\,x^{n+a})$

$$=n\,!\,e^{-x}\,x^{a}\sum_{r=0}^{n}\binom{n+a}{r}\frac{(-x)^{n-r}}{(n-r)\,!},\ a>-1.$$

***26.** If $f(x)=\tan x$, prove that

$$f^n(0)-{}^nc_2\,f^{n-2}(0)+{}^nc_4\,f^{n-4}(0)-\cdots=\sin\tfrac{1}{2}n\pi.$$

***27.** Show that the n^{th} differential coefficient of

$$\frac{1}{1+x+x^2+x^3}\text{ is }\tfrac{1}{2}(-1)^n\,n\,!\,\sin^{n+1}\theta\ [\sin(n+1)\theta$$
$$-\cos(n+1)\theta+(\sin\theta+\cos\theta)^{-n-1}],$$

where $\theta=\cot^{-1}x$.

ANSWERS

1. (i) $e^{ax}.\,a^{n-2}\,\{a^2x^2+2nax+n(n-1)\}$.

(ii) $x^3\sin(\tfrac{1}{2}n\pi+x)+3nx^2\sin\{\tfrac{1}{2}(n-1)\pi+x\}+3n(n-1)x$
$\times\sin\{\tfrac{1}{2}(n-2)\pi+x\}+n(n-1)(n-2)\sin\{\tfrac{1}{2}(n-3)\pi+x\}$.

(iii) $(-1)^n.\,6(n-4)\,!/x^{n-3}$.

(iv) $(-1)^{n-1}(n-3)\,!\,\sin^{n-2}\theta\,\{(n-1)(n-2)\sin n\theta\cos^2\theta$
$-2n(n-2)\sin(n-1)\theta\cos\theta+n(n-1)\sin(n-2)\theta\}$,
where $\cot\theta=x$.

(v) $e^{ax}(a^n\cos bx+{}^nc_1\,a^{n-1}\,b\cos(bx+\tfrac{1}{2}\pi)$
$+{}^nc_2\,a^{n-2}b^2\cos(bx+2.\tfrac{1}{2}\pi)+\cdots+b^n\cos(bx+n.\tfrac{1}{2}\pi)\}$.

(vi) $(-1)^{n-1}(n-1)\,!\left[\dfrac{1}{x^n}+\dfrac{1}{(x+a)^n}\right]$.

(vii) $n\,!\,\{(1-x)^n-({}^nc_1)^2.(1-x)^{n-1}.x+({}^nc_2)^2(1-x)^{n-2}.x^2-\cdots\}$.

(viii) $n\,!/(1+x)^{n+1}$.

7. 0, or $(-1)^{\frac{1}{2}(n-1)}(n-1)\,!$ according as n is even or odd.

8. 0, or $\{1, 3, 5\ldots(n-2)\}^2$ according as n is even or odd.

CHAPTER VI

EXPANSION OF FUNCTIONS

6·1. Rolle's Theorem.

If (i) $f(x)$ *is continuous in the closed interval* $a \leqslant x \leqslant b$,

(ii) $f'(x)$ *exists in the open interval* $a < x < b$,

and (iii) $f(a) = f(b)$,

then there exists at least one value of x (*say* ξ) *between* a *and* b [*i.e.*, $a < \xi < b$], *such that* $f'(\xi) = 0$.

Since $f(a) = f(b)$, if $f(x)$ be constant throughout the interval (a, b), being equal to $f(a)$ or $f(b)$, then evidently $f'(x) = 0$ at every point in the interval.

If $f(x)$ be not constant throughout, then it must have values either greater than or less than $f(a)$ or both, in the interval. Suppose $f(x)$ has values greater than $f(a)$. Now, since $f(x)$ is continuous in the interval, it must be bounded and M being its upper bound [which is $> f(a)$ in this case], there must be a value ξ of x in the interval $a < x < b$ for which $f(\xi) = M$.

$\therefore$ $f(\xi + h) - f(\xi) \leqslant 0$, for positive as well as negative values of h.

$\therefore$ $\dfrac{f(\xi + h) - f(\xi)}{h} \leqslant 0$ if h be positive, and $\geqslant 0$ if h be negative.

Hence $\underset{h \to 0+}{Lt} \dfrac{f(\xi + h) - f(\xi)}{h} \leqslant 0$, and $\underset{h \to 0-}{Lt} \dfrac{f(\xi + h) - f(\xi)}{h} \geqslant 0$

provided the limits exist.

Now since $f'(x)$ exists for every value of x in $a < x < b$, $f'(\xi)$ also exists, and so the above two limits must both exist and be equal, and the only equal value they can have is zero. Hence $f'(\xi)=0$.

If $f(x)$ has values less than $f(a)$ in the interval, we can similarly show that $f'(\xi)=0$, where $f(\xi)=m$, the lower bound of $f(x)$ in the interval.

6·2. Mean Value Theorem. [*Lagrange's form*]

If (i) $f(x)$ *is continuous in the closed interval* $a \leqslant x \leqslant b$, *and* (ii) $f'(x)$ *exists in the open interval* $a < x < b$, *then there is at least one value of* x *(say* ξ*) between* a *and* b [*i.e.*, $a < \xi < b$], *such that*

$$\mathbf{f(b)-f(a)=(b-a)f'(\xi).}$$

Consider the function $\psi(x)$ defined in (a, b) by

$$\psi(x)=f(b)-f(x)-\frac{b-x}{b-a}\{f(b)-f(a)\}$$

Here $\psi(x)$ is continuous in $a \leqslant x \leqslant b$, since $f(x)$ and $b-x$ are so.

$$\psi'(x)= -f'(x)+\frac{f(b)-f(a)}{b-a} \quad \text{exists in} \quad a < x < b,$$

since $f'(x)$ exists in $a < x < b$.

Also, $\psi(a)=0$, $\psi(b)=0$. $\quad\therefore\quad \psi(a)=\psi(b)$.

Hence, by Rolle's Theorem, $\psi'(x)$ vanishes for at least one value of x (say ξ) between a and b, *i.e.*, $\psi'(\xi)=0$,

$$\textit{i.e.,} \quad 0=-f'(\xi)+\frac{f(b)-f(a)}{b-a},$$

whence, $f(b)-f(a)=(b-a)f'(\xi)$, [$a < \xi < b$].

Cor. Since ξ lies between a and b, ξ can be written as $a+\theta(b-a)$ where $0<\theta<1$. Putting $b=a+h$, we get another form of the Mean Value Theorem

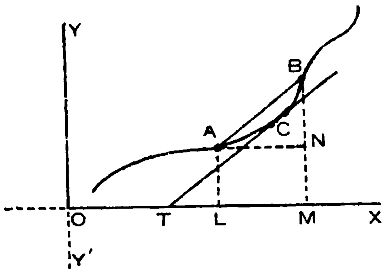
$$f(a+h)=f(a)+hf'(a+\theta h), \qquad \text{where } 0<\theta<1$$

or,
$$\mathbf{f(x+h)=f(x)+hf'(x+\theta h)}, \qquad \text{where } 0<\theta<1.$$

Note. The value of θ usually depends upon both x and h, but there are cases where it is not so dependent. [See *Ex. 2 and 13, Examples VI (A)*]. Also θ may have more than one value in a given range in some cases [See *Ex. 12, Examples VI(A)*]

6·3. Geometrical Interpretation of Mean Value Theorem.

Let ABC be the graph of $f(x)$ in the interval (a, b) and let a, ξ, b be the abscissæ of the points A, C, B on the curve $y=f(x)$, such that the relation $f(b)-f(a)=(b-a)f'(\xi)$ is satisfied.



Draw AL, BM perpendiculars on OX, and AN perpendicular on BM. Then $AL=f(a)$, $BM=f(b)$. Let CT be the tangent at C.

$$\text{Then, } \frac{f(b)-f(a)}{b-a}=\frac{BM-AL}{LM}=\frac{BN}{AN}=\tan\angle BAN.$$

Since, $f'(\xi) = \tan \angle CTX$ (*as explained in* § *4·14*), it follows from the Mean Value Theorem, that $\tan \angle BAN = \tan \angle CTX$, *i.e.*, $\angle BAN = \angle CTX$, *i.e.*, AB is parallel to CT.

Hence, we have the following geometrical interpretation of the Mean Value Theorem :

If the graph ACB of $f(x)$ is a continuous curve having everywhere a tangent then there must be at least one point C intermediate between A and B at which the tangent is parallel to the chord AB.

6·4. Taylor's Series in finite form. (Generalized Mean Value Theorem).

If $f(x)$ possesses differential coefficients of the first $(n-1)$ orders for every value of x in the closed interval (a, b) and the nth derivative of $f(x)$ exists in the open interval (a, b) [i.e., if $f^{n-1}(x)$ is continuous in $a \leqslant x \leqslant b$ and $f^n(x)$ exists in $a < x < b$], then

$$f(b) = f(a) + (b-a) f'(a) + \frac{(b-a)^2}{2\,!} f''(a) + \cdots\cdots$$

$$+ \frac{(b-a)^{n-1}}{(n-1)\,!} f^{n-1}(a) + \frac{(b-a)^n}{n\,!} f^n(\xi),$$

where $a < \xi < b$ $\qquad \cdots \qquad \cdots \qquad$ (A)

and if $b = a + h$, so that $b - a = h$, then

$$f(a+h) = f(a) + hf'(a) + \frac{h^2}{2\,!} f''(a) + \cdots\cdots$$

$$+ \frac{h^{n-1}}{(n-1)\,!} f^{n-1}(a) + \frac{h^n}{n\,!} f^n(a + \theta h)$$

where $0 < \theta < 1$, $\qquad \cdots \qquad \cdots \qquad$ (B) or writing x for a,

$$\mathbf{f(x+h)=f(x)+hf'(x)+\frac{h^2}{2!}f''(x)+\cdots\cdots+\frac{h^{n-1}}{(n-1)!}f^{n-1}(x)}$$
$$\mathbf{+\frac{h^n}{n!}f^n(x+\theta h),}$$

where $0 < \theta < 1$. $\qquad\cdots\qquad$ (C)

Consider the function $\psi(x)$ defined in (a, b), by

$$\psi(x)=\phi(x)-\frac{(b-x)^n}{(b-a)^n}\phi(a), \qquad \cdots \quad (1)$$

where

$$\phi(x)\equiv f(b)-f(x)-(b-x)f'(x)-\frac{(b-x)^2}{2!}f''(x)$$
$$\cdots\cdots-\frac{(b-x)^{n-1}}{(n-1)!}f^{n-1}(x). \qquad \cdots \quad (2)$$

Then, evidently $\psi(a)=0$, and $\psi(b)=0$.

[since $\phi(b)=0$ identically]

Now, $\phi'(x)= -f'(x)+\{f'(x)-(b-x)f''(x)\}$

$$+\left\{(b-x)f''(x)-\frac{(b-x)^2}{2!}f'''(x)\right\}+\cdots\cdots$$
$$+\left\{\frac{(b-x)^{n-2}}{(n-2)!}f^{n-1}(x)-\frac{(b-x)^{n-1}}{(n-1)!}f^n(x)\right\}$$
$$=-\frac{(b-x)^{n-1}}{(n-1)!}f^n(x).$$

Hence, from (1),

$$\psi'(x)=-\frac{(b-x)^{n-1}}{(n-1)!}f^n(x)+\frac{n(b-x)^{n-1}}{(b-a)^n}\phi(a) \quad \cdots \quad (3)$$

Since $\psi(a)=\psi(b)$, and $\psi'(x)$ exists in (a, b),

by Rolle's Theorem, $\psi'(\xi)=0$, where $a < \xi < b$.

$\therefore$ substituting ξ for x in (3), and cancelling the common factor $(b-\xi)^{n-1}$, we get ultimately,

$$\phi(a)=\frac{(b-a)^n}{n!}f^n(\xi), \text{ and since from (2),}$$

$$\phi(a)=f(b)-f(a)-(b-a)f'(a)-\cdots-\frac{(b-a)^{n-1}}{(n-1)!}f^{n-1}(a),$$

the required result follows by transposition.

Since, $a<\xi<b$, we can write $\xi=a+(b-a)\theta$,

i.e., $\xi=a+h\theta$, where $0<\theta<1$, and $b-a=h$,

and hence the form (B) follows.

Note 1. The series (A), (B), or (C) is called **Taylor's series with the remainder in Lagrange's form**, the remainder (after n terms) being

$$\frac{(b-a)^n}{n!}f^n(\xi), \text{ or, } \frac{h^n}{n!}f^n(a+\theta h), \text{ or, } \frac{\mathbf{h^n}}{\mathbf{n!}}\mathbf{f^n(x+\theta h)}, \qquad 0<\theta<1$$

which is generally denoted by $\mathbf{R_n}$.

Note 2. Putting $n=1$ in Taylor's series, we get

$$f(a+h)=f(a)+hf'(a+\theta h),\ 0<\theta<1$$

which is the Mean Value Theorem.

So, Taylor's theorem is sometimes called *Mean Value Theorem of the nth order*. Putting $n=2$ in Taylor's theorem, we get

$$f(a+h)=f(a)+hf'(a)+\frac{h^2}{2!}f''(a+\theta h),\ 0<\theta<1$$

which is often called the Mean Value Theorem of the second order, and so on.

Note 3. Yet another form of Taylor's series which is found sometimes useful is obtained by putting x for b in (A). Thus,

$$f(x)=f(a)+(x-a)f'(a)+\frac{(x-a)^2}{2!}f''(a)+\cdots\cdots$$

$$+\frac{(x-a)^{n-1}}{(n-1)!}f^{n-1}(a)+\frac{(x-a)^n}{n!}f^n\{a+\theta(x-a)\},\ 0<\theta<1,$$

and the function $f(x)$ *is said to be expanded about or in the neighbourhood of* $x=a$.

6·5. Maclaurin's series in finite form.

Putting $x=0$, $h=x$ in Taylor's series in finite form (C), we get

$$f(x)=f(0)+xf'(0)+\frac{x^2}{2!}f''(0)+\cdots\cdots+\frac{x^{n-1}}{(n-1)!}f^{n-1}(0)+\frac{x^n}{n!}f^n(\theta x),$$
$$0<\theta<1,$$

the corresponding form of *the remainder* R_n *being*

$$\frac{x^n}{n!}f^n(\theta x).$$

The above is known as Maclaurin's series for $f(x)$, and $f(x)$ is said to be *expanded in the neighbourhood of* $x=0$.

Note. Putting $n=1, 2$, we get Maclaurin's series of the first and second orders *viz.*,

$$f(x)=f(0)+xf'(\theta x) \text{ and } f(x)=f(0)+xf'(0)+\frac{x^2}{2!}f''(\theta x),\ 0<\theta<1.$$

6·6. Cauchy's form of Remainder R_n.

In Art. 6·4, if we take $\psi(x)=\phi(x)-\dfrac{b-x}{b-a}\phi(a)$, the other conditions remaining the same, and carry out the investigation as in that Art., we get

$$\psi'(x)=-\frac{(b-x)^{n-1}}{(n-1)!}f^n(x)+\frac{1}{b-a}\phi(a). \quad \cdots \quad (4)$$

Since $\psi(a)=\psi(b)$ and $\psi'(x)$ exists in (a, b), by Rolle's theorem, we have $\psi'(\xi)=0$, $a<\xi<b$.

$\therefore$ substituting ξ for x in (4), we get

$$\phi(a)=\frac{(b-a)(b-\xi)^{n-1}}{(n-1)!}f^n(\xi). \quad \cdots \quad (5)$$

Writing $\xi=a+(b-a)\theta$, where $0<\theta<1$, we have, $b-\xi=b-a-b\theta+a\theta=(1-\theta)(b-a)$.

$\therefore \quad (b-\xi)^{n-1}=(1-\theta)^{n-1}(b-a)^{n-1}=(1-\theta)^{n-1}\ h^{n-1},$
since $b-a=h$.

$\therefore$ from (5), we get

$$\phi(a)=\frac{h^n(1-\theta)^{n-1}}{(n-1)\,!}f^n(a+\theta h).$$

Now replacing a by x, the required expression for the remainder R_n would come out as

$$\mathbf{R_n=\frac{h^n\,(1-\theta)^{n-1}}{(n-1)\,!}\,f^n\,(x+\theta h),}\qquad 0<\theta<1.$$

This is known as *Cauchy's form of remainder* in Taylor's expansion.

The corresponding form in Maclaurin's expansion is

$$\mathbf{R_n=\frac{x^n\,(1-\theta)^{n-1}}{(n-1)\,!}\,f^n\,(\theta x),}\qquad 0<\theta<1.$$

This form of remainder is sometimes more useful than that of Lagrange. It should be noted that the value of θ in the two forms of the remainder for the same function need not be the same.

6·7. Illustrative Examples.

Ex. 1. (i) *If $f'(x)=0$ for all values of x in an interval, then $f(x)$ is constant in that interval.*

(ii) *If $\phi'(x)=\psi'(x)$ in an interval, then $\phi(x)$ and $\psi(x)$ differ by a constant in that interval.*

(i) Suppose, $f'(x)=0$ at every point in (a, b).

Let us take any two points x_1, x_2 in (a, b), such that $x_2 > x_1$.

By Mean Value Theorem,

$$f(x_2)-f(x_1)=(x_2-x_1)f'(c),\text{ where } x_1<c<x_2$$
$$=0,\text{ since } f'(c)=0 \text{ by hypothesis.}$$

$\therefore \quad f(x_2)=f(x_1).$

Since x_1, x_2 are any two points in (a, b), it follows that $f(x)$ must be constant throughout (a, b).

(ii) Let $f(x)=\phi(x)-\psi(x)$.

$\therefore \quad f'(x)=\phi'(x)-\psi'(x)=0$, everywhere in (a, b).

$\therefore \quad f(x)=\text{constant}=k$ say, by (i).

$\therefore \quad \phi(x)-\psi(x)=k.$

Note. The result (ii) is fundamental in the theory of integration.

Ex. 2. *If* $f(h)=f(0)+hf'(0)+\dfrac{h^2}{2!}f''(\theta h),\ 0<\theta<1$, *find* θ, *when* $h=1$ *and* $f(x)=(1-x)^{\frac{5}{2}}$. [*C. P. 1944*]

We have, $f(h)=(1-h)^{\frac{5}{2}}$, since, $f(x)=(1-x)^{\frac{5}{2}}$.

$$\therefore \quad f'(h)=-\tfrac{5}{2}(1-h)^{\frac{3}{2}};\ f''(h)=\tfrac{15}{4}(1-h)^{\frac{1}{2}}.$$

$$\therefore \quad f(0)=1,\ f'(0)=-\tfrac{5}{2},$$

$\therefore$ from the given relation,

$$(1-h)^{\frac{5}{2}}=1-\frac{5}{2}h+\frac{h^2}{2!}\cdot\frac{15}{4}(1-\theta h)^{\frac{1}{2}}.$$

Putting $h=1$, $0=1-\frac{5}{2}+\frac{15}{8}(1-\theta)^{\frac{1}{2}}$, whence $(1-\theta)^{\frac{1}{2}}=\frac{4}{5}$.

$\therefore \quad 1-\theta=\frac{16}{25}. \qquad \therefore \quad \theta=\frac{9}{25}.$

Ex. 3. *Prove that the Lagrange's remainder after n terms in the expansion of* $e^{ax}\cos bx$ *in powers of x is*

$$\frac{(a^2+b^2)^{\frac{1}{2}n}}{n!}\, x^n e^{a\theta x} \cos\left(b\theta x+n\tan^{-1}\frac{b}{a}\right),\ 0<\theta<1.$$

[*C. P. 1942*]

Lagrange's remainder after n terms in the expansion of $f(x)$ is

$$\frac{x^n}{n!}f^n(\theta x), \quad 0<\theta<1. \quad (\textit{by Art. } 6\cdot5) \quad \cdots \quad (1)$$

Here, since $f(x)=e^{ax}\cos bx$,

$$\therefore \quad f^n(x)=(a^2+b^2)^{\frac{1}{2}n}e^{ax}\cos\left\{bx+n\tan^{-1}\left(\frac{b}{a}\right)\right\} \quad \cdots \quad (2)$$

[See *Ex. 3* § *5·4*]

$\therefore$ writing θx for x in (2), we get $f^n(\theta x)$, and substituting this value of $f^n(\theta x)$ in (1), the required remainder is obtained.

Ex. 4. *Prove that the Cauchy's remainder after n terms in the expansion of* $(1+x)^m$, *(m being a negative integer or fraction) in powers of x is*

$$\frac{m(m-1)\cdots(m-n+1)}{(n-1)!}x^n(1+\theta x)^{m-1}\left(\frac{1-\theta}{1+\theta x}\right)^{n-1}, 0<\theta<1.$$

Cauchy's remainder after n terms in the expansion of $f(x)$ is

$$\frac{x^n(1-\theta)^{n-1}}{(n-1)!}f^n(\theta x), 0<\theta<1. \quad (\textit{by Art. } 6\,6) \quad \cdots \quad (1)$$

Here, $\quad f(x)=(1+x)^m.$

$\therefore \quad f^n(x)=m(m-1)(m-2)\cdots\cdots(m-n+1)(1+x)^{m-n}.$

So, the expression (1) is equivalent to

$$\frac{m(m-1)(m-2)\cdots(m-n+1)}{(n-1)!}x^n(1-\theta)^{n-1}(1+\theta x)^{m-n}$$

which is the required remainder.

Ex. 5. *Show that the Cauchy's remainder after n terms in the expansion of* $\log(1+x)$ *in powers of x is*

$$(-1)^{n-1}\frac{x^n}{1+\theta x}\left(\frac{1-\theta}{1+\theta x}\right)^{n-1}, 0<\theta<1.$$

Here, $f(x)=\log(1+x)$. $\quad \therefore \quad f^n(x)=\dfrac{(-1)^{n-1}(n-1)!}{(1+x)^n}$.

Hence, $\quad \dfrac{x^n(1-\theta)^{n-1}}{(n-1)!}f^n(\theta x)=(-1)^{n-1}x^n(1-\theta)^{n-1}\cdot\dfrac{1}{(1+\theta x)^n}$,

which is the required remainder in Cauchy's form.

Ex. 6. *If (i)* $f'(x)$ *exists in* $a\leqslant x\leqslant b$, *(ii)* $f'(a)=\alpha$, $f'(b)=\beta$, $\alpha\neq\beta$, *and (iii)* γ *lies between* α *and* β, *then there exists a value* ξ *of x between* a *and* β *such that* $f'(\xi)=\gamma$. **[Darboux's Theorem]**

Suppose, $\alpha<\gamma<\beta$ and let $\phi(x)=f(x)-\gamma(x-a)$.

$\therefore \quad \phi'(x)=f'(x)-\gamma.$

Since, $\phi'(x)$ exists in (a, b), $\phi(x)$ is continuous in (a, b) and therefore attains its lower bound at some point ξ in the interval [§ *3·4 (viii)*].

Now this point cannot be a or b, since $\phi'(a)=f'(a)-\gamma=\alpha-\gamma$ which is negative and $\phi'(b)=f'(b)-\gamma=\beta-\gamma$ which is positive. Hence, the point ξ is between a and b, and $\phi'(\xi)=0$. $\therefore$ $f'(\xi)-\gamma=0$. $\therefore$ $f'(\xi)=\gamma$ for $a<\xi<b$.

Ex. 7. (*a*) *If (i) $\phi(x)$ and $\psi(x)$ are both continuous in $a \leq x \leq b$,*

(ii) $\phi'(x)$ and $\psi'(x)$ exist in $a<x<b$,

and (iii) $\psi'(x) \neq 0$ anywhere in $a<x<b$,

then there is a value ξ of x between a and b for which

$$\frac{\phi(b)-\phi(a)}{\psi(b)-\psi(a)}=\frac{\phi'(\xi)}{\psi'(\xi)}$$ [**Cauchy's Mean Value Theorem**]

(*b*) *If further $\phi(a)=\psi(a)=0$ and $\psi'(x) \neq 0$ in the neighbourhood of a,*

then $\underset{x\to a}{Lt} \dfrac{\phi(x)}{\psi(x)}=\underset{x\to a}{Lt} \dfrac{\phi'(x)}{\psi'(x)}$, *if the latter limit exists.*

[**L' Hospital's Theorem**]

(*a*) Consider the function $f(x)$ defined by the equation,

$$f(x)=\phi(b)-\phi(x)-\frac{\phi(b)-\phi(a)}{\psi(b)-\psi(a)}\{\psi(b)-\psi(x)\}.$$

Now, $f(a)=f(b)$, since each $=0$ identically.

Also, $f'(x)=-\phi'(x)+\dfrac{\phi(b)-\phi(a)}{\psi(b)-\psi(a)}\psi'(x)$.

Since $f(x)$ satisfies all the conditions of Rolle's theorem,

$\therefore$ $f'(\xi)=0$, for some ξ, where $a<\xi<b$,

whence the required result follows.

Note 1. The condition $\psi'(x) \neq 0$, anywhere in (a, b) ensures that $\psi(a) \neq \psi(b)$; for if $\psi(a)=\psi(b)$, then $\psi(x)$ satisfying all the conditions of Rolle's theorem, $\psi'(x)$ would vanish at some point x in (a, b).

Note 2. Putting $b-a=h$, we get

$$\frac{\phi(a+h)-\phi(a)}{\psi(a+h)-\psi(a)}=\frac{\phi'(a+\theta h)}{\psi'(a+\theta h)},\ 0<\theta<1,$$

or, writing x for a, $\dfrac{\phi(x+h)-\phi(x)}{\psi(x+h)-\psi(a)}=\dfrac{\phi'(x+\theta h)}{\psi'(x+\theta h)}$, $0<\theta<1$.

Note 3. Mean Value Theorem can be deduced from this theorem by putting $\psi(x)=x$.

(b) We have for $a < x < b$,

$$\frac{\phi(x)}{\psi(x)} = \frac{\phi(x)-\phi(a)}{\psi(x)-\psi(a)}, \text{ since } \phi(a)=\psi(a)=0 \text{ here,}$$

$$= \frac{\phi'(\xi)}{\psi'(\xi)}, \; a < \xi < x, \text{ by Cauchy's theorem.}$$

Taking limits, and noting that $\xi \to a$ as $x \to a$,

we get $\underset{x\to a+0}{Lt} \dfrac{\phi(x)}{\psi(x)} = \underset{\xi\to a+0}{Lt} \dfrac{\phi'(\xi)}{\psi'(\xi)} = \underset{x\to a+0}{Lt} \dfrac{\phi'(x)}{\psi'(x)}$.

Again, when $b_1 < x < a$, [assuming b_1 sufficiently close to a, such that $\phi'(x)$ and $\psi'(x)$ exist at every point in the interval, and $\psi'(x) \neq 0$ in it], we may similarly write

$$\frac{\phi(x)}{\psi(x)} = \frac{\phi(a)-\phi(x)}{\psi(a)-\psi(x)} = \frac{\phi'(\xi_1)}{\psi'(\xi_1)}, \text{ where } x < \xi_1 < a,$$

by Cauchy's Theorem, and making $x \to a$, we get

$$\underset{x\to a-0}{Lt} \frac{\phi(x)}{\psi(x)} = \underset{\xi_1\to a-0}{Lt} \frac{\phi'(\xi_1)}{\psi'(\xi_1)} = \underset{x\to a-0}{Lt} \frac{\phi'(x)}{\psi'(x)}.$$

Combining the two cases, L'Hospital's Theorem follows.

Examples VI(A)

1. Find the value of ξ in the Mean Value Theorem

$$f(b)-f(a)=(b-a)\,f'(\xi)$$

(i) if $f(x)=x^2$, $a=1$, $b=2$,

(ii) if $f(x)=\sqrt{x}$, $a=4$, $b=9$,

(iii) if $f(x)=x(x-1)(x-2)$, $a=0$, $b=\frac{1}{2}$,

(iv) if $f(x)=Ax^2+Bx+C$ in (a, b).

2. In the Mean Value Theorem

$$f(x+h)=f(x)+hf'(x+\theta h),$$

show that if $f(x)=Ax^2+Bx+C$, $A \neq 0$, then $\theta=\frac{1}{2}$.

Give a geometrical interpretation of the result.

3. In the Mean Value Theorem

$$f(a+h)=f(a)+hf'(a+\theta h),$$

if $a=1$, $h=3$, and $f(x)=\sqrt{x}$, find θ.

4. (i) In the Mean Value Theorem

$$f(h)=f(0)+hf'(\theta h),\ 0<\theta<1,$$

show that the limiting value of θ as $h\to 0+$ is $\dfrac{1}{2}$ or $\dfrac{1}{\sqrt{3}}$ according as $f(x)=\cos x$ or $\sin x$.

(ii) In the Mean Value Theorem

$$f(x+h)=f(x)+hf'(x+\theta h),\ 0<\theta<1,$$

show that the limiting value of θ as $h\to 0+$ is $\dfrac{1}{2}$ whether $f(x)$ is $\sin x$ or $\cos x$.

5. If $f(h)=f(0)+hf'(0)+\dfrac{h^2}{2!}f''(\theta h),\ 0<\theta<1$, find θ when $h=7$ and $f(x)=1/(1+x)$.

6. From the relation

$$f(x)=f(0)+xf'(0)+\frac{x^2}{2!}f''(\theta x),\ 0<\theta<1,$$

show that $\log(1+x)>x-\frac{1}{2}x^2$, if $x>0$,

and $\cos x>1-\frac{1}{2}x^2$, if $0<x<\frac{1}{2}\pi$.

7. Show that $\sin x>x-\frac{1}{6}x^3$, if $0<x<\frac{1}{2}\pi$.

8. If $f(x)=\tan x$, then $f(0)=0$ and $f(\pi)=0$.

Is Rolle's theorem applicable to $f(x)$ in $(0,\pi)$?

9. Is Mean Value Theorem applicable to the functions (i) and (ii) in the intervals $(-1, 1)$ and $(5, 7)$ respectively ?

(i) $f(x)=x\cos(1/x)$ for $x\neq 0$

$=0$ for $x=0$.

(ii) $f(x)=4-(6-x)^{\frac{2}{3}}$.

10. If $f'(x)$ exists and > 0 everywhere in (a, b), then $f(x)$ is an increasing function in (a, b); if $f'(x) < 0$ everywhere in (a, b), then $f(x)$ is a decreasing function in (a, b).

11. Show that $2x^3 + 2x^2 - 10x + 6$ is positive if $x > 1$.

12. (i) In the Mean Value Theorem

$$f(a+h) - f(a) = hf'(a + \theta h),\ 0 < \theta < 1,$$

if $f(x) = \frac{1}{3}x^3 - \frac{3}{2}x^2 + 2x$, and $a = 0$, $h = 3$, show that θ has got two values, and find them.

(ii) In the Mean Value Theorem

$$f(b) - f(a) = (b-a)f'(\xi),\ a < \xi < b,$$

find ξ, if $f(x) = x^3 - 3x - 1$, $a = -11/7$, $b = 13/7$ and give a geometrical interpretation of the result.

13. (i) Find the value of θ in the Mean Value Theorem

$$f(x+h) = f(x) + hf'(x + \theta h),$$

where, (a) $f(x) = \dfrac{1}{x}$, (b) $f(x) = e^x$, (c) $f(x) = \log x$.

(ii) If $f(x) = a + bx + cm^x$, show that θ is independent of x.

14. Show that

$$(x+h)^{\frac{3}{2}} = x^{\frac{3}{2}} + \frac{3}{2}x^{\frac{1}{2}}h + \frac{3.1}{2\ 2}\frac{h^2}{2!}\frac{1}{\sqrt{(x+\theta h)}},\ 0 < \theta < 1.$$

Find θ when $x = 0$.

15. Expand in a finite series in powers of h, and find the remainder in each case :

(i) $\log(x+h)$. (ii) $\sin(x+h)$. (iii) $(x+h)^m$.

16. (i) Apply Taylor's Theorem to obtain the Binomial expansion of $(a+h)^n$, where n is a positive integer.

(ii) If $f(x)$ is a polynomial of degree r, show that

$$f(a+h)=f(a)+hf'(a)+\frac{h^2}{2\,!}f''(a)+\cdots\cdots+\frac{h^r}{r\,!}f^r(a).$$

(iii) Expand $5x^2+7x+3$ in powers of $(x-2)$.

17. Expand the following functions in a finite series in powers of x, with the remainder in Lagrange's form in each case :

(i) e^x. (ii) a^x. (iii) $\sin x$.

(iv) $\cos x$. (v) $\log(1+x)$. (vi) $\log(1-x)$.

(vii) $(1+x)^m$. (viii) $\tan^{-1}x$. (ix) $e^x \cos x$.

(x) $e^{ax}\sin bx$.

18. Find the value of θ in the Lagrange's form of the remainder R_n for the expansion of $\dfrac{1}{1-x}$ in powers of x.

19. Expand the following functions in the neighbourhood of $x=0$ to three terms plus remainder (in Lagrange's form)

(i) $\sin^2 x$. (ii) $\cos^3 x$. (iii) e^{-x^2}.

20. Expand the following functions in a finite series in powers of x, with the remainder in Cauchy's form in each case :

(i) e^x. (ii) $\cos x$. (iii) $(1-x)^{-1}$.

21. (i) Prove that $\underset{h\to 0}{Lt}\dfrac{f(a+h)-f(a-h)}{2h}=f'(a)$

provided $f'(x)$ is continuous.

(ii) Prove that $\underset{h\to 0}{Lt}\dfrac{f(a+h)-2f(a)+f(a-h)}{h^2}=f''(a)$

provided $f''(x)$ is continuous.

22. (i) If $f''(x)$ is continuous in the interval $(a, a+h)$ and $f''(a) \neq 0$, prove that $\underset{h \to 0}{Lt}\ \theta = \frac{1}{2}$,

where θ is given by

$$f(a+h) = f(a) + hf'(a+\theta h),\ 0 < \theta < 1.$$

(ii) Show that the limit when $h \to 0$ of θ which occurs in Lagrange's form of remainder $\frac{h^n}{n!} f^n(x+\theta h)$ in the expansion of $f(x+h)$, is $\frac{1}{n+1}$, provided $f^{n+1}(x)$ is continuous and $\neq 0$.

23. In Cauchy's Mean Value Theorem, (i) if $\phi(x) = \sin x$ and $\psi(x) = \cos x$, or (ii) if $\phi(x) = e^x$ and $\psi(x) = e^{-x}$, or (iii) if $\phi(x) = x^2 + x + 1$ and $\psi(x) = 2x^2 + 3x + 4$, show that θ is independent of both x and h, and is equal to $\frac{1}{2}$.

24. If $f(x) = x^2$, $\phi(x) = x$ then find a value of ξ in terms of a and b in Cauchy's Mean Value Theorem.

25. If $f(x)$ and $\phi(x)$ are continuous in $a \leqslant x \leqslant b$ and differentiable in $a < x < b$ and $f'(x)$ and $\phi'(x)$ never vanish for the same value of x, then,

$$\frac{f(\xi) - f(a)}{\phi(b) - \phi(\xi)} = \frac{f'(\xi)}{\phi'(\xi)}, \text{ where } a < \xi < b.$$

26. If $\psi''(x) \neq 0$ for $a < x < b$, then

$$\frac{\phi(b) - \phi(a) - (b-a)\phi'(a)}{\psi(b) - \psi(a) - (b-a)\psi'(a)} = \frac{\phi''(\xi)}{\psi''(\xi)}, \text{ where } a < \xi < b.$$

27. If $f(x)$ and $g(x)$ are differentiable in the interval (a, b), then there is a number ξ, $a < \xi < b$ such that

$$\begin{vmatrix} f(a) & f(b) \\ g(a) & g(b) \end{vmatrix} = (b-a) \begin{vmatrix} f(a) & f'(\xi) \\ g(a) & g'(\xi) \end{vmatrix}$$

28. (i) If $f(x)$, $\phi(x)$, $\psi(x)$ are continuous in $a \leqslant x \leqslant b$ and differentiable in $a < x < b$, then

$$\begin{vmatrix} f(a) & \phi(a) & \psi(a) \\ f(b) & \phi(b) & \psi(b) \\ f'(\xi) & \phi'(\xi) & \psi'(\xi) \end{vmatrix} = 0.$$

(ii) If $F(x)$, $G(x)$, $H(x)$ are continuous in $a \leqslant x \leqslant b$ and differentiable in $a < x < b$, then

$$\begin{vmatrix} 1 & F(b) - F(a) & F'(\xi) \\ 1 & G(b) - G(a) & G'(\xi) \\ 1 & H(b) - H(a) & H'(\xi) \end{vmatrix} = 0.$$

29. If $f(x) = \begin{vmatrix} \sin x & \sin \alpha & \sin \beta \\ \cos x & \cos \alpha & \cos \beta \\ \tan x & \tan \alpha & \tan \beta \end{vmatrix}$ $0 < \alpha < \beta < \frac{1}{2}\pi$,

show that $f'(\xi) = 0$, where $\alpha < \xi < \beta$.

[*C. H. 1955*]

30. Deduce Taylor's Theorem from Cauchy's Mean Value Theorem.

[*C. H. 1961*]

[Assume $\phi(x) = f(b) - f(x) - (b-x) f'(x) - \cdots - \frac{(b-x)^{n-1}}{(n-1)!} f^{n-1}(x)$ and $\psi(x) = (b-x)^n$.]

31. If $f(a) = f(c) = f(b) = 0$ where $a < c < b$, and if $f'(x)$ satisfies the conditions of Rolle's Theorem in (a, b), prove that there exists at least one number ξ such that $a < \xi < b$, $f''(\xi) = 0$.

32. Given that

$$\phi(x)=\frac{(x-b)(x-c)}{(a-b)(a-c)}f(a)+\frac{(x-c)(x-a)}{(b-c)(b-a)}f(b)$$
$$+\frac{(x-a)(x-b)}{(c-a)(c-b)}f(c)-f(x),$$

where $a<c<b$ and $f''(x)$ exists at all points in (a, b). Prove by considering the function $\phi(x)$, that there exists a number ξ, $a<\xi<b$, such that

$$\frac{f(a)}{(a-b)(a-c)}+\frac{f(b)}{(b-c)(b-a)}+\frac{f(c)}{(c-a)(c-b)}=\tfrac{1}{2}f''(\xi).$$

33. Given that

$$\phi(x)=\begin{vmatrix} f(x) & x^2 & x & 1 \\ f(b) & b^2 & b & 1 \\ f(a) & a^2 & a & 1 \\ f'(a) & 2a & 1 & 0 \end{vmatrix}$$

and $f''(x)$ exists at all points in (a, b), deduce

$$f(b)=f(a)+(b-a)f'(a)+\tfrac{1}{2}(b-a)^2 f''(\xi),\ a<\xi<b.$$

34. If $f''(x)$ exists for all points in (a, b) and

$$\frac{f(c)-f(a)}{c-a}=\frac{f(b)-f(c)}{b-c},$$

where $a<c<b$, then there is a number ξ such that $a<\xi<b$ and $f''(\xi)=0$.

35. Given that

$$\phi(x)=\begin{vmatrix} f(a) & f(b) & f(x) \\ g(a) & g(b) & g(x) \\ h(a) & h(b) & h(x) \end{vmatrix}$$

and $F(x)=\phi(x)-\dfrac{(x-a)(x-b)}{(c-a)(c-b)}\phi(c)$, where $a < c < b$ and $f''(x)$, $g''(x)$, $h''(x)$ exist throughout the interval (a, b), show that by considering the function $F(x)$

$$\phi(c) = \tfrac{1}{2}(c-a)(c-b)\,\phi''(\xi),\ a < \xi < b.$$

36. If $f''(x)$ exists at all points in (a, b) and if $f(a) = f(b) = 0$ and if $f(c) > 0$ where $a < c < b$, prove that there is at least one value ξ such that $f''(\xi) < 0$, $a < \xi < b$.

ANSWERS

1. (i) $1\cdot5$. (ii) 6.25. (iii) $1 - \sqrt{\frac{7}{12}}$. (iv) $\frac{1}{2}(a+b)$.

2. The tangent at the middle point of a parabolic arc is parallel to the chord of the arc.

3. $\frac{5}{12}$. **5.** $\frac{1}{7}$. **8.** No. **9.** (i) No. (ii) No.

12. (i) $\theta = \frac{1}{6}(3 \pm \sqrt{3})$. (ii) $\xi = \pm 1$; the tangents at these two points are parallel to the line joining the points $a, f(a)$ and $b, f(b)$ which is parallel to x-axis in this case.

13. (i) (*a*) $\dfrac{\sqrt{x^2+xh}-x}{h}$. (*b*) $\dfrac{1}{h}\log\dfrac{e^h-1}{h}$.

(*c*) $\dfrac{1}{\log(1+h/x)} - \dfrac{x}{h}$. **14.** $\frac{9}{64}$.

15. (i) $\log x + \dfrac{h}{x} - \dfrac{h^2}{2x^2} + \cdots\cdots + (-1)^{n-1}\dfrac{h^n}{n\,(x+\theta h)^n}$.

(ii) $\sin x + h\cos x - \dfrac{h^2}{2!}\sin x - \dfrac{h^3}{3!}\cos x + \cdots + \dfrac{h^n}{n!}\sin\left(x + \theta h + \dfrac{n\pi}{2}\right)$.

(iii) $x^m + mx^{m-1}h + \dfrac{m(m-1)}{2!}x^{m-2}h^2 + \cdots\cdots$

$$+ \frac{m(m-1)(m-2)\cdots(m-n+1)}{n!}\,h^n\,(x+\theta h)^{m-n}.$$

16. (i) $a^n + {}^nc_1 a^{n-1}h + {}^nc_2 a^{n-2}h^2 + \cdots + {}^nc_r a^{n-r}h^r + \cdots + h^n$.

(ii) $37 + 27(x-2) + 5(x-2)^2$.

In the following series in every case $0 < \theta < 1$.

17. (i) $1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\cdots+\frac{x^n}{n!}\,e^{\theta x}$

(ii) $1+x\log a+\frac{x^2}{2!}(\log a)^2+\cdots\cdots+\frac{x^n}{n!}(\log a)^n a^{\theta x}$.

(iii) $x-\frac{x^3}{3!}+\frac{x^5}{5!}-\cdots\cdots+\frac{x^n}{n!}\sin\left(\frac{n\pi}{2}+\theta x\right)$.

(iv) $1-\frac{x^2}{2!}+\frac{x^4}{4!}-\cdots\cdots+\frac{x^n}{n!}\cos\left(\frac{n\pi}{2}+\theta x\right)$.

(v) $x-\frac{x^2}{2}+\frac{x^3}{3}-\cdots\cdots+\frac{(-1)^{n-1}}{n}\,\frac{x^n}{(1+\theta x)^n}$.

(vi) $-x-\frac{x^2}{2}-\cdots\cdots-\frac{1}{n}\,\frac{x^n}{(1-\theta x)^n}$.

(vii) $1+mx+\frac{m(m-1)}{2!}x^2+\cdots\cdots$

$$+\frac{m(m-1)(m-2)\cdots(m-n+1)}{n!}x^n(1+\theta x)^{m-n}.$$

(viii) $x-\frac{x^3}{3}+\frac{x^5}{5}-\cdots$

$$+\frac{(-1)^{n-1}x^n}{n}\sin^n(\cot^{-1}\theta x)\sin n(\cot^{-1}\theta x).$$

(ix) $1+\frac{x}{1!}\sqrt{2}\cos\left(\frac{\pi}{4}\right)+\frac{x^2}{2!}(\sqrt{2})^2\cos\left(2\cdot\frac{\pi}{4}\right)+\cdots\cdots$

$$+\frac{x^n}{n!}(\sqrt{2})^n e^{\theta x}\cos\left(\theta x+\frac{n\pi}{4}\right).$$

(x) $\frac{x}{1!}r\sin\phi+\frac{x^2}{2!}r^2\sin 2\phi+\cdots\cdots+\frac{x^n}{n!}r^n e^{a\theta x}\sin(b\theta x+n\phi)$,

where $r=\sqrt{a^2+b^2}$, and $\phi=\tan^{-1}(b/a)$.

18. $\theta=1-(1-x)^{1/(n+1)}$.

19. (i) $x^2-\frac{x^4}{3}+\frac{2x^6}{45}-\frac{4}{315}x^7\sin 2\theta x$.

(ii) $1-\frac{3}{2}x^2+\frac{7}{8}x^4-\frac{1}{160}x^5(81\sin 3\theta x+\sin\theta x)$.

(iii) $1-x^2+\frac{1}{2}x^4-\frac{1}{15}x^6\,e^{-\theta^2x^2}(4\theta^5x^5-20\theta^3x^3+15\theta x)$.

20. (i) $1+\frac{x}{1!}+\frac{x^2}{2!}+\frac{x^3}{3!}+\cdots\cdots+\frac{x^n(1-\theta)^{n-1}}{(n-1)!}e^{\theta x}$.

(ii) $1-\frac{x^2}{2!}+\frac{x^4}{4!}-\cdots\cdots+\frac{x^n(1-\theta)^{n-1}}{(n-1)!}\cos\left(\frac{n\pi}{2}+\theta x\right)$.

(iii) $1+x+x^2+\cdots\cdots+\frac{nx^n(1-\theta)^{n-1}}{(1-\theta x)^{n+1}}$.

24. $\frac{1}{2}(b+a)$.

6·8. Expansion of functions in infinite power series.

Taylor's series (extended to infinity).

If $f(x)$, $f'(x)$, $f''(x)$,...... $f^n(x)$ exist *finitely* however large n may be, in any interval $(x-\delta,\ x+\delta)$ enclosing the point x and if in addition, *R_n tends to zero as n tends to infinity,* then Taylor's series extended to infinity is valid, and we have

$$\mathbf{f(x+h)=f(x)+hf'(x)+\frac{h^2}{2!}f''(x)+\cdots\ to\ \infty}.\quad [|h|<\delta]$$

Denoting the first n terms of the expansion of $f(x+h)$ by S_n and the remainder by R_n, we have by Art. 6·4,

$$f(x+h)=S_n+R_n,\ \textit{i.e.},\ f(x+h)-R_n=S_n.$$

Now, let $n\to\infty$; then if $R_n\to 0$, we have

$$f(x+h)=\underset{n\to\infty}{Lt}\ S_n=f(x)+hf'(x)+\frac{h^2}{2!}f''(x)+\cdots\text{ to }\infty.$$

Again, since $f(x+h)-S_n=R_n$, if $(x+h)=\underset{n\to\infty}{Lt}\ S_n$,

then, $\underset{n\to\infty}{Lt}\ R_n=0$.

Thus, $\underset{n\to\infty}{Lt}\ R_n=0$ is both a *necessary* and *sufficient* condition that $f(x+h)$ can be expanded in an infinite series.

Cor. Another form of Taylor's series which is found often useful is obtained by putting $x-a$ for h in the form (B), Art. 6·4.

Thus, $f(x)=f(a)+(x-a)f'(a)+\{(x-a)^2/2!\}f''(a)+\cdots$ to ∞.

Maclaurin's series (extended to infinity).

If $f(x), f'(x), f''(x)......f^n(x)$ exist *finitely* however large n may be in any interval $(-\delta, \delta)$, and if R_n *tends to zero as* n *tends to infinity*, then Maclaurin's series extended to infinity is valid, and we have

$$\mathbf{f(x)=f(0)+xf'(0)+\frac{x^2}{2!}f''(0)+\cdots\cdots \text{ to } \infty}, \text{ where } |x|<\delta.$$

Ex. *Expand the following functions in powers of x in infinite series stating in each case the conditions under which the expansion is valid.*

(i) $\sin x$, (ii) $\cos x$, (iii) e^x, (iv) $\log(1+x)$, (v) $(1+x)^m$.

(i) Let $f(x)=\sin x$.

$\therefore$ $f^n(x)=\sin(\frac{1}{2}n\pi+x)$, so that $f(x)$ possesses derivatives of every order for every value of x. Also, $f^n(0)=\sin(\frac{1}{2}n\pi)$ which is 0 or ± 1 according as n is even or odd.

$$\therefore \quad R_n=\frac{x^n}{n!}f^n(\theta x)=\frac{x^n}{n!}\sin\left(\theta x+\frac{n\pi}{2}\right).$$

$$\therefore \quad |R_n|=\left|\frac{x^n}{n!}\right|\left|\sin\left(\theta x+n\frac{\pi}{2}\right)\right|\leqslant\left|\frac{x^n}{n!}\right|$$

since $|\sin(\theta x+\frac{1}{2}n\pi)|\leqslant 1$.

$R_n\to 0$, as $n\to\infty$, since $\frac{x^n}{n!}\to 0$ as $n\to\infty$ for all values of x.

[*Ex. 8. § 2·11*]

Thus the conditions for Maclaurin's infinite expansion are satisfied.

$$\therefore \quad \sin x=x-\frac{x^3}{3!}+\frac{x^5}{5!}-\frac{x^7}{7!}+\cdots\cdots \text{ to } \infty, \text{ for all values of } x.$$

(ii) Let $f(x)=\cos x$.

$\therefore$ $f^n(x)=\cos(\frac{1}{2}n\pi+x)$. $\therefore$ $f^n(0)=\cos(n.\frac{1}{2}\pi)$, which is 0 or ± 1 according as n is odd or even.

$$\therefore \quad R_n=\frac{x^n}{n!}f^n(\theta x)=\frac{x^n}{n!}\cos\left(\theta x+\frac{n\pi}{2}\right).$$

Now proceeding as in the case of $\sin x$, we can show that

$$R_n \to 0 \text{ as } n \to \infty, \text{ for all values of } x.$$

$\therefore \quad \cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots\cdots$ to ∞ for all values of x.

(iii) Let $f(x) = e^x$.

$\therefore \quad f^n(x) = e^x$. $\therefore$ $f^n(0) = 1$, thus $f^n(0)$ exists and is finite, however large n may be.

$$\therefore \quad R_n = \frac{x^n}{n!} f^n(\theta x) = \frac{x^n}{n!} e^{\theta x}.$$

Now since $e^{\theta x} < e^{|x|}$ (a finite quantity for a given x)

and $\frac{x^n}{n!} \to 0$ as $n \to \infty$, $R_n \to 0$ as $n \to \infty$.

$\therefore \quad e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots\cdots$ to ∞, for all values of x.

(iv) Let $\quad f(x) = \log(1+x)$.

$$f^n(x) = \frac{(-1)^{n-1}(n-1)!}{(1+x)^n}$$ which exists for every value of n for $x > -1$.

$$f^n(0) = (-1)^{n-1}(n-1)!$$

If R_n denotes Lagrange's form of remainder, we have

$$R_n = \frac{x^n}{n!} f^n(\theta x) = (-1)^{n-1} \frac{1}{n}\left(\frac{x}{1+\theta x}\right)^n.$$

(i) Let $0 \leqslant x \leqslant 1$, so that $\left(\frac{x}{1+\theta x}\right)^n \to 0$ as $n \to \infty$,

since $\frac{x}{1+\theta x}$ is positive and less than 1.

Also $\frac{1}{n} \to 0$ as $n \to \infty$. $\therefore$ $R_n \to 0$ as $n \to \infty$.

(ii) Let $-1 < x < 0$; in this case $\frac{x}{1+\theta x}$ may not be numerically less than unity and hence $\left(\frac{x}{1+\theta x}\right)^n$ may not tend to 0 as $n \to \infty$.

Thus, we fail to draw any definite conclusion from Lagrange's form of remainder. Using Cauchy's form of remainder, we have

$$R_n = \frac{x^n (1-\theta)^{n-1}}{(n-1)\,!} f^n(\theta x) = (-1)^{n-1} \cdot \frac{x^n}{1+\theta x}\left(\frac{1-\theta}{1+\theta x}\right)^{n-1}.$$

Now, $\dfrac{1-\theta}{1+\theta x}$ is positive and less than 1; hence $\left(\dfrac{1-\theta}{1+\theta x}\right)^{n-1} \to 0$ as $n \to \infty$.

Also, $x^n \to 0$, as $n \to \infty$, since $-1 < x < 0$; $\dfrac{1}{1+\theta x}$ is bounded.

$\therefore \quad R_n \to 0$ as $n \to \infty$.

$\therefore \quad \log(1+x) = x - \dfrac{x^2}{2} + \dfrac{x^3}{3} - \cdots\cdots$

is valid for $-1 < x \leqslant 1$.

Proceeding similarly we can show that

$$\log(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \cdots\cdots$$

is valid for $-1 \leqslant x < 1$.

(v) Let $f(x) = (1+x)^m$, where *m is any real number*

[*Binomial expansion*].

(i) When *m is a positive integer*, $f^n(x) = 0$, when $n > m$, for every value of x. Hence the expansion stops after the $(m+1)$th term and the binomial expansion being a finite series, is *valid for all values of x*.

(ii) When m is a *negative integer* or a *fraction*,

$$f^n(x) = m(m-1)\cdots(m-n+1)(1+x)^{m-n}, \text{ for } x > -1.$$

Hence Cauchy's form of remainder R_n is

$$R_n = \frac{m(m-1)\cdots(m-n+1)}{(n-1)\,!} x^n (1+\theta x)^{m-1} \left(\frac{1-\theta}{1+\theta x}\right)^{n-1}.$$

Let $\quad -1 < x < 1$ *i.e.*, $|x| < 1$; also $0 < \theta < 1$

$\therefore \quad 0 < 1-\theta < 1+\theta x.$

$$0 < \frac{1-\theta}{1+\theta x} < 1. \qquad \therefore \quad 0 < \left(\frac{1-\theta}{1+\theta x}\right)^{n-1} < 1.$$

(n being a positive integer > 1)

$\therefore \quad \left(\frac{1-\theta}{1+\theta x}\right)^{n-1} \to 0$ as $n \to \infty$. Also $(1+\theta x)^{m-1}$ is finite whether $(m-1)$ is positive or negative.

Again if $|x| < 1$, $\dfrac{m(m-1)\cdots(m-n+1)}{(n-1)!} x^n \to 0$.

[See *Art. 2·11, Ex. 8(iv)*]

$\therefore$ when $|x| < 1$, $R_n \to 0$ as $n \to \infty$.

$\therefore$ for $|x| < 1$, Maclaurin's infinite expansion for $(1+x)^m$ is valid, m being a negative integer or a fraction.

6·9. Determination of the coefficients in the expansion of f(x) and f(x+h). *(Alternative method)*

(i) Assuming that $f(x)$ admits of expansion in a convergent power series in x for all values of x within a certain range, and that the expansion can be differentiated term by term any number of times within this range, we can easily get the coefficients of the different powers of x as follows :

Let $\quad f(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots\cdots \qquad \cdots \quad (1)$

where $a_0, a_1, a_2, \ldots$ are constants.

We have by successive differentiations,

$$f'(x) = a_1 + 2a_2 x + 3a_3 x^2 + 4a_4 x^3 + \cdots\cdots \qquad (2)$$
$$f''(x) = 2.1a_2 + 3.2a_3 x + 4.3a_4 x^2 + \cdots\cdots \qquad (3)$$
$$f'''(x) = 3.2.1a_3 + 4.3.2a_4 x + \cdots\cdots \qquad (4)$$

etc. etc. etc.

Putting $x = 0$ in (1), (2), (3), (4),......, we get

$$f(0) = a_0, f'(0) = a_1, f''(0) = 2!\ a_2, f'''(0) = 3!\ a_3 \ldots$$

Hence, $f(x) = f(0) + xf'(0) + \dfrac{x^2}{2!} f''(0) + \dfrac{x^3}{3!} f'''(0) + \cdots\cdots$

(ii) Let $f(x+h)$ be a function of h (x being independent of h), and let us assume that it can be expanded in powers of h, and that the expansion can be differentiated with respect to h term by term any number of times within a certain range of values of h. We can easily obtain the coefficients of the various powers of h as follows :

Let $f(x+h)=a_0+a_1h+a_2h^2+a_3h^3+\cdots\cdots$ (1)
where a_0, a_1, a_2,...... are functions of x, and independent of h.

$$\text{Since,}\quad \frac{d}{dh}f(x+h)=\frac{d}{dz}f(z)\frac{dz}{dh}\ [\text{ where } z=x+h\ ]$$
$$=f'(z)=f'(x+h),$$

differentiating (i) successively with respect to h, we get

$$f'(x+h)=a_1+2a_2h+3a_3h^2+4a_4h^3+\cdots\cdots \quad (2)$$
$$f''(x+h)=2.1.a_2+3.2\ a_3h+4.3.a_4h^2+\cdots\cdots \quad (3)$$
$$f'''(x+h)=3.2.1.a_3+4.3.2.a_4h+\cdots\cdots \quad (4)$$
etc. etc. etc.

Putting $h=0$ in (1), (2), (3), (4),... we have

$$f(x)=a_0,\ f'(x)=a_1,\ f''(x)=2\,!\ a_2,\ f'''(x)=3\,!\ a_3,\ldots$$

$$\therefore\quad f(x+h)=f(x)+hf'(x)+\frac{h^2}{2\,!}f''(x)+\frac{h^3}{3\,!}f'''(x)+\cdots$$

Note. Although the *forms* of the series obtained above for $f(x+h)$ and $f(x)$ are identical with the Taylor's and Maclaurin's infinite series for the expansions of these two functions, the above method of proof, if used, for establishing these two series, is considered as *defective* in as much as it does not enable us to determine exactly the values of x for which the infinite series obtained from each of the functions converges to the value of the function. In fact, Taylor's and Maclaurin's expansions in infinite series do not converge to the functions from which they are developed unless $R_n \to 0$ as $n \to \infty$, even though the function might possess finite differential coefficients of all order, and the infinite series may be convergent ; *e.g.*, $f(x) = e^{-1/x^2}$ ($x \neq 0$), $f(0) = 0$.

Here, $f^r(0) = 0$ for every value of r. But Maclaurin's infinite series for this function, though convergent for all values of x, is not equal to $f(x)$.

6·10. Other methods of Expansion.

The use of Maclaurin's (as also of Taylor's) theorem in expanding a given function in infinite power series is limited in applications because of the unwieldy form of the remainder (*i.e.*, of the nth derivative of the function) in many cases. So we employ other methods for expansion. Now, in this connection it should be noted that the operations of algebra like addition, subtraction, multiplication, division and operations of calculus like term by term limit and term by term differentiation, though applicable to the sum of a finite number of functions, are not applicable without further examination to the case when the number of terms is infinite, and hence to the infinite power series $\Sigma a_n x^n$. If a power series in x converges (*i.e.*, has finite sum) for value of x lying within a certain range (called the *interval of convergence*)*, then for values of x within that range, operations of algebra and calculus referred to above are applicable, as in the case of polynomials, and the series obtained by such operations would represent the function for which it stands only for those values of x which lie within the interval of convergence. We illustrate below some of these methods.

* See *Appendix*.

A. Algebraical Method.

Ex. 1. *Expand* $\tan x$ *in powers of* x *as far as* x^5.

Since, $\tan x = \dfrac{\sin x}{\cos x} = \dfrac{x - \dfrac{x^3}{3!} + \dfrac{x^5}{5!} - \cdots}{1 - \dfrac{x^2}{2!} + \dfrac{x^4}{4!} - \cdots}$

we may by actual division, show that

$$\tan x = x + \tfrac{1}{3}x^3 + \tfrac{2}{15}x^5 + \cdots$$

Ex. 2. *Expand* $\dfrac{\log(1+x)}{1+x}$ *in powers of* x *as far as* x^4.

Multiplying the two series,

$$\log(1+x) = x - \tfrac{1}{2}x^2 + \tfrac{1}{3}x^3 - \tfrac{1}{4}x^4 + \cdots \quad (-1 < x < 1)$$

and $$(1+x)^{-1} = 1 - x + x^2 - x^3 + x^4 - \cdots \quad (-1 < x < 1)$$

and collecting together the coefficients of like powers of x, we have $\dfrac{\log(1+x)}{1+x} = x - (1+\tfrac{1}{2})x^2 + (1+\tfrac{1}{2}+\tfrac{1}{3})x^3 - (1+\tfrac{1}{2}+\tfrac{1}{3}+\tfrac{1}{4})x^4 + \cdots\cdots\cdots$ for $|x| < 1$, (the common interval of convergence).

B. Method of Undetermined Coefficients.

Ex. 3. *Expand* $\log(1+x)$ *in ascending powers of* x.

Let $\log(1+x) = a_0 + a_1x + a_2x^2 + a_3x^3 + \cdots \qquad \cdots (1)$

$\therefore$ differentiating with respect to x,

$$\frac{1}{1+x} = a_1 + 2a_2x + 3a_3x^2 + \cdots \qquad \cdots (2)$$

$\therefore \quad (1+x)(a_1 + 2a_2x + 3a_3x^2 + \cdots\cdots) = 1. \qquad \cdots (3)$

Equating coefficients of x^n on both sides,

$$na_n + (n+1)a_{n+1} = 0. \qquad \cdots (4)$$

Putting $x=0$ in (1) and (2), $a_0 = \log 1 = 0$, $a_1 = 1$.

Putting $n = 1, 2, 3, \ldots\ldots$ in (4), we get $a_2 = -\tfrac{1}{2}$, $a_3 = \tfrac{1}{3}$, $a_4 = -\tfrac{1}{4}$, etc.

$\therefore \quad \log(1+x) = x - \tfrac{1}{2}x^2 + \tfrac{1}{3}x^3 - \cdots \qquad \cdots (5)$

Alternatively

For $|x| < 1$, $(1+x)^{-1} = 1 - x + x^2 - x^3 + \cdots\cdots$

Hence, comparing coefficients of like powers of x on both sides of (2), we obtain $a_1=1$, $a_2=-\frac{1}{2}$, $a_3=\frac{1}{3}$, etc.

Note. We shall have now to find for which values of x the series is convergent, and hence represents the function.

It can be shown that the series is convergent for $-1 < x < 1$.

Ex. 4. *Show that*

$$\sin^{-1}x=x+\frac{1}{2}\frac{x^3}{3}+\frac{1.3}{2.4}\frac{x^5}{5}+\frac{1.3.5}{2.4.6}\frac{x^7}{7}+\cdots\cdots \qquad [\textit{C. P. 1947}]$$

Let $y=\sin^{-1}x=a_0+a_1x+a_2x^2+\cdots\cdots+a_nx^n+\cdot$ (1)

Since $y=\sin^{-1}x$, $\therefore$ differentiating, $y_1=\dfrac{1}{\sqrt{1-x^2}}$ (2)

$$\therefore \quad y_1=(1-x^2)^{-\frac{1}{2}}=1+\tfrac{1}{2}x^2+\frac{1.3}{2.4}x^4+\frac{1.3.5}{2.4.6}x^6+\cdots\cdots \qquad (3)$$

for $-1 < x < 1$ by Binomial expansion.

Also, $y_1=a_1+2a_2x+3a_3x^2+\cdots\cdots+na_nx^{n-1}+\cdots\cdots$ (4)

Hence, comparing the coefficients of (3) and (4), we get

$$a_1=1,\ a_2=0,\ a_3=\frac{1}{2\ 3},\ a_4=0,\ a_5=\frac{1.3}{2.4\ 5}, \text{ etc.}$$

Also, putting $x=0$ in (1), $a_0=\sin^{-1}0=0$.

Hence, the result.

C. Method of formation of Differential equation.

Ex. 5. *Expand* $(\sin^{-1}x)^2$ *in a series of ascending powers of* x.

Let $y=(\sin^{-1}x)^2$. ... ... (1)

Differentiating, $y_1=\dfrac{2\sin^{-1}x}{\sqrt{1-x^2}}$ ... ... (2)

or, $y_1^2(1-x^2)=4(\sin^{-1}x)^2=4y$.

Differentiating again, and dividing by $2y_1$,

$$(1-x^2)\,y_2-xy_1-2=0. \quad \cdots \quad \cdots \quad (3)$$

Differentiating this n times by Leibnitz theorem, we get

$$(1-x^2)\,y_{n+2}-2nxy_{n+1}-n(n-1)\,y_n-xy_{n+1}-ny_n=0,$$

or, $(1-x^2)\,y_{n+2}-(2n+1)\,xy_{n+1}-n^2y_n=0.$ ... (4)

From (1), (2) and (3), we get $y_0=0$, $(y_1)_0=0$, $(y_2)_0=2$, and from (4), putting $x=0$, we get

$$(y_{n+2})_0=n^2(y_n)_0. \qquad \cdots \qquad \cdots \quad (5)$$

$\therefore$ putting $n=1, 3, 5, \ldots\ldots$ in (5), we get

$$(y_3)_0=(y_5)_0=(y_7)_0=\cdots\cdots=0$$

and putting $n=2, 4, 6, \ldots\ldots$ in (5), we have

$$(y_4)_0=2^2(y_2)_0=2^2 2$$

$$(y_6)_0=4^2(y_4)_0=4^2.2^2.2.$$

Similarly, $(y_8)_0=6^2.4^2.2^2.2$, etc.

Assuming that a Maclaurin's series exists for this function, the coefficients are the values of $y, y_1, y_2, \ldots\ldots, y_n, \ldots\ldots$ when $x=0$.

Hence,

$$(\sin^{-1}x)^2=\frac{1}{2!}\cdot 2x^2+\frac{2^2}{4!}\cdot 2x^4+\frac{2^2.4^2}{6!}\cdot 2x^6+\frac{2^2.4^2.6^2}{8!}\cdot 2x^8+\cdots\cdots$$

Note. It can be shown that this series converges for $x^2 \leq 1$.

D. Differentiation of known series.

Ex. 6. *Assuming expansion of sin x, prove that*

$$\cos x=1-\frac{x^2}{2!}+\frac{x^4}{4!}-\frac{x^6}{6!}+\cdots\cdots$$

From the series $\sin x=x-\frac{x^3}{3!}+\frac{x^5}{5!}-\frac{x^7}{7!}+\cdots\cdots$ which converges for all values of x, we get the required result by differentiation.

Ex. 7. *Show that* $\sec^2 x=1+x^2+\frac{2}{3}x^4+\cdots\cdots$

Since, $\tan x=x+\frac{1}{3}x^3+\frac{2}{15}x^5+\cdots$
we get the required result by differentiation.

Examples VI(B)

1. Expand in infinite series in powers of h :

(i) e^{x+h}. (ii) $\cos(x+h)$. (iii) $e^h \sin(x+h)$.

2. Expand the following functions in powers of x in infinite series, stating in each case the condition under which the expansion is valid :

(i) a^x. (ii) $\sinh x$. (iii) $\cosh x$.

(iv) $\tan^{-1}x$. (v) $\cot^{-1}x$. (vi) $e^{ax}\sin bx$.

(vii) $e^{ax}\cos bx$. (viii) $e^{x}\sin x$. (ix) $e^{x}\cos x$.

(x) $\dfrac{1}{1+x}$. (xi) $\dfrac{1}{1+x^2}$.

3. Show that

$$\tan^{-1}(x+h)=\tan^{-1}x+(h\sin\theta).\sin\theta-\tfrac{1}{2}(h\sin\theta)^2\sin 2\theta$$
$$+\tfrac{1}{3}(h\sin\theta)^3\sin 3\theta+\cdots,\text{ where }\theta=\cot^{-1}x.$$

4. Find approximately the value of $\sin 60^\circ\,34'\,23''$ to 4 places of decimals from the expansion of $\sin(x+h)$ in a series of ascending powers of h by putting

$x=\frac{1}{3}\pi\ (=60^\circ)$ and $h=\frac{1}{100}$ of a radian ($=34'\,23''$ nearly).

5. Show that

(i) $\log x=(x-1)-\frac{1}{2}(x-1)^2+\frac{1}{3}(x-1)^3-\cdots$

is true for $0<x\leqslant 2$.

(ii) $\dfrac{1}{x}=\dfrac{1}{2}-\dfrac{1}{2^2}(x-2)+\dfrac{1}{2^3}(x-2)^2-\cdots$

is true for $0<x<4$.

6. Expand e^x in powers of $(x-1)$.

Verify the following series (*Ex. 7-19*) :

7. $\sec x=1+\frac{1}{2}x^2+\frac{5}{24}x^4+\cdots$

8. $\log(1+x)^{1+x}=x+\dfrac{x^2}{2}-\dfrac{x^3}{6}+\cdots$

9. $\log(1-x+x^2)=-x+\dfrac{x^2}{2}+\dfrac{2}{3}x^3+\dfrac{x^4}{4}-\cdots$

10. $e^x\sin x=x+x^2+\dfrac{x^3}{3}-\cdots$

11. $e^x\log(1+x)=x+\dfrac{x^2}{2!}+\dfrac{2x^3}{3!}+\dfrac{9x^5}{5!}+\cdots$

[*C. P. 1933*]

12. $\dfrac{x}{e^x - 1} = 1 - \dfrac{1}{2}x + \dfrac{1}{12}x^2 - \dfrac{1}{720}x^4 + \cdots$

13. $x \cot x = 1 - \frac{1}{3}x^2 - \frac{1}{45}x^4 + \cdots$

14. $\log(1 + \sin x) = x - \dfrac{x^2}{2} + \dfrac{x^3}{6} - \cdots$

15. $\log \sec x = \dfrac{x^2}{2\,!} + \dfrac{2x^4}{4\,!} + 16\dfrac{x^6}{6\,!} + \cdots$

16. $e^{\sin x} = 1 + x + \frac{1}{2}x^2 - \frac{1}{8}x^4 - \cdots$ [*C. P. 1940*]

17. $e^{\tan^{-1}x} = 1 + x + \dfrac{x^2}{2} - \dfrac{x^3}{6} + \cdots$

18. $\log\{x + \sqrt{(x^2+1)}\} = x - \dfrac{1}{2}\cdot\dfrac{x^3}{3} + \dfrac{1.3}{2.4}\,\dfrac{x^5}{5} - \cdots$

19. $(1 + x)^{\frac{1}{x}} = e[1 - \frac{1}{2}x + \frac{11}{24}x^2 - \frac{7}{16}x^3 + \cdots].$

20. (i) By differentiating the identity

$$(1 - x)^{-1} = 1 + x + x^2 + x^3 + \cdots, \qquad |x| < 1,$$

show that

$$(1 - x)^{-3} = 1 + 3x + \frac{3.4}{1.2}x^2 + \frac{3.4.5}{1.2.3}x^3 + \cdots$$

(ii) Differentiating the expansion for $\log(1 + \sin x)$, obtain the expansion for $\sec x - \tan x$.

21. (i) Show that $\sqrt{x}$, and $x^{\frac{5}{2}}$, cannot be expanded in Maclaurin's infinite series.

(ii) Given $f(x) = x^{\frac{3}{2}}$ show that for this function the expansion of $f(x+h)$ fails when $x = 0$, but that there exists a proper fraction θ such that

$f(x+h) = f(x) + hf'(x) + \frac{1}{2}h^2 f''(x + \theta h)$ holds when $x = 0$. Find θ. [*C. P. 1949*]

22. If $y=(1+x)^n=a_0+a_1x+a_2x^2+\cdots$
show that $(1+x)\,y_1=ny$
and hence obtain the expansion of $(1+x)^n$.

23. If $y=e^{a\sin^{-1}x}=a_0+a_1x+a_2x^2+a_3x^3+\cdots$
prove that (i) $(1-x^2)\,y_2=xy_1+a^2y$,

(ii) $(n+1)(n+2)\,a_{n+2}=(n^2+a^2)\,a_n$,

and hence obtain the expansion of $e^{a\sin^{-1}x}$.

[*C. P. 1945*]

Deduce from the expansion of $e^{a\sin^{-1}x}$, the expansion of $\sin^{-1}x$.

24. If $y=e^{m\tan^{-1}x}=a_0+a_1x+a_2x^2+\cdots$
show that (i) $(1+x^2)\,y_1=my$,

(ii) $(n+1)\,a_{n+1}+(n-1)\,a_{n-1}=ma_n$,

and hence obtain the expansion of $e^{m\tan^{-1}x}$.

25. If $y=\sin(m\sin^{-1}x)$, show that

$$(1-x^2)\,y_{n+2}-(2n+1)\,xy_{n+1}+(m^2-n^2)\,y_n=0,$$

and hence obtain the expansion of $\sin(m\sin^{-1}x)$.

[*C. P. 1938*]

26. If $y=e^{ax}\cos bx$, prove that

$$y_2-2ay_1+(a^2+b^2)\,y=0,$$

and hence obtain the expansion of $e^{ax}\cos bx$.

Deduce the expansion of e^{ax} and $\cos bx$. [*C. P. 1937*]

ANSWERS

1. (i) $e^x\left\{1+h+\frac{h^2}{2!}+\frac{h^3}{3!}+\cdots\cdots\right\}$.

(ii) $\cos x-h\sin x-\frac{h^2}{2!}\cos x+\frac{h^3}{3!}\sin x+\cdots$

(iii) $\sin x+\sqrt{2}h\sin\left(x+\frac{\pi}{4}\right)+\frac{(\sqrt{2}h)^2}{2!}\sin\left(x+\frac{2\pi}{4}\right)+\cdots$

2. (i) $1+x\log a+\frac{x^2}{2!}(\log a)^2+\frac{x^3}{3!}(\log a)^3+\cdots$

(ii) $x+\frac{x^3}{3!}+\frac{x^5}{5!}+\cdots$

(iii) $1+\frac{x^2}{2!}+\frac{x^4}{4!}+\cdots$

for all values of x.

(iv) $x-\frac{x^3}{3}+\frac{x^5}{5}-\cdots$

(v) $\frac{\pi}{2}-\left(x-\frac{x^3}{3}+\frac{x^5}{5}-\cdots\right)$

for $-1 \leqslant x \leqslant 1$.

(vi) $\frac{x}{1!}r\sin\phi+\frac{x^2}{2!}r^2\sin 2\phi+\cdots$

(vii) $1+\frac{x}{1!}r\cos\phi+\frac{x^2}{2!}r^2\cos 2\phi+\cdots$

where $r=\sqrt{(a^2+b^2)}$ and $\phi=\tan^{-1}(b/a)$

(viii) $x\sqrt{2}\sin\left(\frac{\pi}{4}\right)+\frac{x^2}{2!}(\sqrt{2})^2\sin\left(2\frac{\pi}{4}\right)+\cdots+\frac{x^n}{n!}2^{\frac{n}{2}}\sin\left(\frac{n\pi}{4}\right)+\cdots$

(ix) $1+x\sqrt{2}\cos\left(\frac{\pi}{4}\right)+\frac{x^2}{2!}(\sqrt{2})^2\cos\left(2\frac{\pi}{4}\right)+\cdots$

$$+\frac{x^n}{n!}2^{\frac{n}{2}}\cos\left(\frac{n\pi}{4}\right)+\cdots$$

Ex. (vi)-(ix) are valid for all values of x.

(x) $1-x+x^2-x^3+\cdots$ $\quad |x|<1$.

(xi) $1-x^2+x^4-x^6+\cdots$ $\quad -1<x<1$.

4. ·8700. **6.** $e\left\{1+(x-1)+\frac{(x-1)^2}{2!}+\frac{(x-1)^3}{3!}+\cdots\cdots\right\}$.

20. (ii) $1-x+\frac{1}{2}x^2-\frac{1}{3}x^3+\cdots$ **21.** (ii) $\frac{9}{64}$.

22. $1+nx+\frac{n(n-1)}{2}x^2+\cdots$

23. $1+ax+\frac{a^2x^2}{2!}+\frac{a(a^2+1^2)}{3!}x^3+\frac{a^2(a^2+2^2)}{4!}x^4$

$$+\frac{a(a^2+1^2)(a^2+3^2)}{5!}x^5+\cdots$$

$$\sin^{-1}x=x+\frac{x^3}{3!}+\frac{1^2.3^2}{5!}x^5+\cdots$$

24. $1 + mx + \dfrac{m^2}{2\,!}x^2 + \dfrac{m(m^2-2)}{3\,!}x^3 + \dfrac{m^2(m^2-8)}{4\,!}x^4 + \cdots$

25. $mx - \dfrac{m(m^2-1^2)}{3\,!}x^3 + \dfrac{m(m^2-1^2)(m^2-3^2)}{5\,!}x^5 - \cdots$

26. $1 + ax + \dfrac{a^2-b^2}{2\,!}x^2 + \dfrac{a(a^2-3b^2)}{3\,!}x^3 + \cdots$

$$e^{ax} = 1 + ax + \frac{a^2x^2}{2\,!} + \cdots\cdots$$

$$\cos bx = 1 - \frac{b^2x^2}{2\,!} + \frac{b^4x^4}{4\,!} - \cdots\cdots$$

CHAPTER VII

MAXIMA AND MINIMA

(*Function of a Single Variable*)

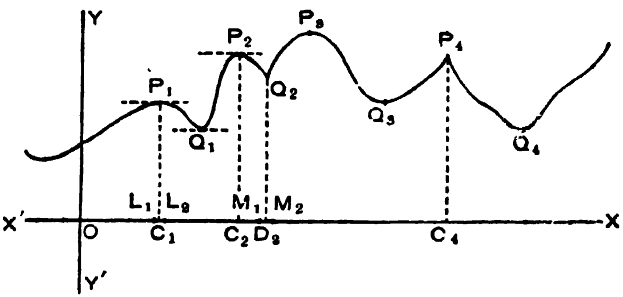
7·1. By the maximum value of a function $f(x)$ in Calculus we do not necessarily mean the absolutely greatest value attainable by the function. A function $f(x)$ is said to be maximum for a value c of x, provided $f(c)$ is greater than every other value assumed by $f(x)$ in the immediate neighbourhood of $x=c$. Similarly, a minimum value of $f(x)$ is defined to be the value which is less than other values in the immediate neighbourhood. A formal definition is as follows :

A function $f(x)$ is said to have a maximum value for $x=c$, provided we can get a positive quantity δ such that for all values of x in the interval $c-\delta < x < c+\delta$, $(x \neq c)$ $f(c) > f(x)$;

i.e., if $f(c+h)-f(c) < 0$, for $|h|$ sufficiently small.

Similarly, *the function $f(x)$ has a minimum value for $x=d$, provided we can get an interval $d-\delta' < x < d+\delta'$ within which $f(d) < f(x)$ $(x \neq d)$;*

i.e., if $f(d+h)-f(d) > 0$, for $|h|$ sufficiently small.



Thus, in the above figure which represents graphically the function $f(x)$ (a continuous function here), the function has a maximum value at P_1, as also at P_2, P_3, P_4, etc. and has minimum values at Q_1, Q_2, Q_3, Q_4, etc. At P_1, for instance, corresponding to $x = OC_1$ ($= c_1$ say), the value of the function, namely, the ordinate P_1C_1 is not necessarily bigger than the value Q_2D_2 at $x = OD_2$, but we can get a range say $L_1C_1L_2$ in the neighbourhood of C_1 on either side of it, (*i.e.*, we can find a $\delta = L_1C_1 = C_1L_2$ say) such that for every value of x within $L_1C_1L_2$ (except at C_1), the value of the function (represented by the corresponding ordinate) is less than P_1C_1 (the value at C_1). Hence, by definition, the function is maximum at $x = OC_1$. Similarly, we can find out an interval $M_1D_2M_2$ ($M_1D_2 = D_2M_2 = \delta'$ say) in the neighbourhood of D_2 within which for every other value of x the function is greater than that at D_2. Hence, the function at D_2 (represented by Q_2D_2) is a minimum.

From the figure the following features regarding maxima and minima of a continuous function will be apparent :

(i) that the function may have several maxima and minima in an interval , (ii) that a maximum value of the function at some point may be less than a minimum value of it at another point ($C_1P_1 < D_2Q_2$) , (iii) maximum and minimum values of the function occur alternately, *i.e.*, between any two consecutive maximum values there is a minimum value, and *vice versa*.

7·2. A necessary condition for maximum and minimum.

If $f(x)$ be a maximum, or a minimum at $x = c$, and if $f'(c)$ exists, then $\mathbf{f'(c) = 0}$.

By definition, $f(x)$ is a maximum at $x = c$, provided we can find a positive number δ, such that

$$f(c+h) - f(c) < 0 \text{ whenever } -\delta < h < \delta, (h \neq 0),$$

$$\therefore \quad \frac{f(c+h) - f(c)}{h} < 0 \text{ if } h \text{ be positive and sufficiently}$$

small, and > 0 if h be negative and numerically sufficiently small.

Thus, $\underset{h\to 0+}{Lt} \dfrac{f(c+h)-f(c)}{h} \leqslant 0$, [See *Ex. 5*, § *2·11*]

and similarly, $\underset{h\to 0-}{Lt} \dfrac{f(c+h)-f(c)}{h} \geqslant 0$.

Now, if $f'(c)$ exists, the above two limits, which represent the right-hand and left-hand derivatives respectively of $f(x)$ at $x=c$, must be equal. Hence, the only common value of the limit is zero. Thus, $f'(c)=0$.

Exactly similar is the proof when $f(c)$ is a minimum.

Note. *In case $f'(c)$ does not exist, $f(c)$ may be a maximum or a minimum*, as is apparent from the figure for points Q_2 and P_4. At the former point $f(x)$ is a minimum, and at the latter it is a maximum. $f'(x)$ is however not zero at these points, for $f'(x)$ does not exist at all at these points.

7·3. Determination of Maxima and Minima.

(A) *If c be a point in the interval in which the function $f(x)$ is defined, and if* **f′(c) = 0,** *and* **f″(c) ≠ 0,** *then* **f(c)** is (i) **a maximum** *if* **f″(c) is negative** *and* (ii) **a minimum** *if* **f″(c) is positive.**

Proof: Suppose $f'(c)=0$, and $f''(c)$ exists, and $\neq 0$.

By the Mean Value Theorem*,

$$f(c+h)-f(c)=hf'(c+\theta h),\ 0<\theta<1,$$

$$=\theta h^2 . \frac{f'(c+\theta h)-f'(c)}{\theta h}.$$

Since $0<\theta<1$, $\theta h\to 0$ as $h\to 0$, and writing $\theta h=k$, the coefficient of θh^2 on the right side $\to \underset{k\to 0}{Lt} \dfrac{f'(c+k)-f'(c)}{k}=f''(c)$. Accordingly, since θh^2 is positive, $f(c+h)-f(c)$ has the same sign as that of $f''(c)$ when $|h|$ is sufficiently small.

---

* Since, $f''(c)$ exists, $f'(x)$ also exists in the neighbourhood of c.

$\therefore$ if $f''(c)$ is positive, $f(c+h)-f(c)$ is positive, whether h is positive or negative, provided $|h|$ is small. Hence $f(c)$ is a *minimum*, by definition.

Similarly, if $f''(c)$ is negative, $f(c+h)-f(c)$ is negative, whether h is positive or negative, when $|h|$ is small, and so $f(c)$ is a *maximum*.

(B) *Let c be an interior point of the interval of definition of the function $f(x)$, and let*

$$\mathbf{f'(c)=f''(c)=\cdots=f^{n-1}(c)=0},\ \textit{and}\ \mathbf{f^n(c)\neq 0}\ ;$$

then (i) *if* **n be even, f (c)** *is a* **maximum** *or a* **minimum** *according as* **$\mathbf{f^n}$ (c)** *is* **negative** *or* **positive,**

and (ii) *if* **n be odd, f (c)** *is* **neither a minimum, nor a maximum.**

Proof: By the Mean Value Theorem of Higher order, here

$$(c+h)-f(c)=\frac{h^{n-1}}{(n-1)!}f^{n-1}(c+\theta h),\ 0<\theta<1,$$

$$=\frac{\theta h^n}{(n-1)!}\cdot\frac{f^{n-1}(c+\theta h)-f^{n-1}(c)}{\theta h}.$$

Since $0<\theta<1$, as $h\to 0$, $\theta h\to 0$ and the coefficient of $\theta h^n/(n-1)!$, on the right side $\to f^n(c)$.

Now, suppose n is *even*; then, $\theta h^n/(n-1)!$ is positive.

$\therefore$ $f(c+h)-f(c)$ has the same sign as of $f^n(c)$, whether h is positive, or negative, provided $|h|$ is sufficiently small. Hence, *if $f^n(c)$ be positive*, $f(c+h)-f(c)$ is positive for either sign of h, when $|h|$ is small, and so *$f(c)$ is a minimum*. Similarly, *if $f^n(c)$ is negative*, $f(c)$ is *a maximum*.

Next suppose *n is odd*; then $\theta h^n/(n-1)!$ is positive or negative according as h is positive or negative. Hence, $f(c+h)-f(c)$ changes in sign with the change of h whatever the sign of $f^n(c)$ may be, and so $f(c)$ cannot be either a maximum or a minimum at $x=c$.

Hence to determine maxima and minima of $f(x)$, we proceed with the following **working rule :**

Equate $f'(x)$ to zero, and let the roots be $c_1, c_2, c_3, \dots$ Now work out the value of $f''(c_1)$. If it is negative, then $x = c_1$ makes $f(x)$ a maximum. If $f''(c_1)$ be positive, then $f(c_1)$ is a minimum of $f(x)$. Similarly test the sign of $f''(x)$ for the other values $c_2, c_3 \dots$ of x for which $f'(x)$ is zero, and determine whether $f(x)$ is a maximum or a minimum at these points.

If in any case above, $f''(c_n) = 0$, use criterion (B).

Note 1. The above criterion for determining maxima and minima of $f(x)$ fails at a point where $f'(x)$ is non-existent, even though $f(x)$ may be continuous there.

In such a case we should bear in mind that if $f(x)$ be a **maximum** at a point, immediately to the left of it the value of $f(x)$ is less, and gradually increases towards the value at the point and so $f'(x)$ [which represents the rate of increase of $f(x)$] is positive. Immediately to the right, the value of $f(x)$ is again less, and so $f(x)$ decreases with x increasing and therefore $f'(x)$ is negative to the right. Thus $f'(x)$ changes sign from positive on the left to negative towards the right of the point. [See point P_4 *in the figure of Art. 7·1*]

Similarly, if $f(x)$ be a minimum at any point, $f(x)$ is larger on the left, and diminishes to the value at the point, and again becomes larger on the right, *i.e.*, $f(x)$ increases to the right. Thus $f'(x)$ changes sign here, being negative on the left, and positive on the right of the point.

Thus we have the following **alternative criterion** for maxima and minima : *At a point where $f(x)$ is a maximum or a minimum,* **f′ (x) changes sign, from positive on the left to negative on the right if f (x) be a maximum, and from negative on the left to positive on the right if f (x) be a minimum.**

If $f'(x)$ exists at such a point, its change of sign from one side to another takes place through the zero value of $f'(x)$, so that $f'(x) = 0$ at the point. If $f'(x)$ be non-existent at the point, the left-hand and right-hand derivatives are of opposite signs at the point.

Even in the case where the successive derivatives exist, instead of proceeding to calculate their values at a point to apply the usual criteria for maxima and minima of $f(x)$ at the point, we may apply effectively in many cases, this simple criterion of changing of sign of $f'(x+h)$ as h is changed from negative to positive values, being numerically small. [For *illustration* see *Ex. 4*, § *7·5*]

Note 2. At points where $f(x)$ is a maximum or a minimum, $f'(x)=0$ when it exists, and accordingly at these points the tangent line to the graph of $f(x)$ will be parallel to the x-axis (as at P_1, Q_1, P_2, Q_2, P_3, Q_3, etc. in figure of § *7·1*). At points where $f(x)$ is a maximum or a minimum, but $f'(x)$ does not exist (*e.g.*, at Q_2 and P_4), the tangent line to the curve changes its direction abruptly while passing through the point. A special case is where the tangent is parallel to the y-axis, the change in the sign of $f'(x)$ taking place through an infinite value.

Note 3. A maximum or minimum is often called an '*extremum*' (extremal) or '*turning value*'. The values of x for which $f'(x)$ or $1/f'(x)=0$ are often called '*critical values*' or, critical points of $f(x)$.

Note 4. The use of the following principles greatly simplifies the solution of problems in maxima and minima.

(i) Since, $f(x)$ and $\log f(x)$ increase and decrease together, $\log f(x)$ is maximum or minimum for any value of x for which $f(x)$ is maximum or minimum.

(ii) When $f(x)$ increases, since $1/f(x)$ decreases, any value of x which renders $f(x)$ a maximum or minimum renders its reciprocal $1/f(x)$ a minimum or a maximum.

(iii) Any value of x which renders $f(x)$ positive and a maximum or a minimum, renders $\{f(x)\}^n$ a maximum or a minimum, n being a positive integer.

For examples on maxima and minima of functions of two variables connected by a relation, see Ex. 7 and Ex. 12 of Art. 7·5.

7·4. Elementary methods *(Algebraical and Trigonometrical).*

Certain types of problems in maxima and minima can be solved very simply by elementary algebra or trigonometry*. The discussion of the maxima and minima of the quadratic functions or the quotient of two quadratic functions, will be found in any text book on algebra. †

In solving simpler problems of maxima and minima of functions of more than one variable, the following elementary results are of great use :

(i) $xy = \{\frac{1}{2}(x+y)\}^2 - \{\frac{1}{2}(x-y)\}^2.$

(ii) $(x+y)^2 = 4xy + (x-y)^2.$

(iii) $x^2 + y^2 = \frac{1}{2}(x+y)^2 + \frac{1}{2}(x-y)^2.$

When *the sum of two positive quantities is given,* it follows from (i) that *their product is greatest,* and from (iii) *that the sum of their squares is least, when they are equal.* When *the product of two quantities is given,* from (ii) *their sum is least when they are equal.*

The above theorems may easily be extended to the cases of more than two quantities.

Thus, *when the sum of any number of positive quantities is given, their product is greatest when they are all equal,* and so on.

For illustrative examples see Art. 7·5, Ex. 9 to 11.

Note. In algebraical or trigonometrical examples, by maximum or a minimum value of a function we usually mean the greatest or the least value attainable by the function out of all its possible values. In Calculus however, as has already been remarked, a maximum or a minimum value indicates a local (or relative) maximum or minimum.

* See Das & Mukherjees' *Higher Trigonometry, Chap. XV, Sec. B.*

† See Ganguly & Mukherjees' *Intermediate Algebra, Chap. VIII, Art. 51.*

7·5. Illustrative Examples.

Ex. 1. *Find for what values of x, the following expression is maximum and minimum respectively :*

$$2x^3-21x^2+36x-20.$$

Find also the maximum and minimum values of the expression.

[*C. P. 1936*]

Let $\quad f(x)=2x^3-21x^2+36x-20.$

$\therefore \quad f'(x)=6x^2-42x+36$, which exists for all values of x.

Now, when $f(x)$ is a maximum or a minimum, $f'(x)=0$.

$\therefore$ we should have $6x^2-42x+36=0$, *i.e.*, $x^2-7x+6=0$,
or, $(x-1)(x-6)=0$; $\quad \therefore \quad x=1$ or 6.

Again, $f''(x)=12x-42=6(2x-7)$.

Now, when $x=1$, $f''(x)=-30$ which is *negative*,
when $x=6$, $f''(x)=30$, which is *positive*.

Hence, the given expression is *maximum* for $x=1$, and *minimum* for $x=6$.

The maximum and minimum values of the given expression are respectively $f(1)$, *i.e.*, -3, and $f(6)$, *i.e.*, -128.

Ex. 2. *Investigate for what values of x,*

$$f(x)=5x^6-18x^5+15x^4-10$$

is a maximum or minimum.

Here, $\quad f'(x)=30(x^5-3x^4+2x^3)$.

Putting $f'(x)=0$, we have $x^3(x^2-3x+2)=0$,

i.e., $x^3(x-1)(x-2)=0$, whence, $x=0$, 1 or 2.

Again, $f''(x)=30(5x^4-12x^3+6x^2)$.

When $x=1$, $f''(x)$ is *negative*, and hence $f(x)$ is a *maximum* for $x=1$.

When $x=2$, $f''(x)$ is *positive*, and hence $f(x)$ is a *minimum* for $x=2$.

When $x=0$, $f''(x)=0$; so the test fails, and we have to examine higher order derivatives.

$f'''(x)=120(5x^3-9x^2+3x)$. $\quad \therefore \quad f'''(0)=0$.

$f^{iv}(x)=360(5x^2-6x+1)$. $\quad \therefore \quad f^{iv}(0)$ is *positive*.

Since *even order* derivative is positive for $x=0$,

$\therefore$ for $x=0$, $f(x)$ is a *minimum*.

Ex. 3. *Show that $f(x)=x^3-6x^2+24x+4$ has neither a maximum nor a minimum.*

Here, $f'(x)=3(x^2-4x+8)=3\{(x-2)^2+4\}$
which is always positive and can never be zero.

$\therefore$ $f(x)$ has neither a maximum nor a minimum.

Ex. 4. *Examine $f(x)=x^3-9x^2+24x-12$ for maximum or minimum values.*

Here, $f'(x)=3(x^2-6x+8)=3(x-2)(x-4)$.

Putting $f'(x)=0$, we find $x=2$ or 4.

Now, $f'(2-h)=3(-h)(-2-h)=+$,

and $f'(2+h)=3(h)(h-2)=-$, since, h is positive and small.

$\therefore$ by § 7·3, Note 1, for $x=2$, $f(x)$ has a *maximum* value, and this is $f(2)=8$.

Again, $f'(4-h)=3.(2-h)(-h)=-$, since h is positive and small,
$f'(4+h)=3.(2+h)(h)=+$.

$\therefore$ by § 7·3, Note 1, for $x=4$, $f(x)$ has a *minimum* value, and this is $f(4)=4$.

Note In this case we could have easily applied rule of Art. 7·3.

Ex. 5. *Find the maxima and minima of*

$$1+2\sin x+3\cos^2 x. \qquad (0 \leqslant x \leqslant \tfrac{1}{2}\pi).$$

Let $f(x)=1+2\sin x+3\cos^2 x$.

Then $f'(x)=2\cos x-6\cos x\sin x$.

$\therefore$ $f'(x)=0$ when $2\cos x(1-3\sin x)=0$, *i.e.*, when $\cos x=0$, and also when $\sin x=\frac{1}{3}$.

$$f''(x)=-2\sin x-6(\cos^2 x-\sin^2 x).$$

When $\cos x=0$, $x=\frac{1}{2}\pi$. $\therefore$ $\sin x=1$. $\therefore$ $f''(x)=-2+6=4$ (*positive*).

$\therefore$ for $\cos x=0$, $f(x)$ is a *minimum*, and the minimum value is 3.

When $\sin x=\frac{1}{3}$,

$$f''(x)=-2\sin x-6(1-2\sin^2 x)=-\tfrac{2}{3}-6(1-\tfrac{2}{9}) \text{ (negative)}.$$

Therefore, for $\sin x = \frac{1}{3}$, $f(x)$ is a *maximum* and the maximum value is $1+2.\frac{1}{3}+3.(1-\frac{1}{9})=4\frac{1}{3}$.

Ex. 6. *Examine whether $x^{\frac{1}{x}}$ possesses a maximum or a minimum, and determine the same.* [*C. P. 1941, '45*]

Let $\quad y = x^{\frac{1}{x}}$. $\quad \therefore \log y = \dfrac{1}{x} \log x$.

$$\therefore \quad \frac{1}{y}\frac{dy}{dx} = \frac{1}{x^2} - \frac{1}{x^2}\log x = \frac{1}{x^2}(1-\log x). \quad \cdots \quad (1)$$

$\therefore$ when $\dfrac{dy}{dx}=0$, $1-\log x=0$. $\therefore \log x = 1 = \log e$. $\therefore x = e$.

Again, differentiating (1) with respect to x,

$$-\frac{1}{y^2}\left(\frac{dy}{dx}\right)^2 + \frac{1}{y}\frac{d^2y}{dx^2} = \frac{x^2.(-1/x)-(1-\log x)\,2x}{x^4} = \frac{-3+2\log x}{x^3}.$$

$$\therefore \text{ when } x=e,\ \frac{d^2y}{dx^2} = e^{\frac{1}{e}}.\frac{-3+2}{e^3} = -\frac{e^{\frac{1}{e}}}{e^3}, \text{ which is } \textit{negative}.$$

$$\left(\because \text{ for } x=e,\ \frac{dy}{dx}=0.\right)$$

$\therefore$ for $x=e$, the function is a *maximum*, and the maximum value is $e^{\frac{1}{e}}$.

Ex. 7 *Find the maximum and minimum values of u where*

$$u = \frac{4}{x} + \frac{36}{y} \text{ and } x+y=2.$$

Eliminating y between the two given relations

$$u = \frac{4}{x} + \frac{36}{2-x}. \quad \therefore \frac{du}{dx} = -\frac{4}{x^2} + \frac{36}{(2-x)^2} = \frac{16\,(2x^2+x-1)}{x^2\,(2-x)^2}.$$

$$\frac{du}{dx}=0, \text{ gives } x=\tfrac{1}{2} \text{ or } -1. \quad \text{Also, } \frac{d^2u}{dx^2} = \frac{8}{x^3} + \frac{72}{(2-x)^3}.$$

When $x=\frac{1}{2}$, $\dfrac{d^2u}{dx^2} = \dfrac{8}{(\frac{1}{2})^3} + \dfrac{72}{(\frac{3}{2})^3}$ which is *positive*.

$\therefore$ for $x=\frac{1}{2}$, u is a *minimum*.

$\therefore$ *minimum value* of $u = \dfrac{4}{\frac{1}{2}} + \dfrac{36}{2-\frac{1}{2}} = 32$.

When $x=-1$, $\dfrac{d^2u}{dx^2} = -8 + \dfrac{72}{27}$, which is *negative*.

$\therefore$ for $x=-1$, u is a *maximum*.

$\therefore$ *maximum value* of $u = \dfrac{4}{-1} + \dfrac{36}{2+1} = 8$.

Ex. 8. *Examine the function* $f(x)=4-3(x-2)^{\frac{2}{3}}$ *for maxima and minima at* $x=2$.

Here, $f'(x)=-\dfrac{2}{(x-2)^{\frac{1}{3}}}$.

For $x=2$, $f'(x)$ does not exist, the left-hand derivative being $+\infty$ and the right-hand derivative $-\infty$; but $1/f'(x)$ is zero for $x=2$. So the test of Art. 7·2 fails. Let us apply the criterion of § 7·3, Note 1. Now, $f'(2-h)$ is *positive* and $f'(2+h)$ is *negative*. Hence, the function has a *maximum* value for $x=2$, and the maximum value is $f(2)$ *i.e.*, 4.

Ex. 9. *A conical tent of given capacity has to be constructed. Find the ratio of the height to the radius of the base for the minimum amount of the canvas required for the tent.*

Let r be the radius of the base, h the height, V the volume, and S the surface-area of the conical tent.

Then, $V=\frac{1}{3}\pi r^2 h \cdots (1)$ and $S=\pi r\sqrt{r^2+h^2}. \cdots (2)$

Here, V is given as constant,

$$\therefore \quad S^2=\pi^2 r^2(r^2+h^2)=\pi^2 r^2\left(r^2+\frac{9V^2}{\pi^2 r^4}\right) \quad [\textit{from (1)}]$$

$$=\pi^2 r^4+9V^2\cdot\frac{1}{r^2}.$$

Now, if S is a max. or min., S^2 is so and hence for max. or min. of S, $\dfrac{d}{dr}(S^2)=0$, *i.e.*, $\dfrac{d}{dr}\left(\pi^2 r^4+9V^2\cdot\dfrac{1}{r^2}\right)=0$,

i.e., $4\pi^2 r^3-18V^2\cdot\dfrac{1}{r^3}=0$. $\therefore$ $r^6=\dfrac{9V^2}{2\pi^2}$. $\therefore$ $r=\left(\dfrac{3V}{\pi\sqrt{2}}\right)^{\frac{1}{3}}$.

Now, $\dfrac{d^2}{dr^2}(S^2)=12\pi^2 r^2+54V^2\dfrac{1}{r^4}$ which is *positive* for $r=\left(\dfrac{3V}{\pi\sqrt{2}}\right)^{\frac{1}{3}}$.

$\therefore$ for *minimum* amount of canvas,

$$r=\left(\frac{3V}{\pi\sqrt{2}}\right)^{\frac{1}{3}}, \text{ i.e., } r^6=\frac{9V^2}{2\pi^2}=\frac{9\cdot\frac{1}{9}\pi^2 r^4 h^2}{2\pi^2} \; [\textit{from (1)}] =\frac{r^4}{2}h^2,$$

i.e., $r^2=\frac{1}{2}h^2$. $\therefore$ $r^2 : h^2=1:2$ or $r:h=1:\sqrt{2}$.

Ex. 10. *Show that for a given perimeter, the area of a triangle is maximum when it is equilateral.*

The area Δ of a triangle $ABC = \sqrt{s(s-a)(s-b)(s-c)}$.

Let $s-a=x$, $s-b=y$, $s-c=z$. $\therefore$ $x+y+z=3s-(a+b+c)$ $=3s-2s=s=$ const. Now, $\Delta = \sqrt{sxyz}$. Since s is constant, Δ will be maximum when xyz will be maximum subject to the condition $x+y+z$ $=$ const. *i.e.*, when $x=y=z$, [See § 7·4]

i.e., when $s-a=s-b=s-c$, *i.e.*, $a=b=c$.

Ex. 11. *Show that the maximum triangle which can be inscribed in a circle is equilateral.*

Area Δ of a triangle ABC inscribed in a circle of radius R

$$=\tfrac{1}{2}bc \sin A = \tfrac{1}{2}.2R \sin B.2R \sin C \,.\, \sin A$$

$$=2R^2 \sin A \sin B \sin C$$

$$=R^2 \{\cos(A-B) - \cos(A+B)\} \sin C.$$

Let us suppose C remains constant, while A and B vary. Since, R is constant, the above expression will be maximum when $A=B$.

Hence, so long as any two of the angles, A, B, C are unequal, the expression $2R^2 \sin A \sin B \sin C$ is not a maximum, that is, it is maximum when $A=B=C$.

Thus, Δ will be maximum when $A=B=C$.

Ex. 12. *Find the maximum and minimum values of*

$a \sin x + b \cos x$.

Let $a=r\cos\theta$, $b=r\sin\theta$,

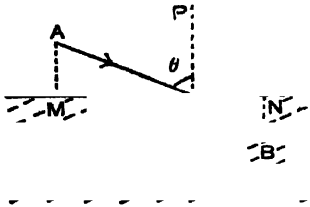
so that $r^2=a^2+b^2$ and $\tan\theta=b/a$.

Thus, $a \sin x+b \cos x = r(\sin x \cos\theta + \cos x \sin\theta) = r\sin(x+\theta)$

$$=\sqrt{a^2+b^2}\, \sin\{x+\tan^{-1}b/a\}.$$

Since the greatest and least values of sine of an angle are 1 and -1, the required maximum and minimum values of the given expression are $\sqrt{a^2+b^2}$ and $-\sqrt{a^2+b^2}$.

Ex. 13. *Assuming Fermat's theorem that a ray of light in passing from a point A in one medium to a point B in another medium takes the path for which the time of description is a minimum, prove the law of refraction.*



Let AOB be a possible path of the ray of light, O being the point where it meets the surface of separation MON of the two media, and let POQ be the normal to the common surface MN at O, and AM, BN the perpendiculars from A and B on MN. Let $\angle AOP=\theta$, $\angle BOQ=\phi$, and let v and v' be velocities of light in the two media. If $AM=a$, $BN=b$, then $AO=a\sec\theta$, $BO=b\sec\phi$. The time taken by the ray of light to travel the path AOB is

$T=\dfrac{a\sec\theta}{v}+\dfrac{b\sec\phi}{v'}$ ... (i), and by Fermat's theorem this is to be a minimum.

Again, since A and B are fixed points, $a\tan\theta+b\tan\phi=MO+ON=MN=$ constant ... (ii), so that θ and ϕ are not independent, and we can thus consider ϕ as a function of θ, which is then the only independent variable.

For T to be minimum, $\dfrac{dT}{d\theta}=0$, giving

$$\frac{a}{v}\sec\theta\tan\theta+\frac{b}{v'}\sec\phi\tan\phi\frac{d\phi}{d\theta}=0.$$

Also, from (ii),

$$a\sec^2\theta+b\sec^2\phi\frac{d\phi}{d\theta}=0.$$

From these two, eliminating $\dfrac{d\phi}{d\theta}$, we easily get

$$\frac{\sin\theta}{v}=\frac{\sin\phi}{v'}, \text{ or, } \frac{\sin\theta}{\sin\phi}=\frac{v}{v'}=\mu \text{ (say)}$$

which is the law of refraction, satisfied for the actual path of the ray of light.

Examples VII

1. Find for which values of x the following functions are maximum and minimum :

(i) $x^3 - 9x^2 + 15x - 3$. (ii) $4x^3 - 15x^2 + 12x - 2$.

(iii) $\dfrac{x^2 - 7x + 6}{x - 10}$. [*C.P. 1939*] (iv) $\dfrac{x^2 + x + 1}{x^2 - x + 1}$.

(v) $x^4 - 8x^3 + 22x^2 - 24x + 5$.

2. Find the maximum and minimum values of (iii), (iv) and (v) of Ex. 1.

3. (i) Show that the maximum value of $x + \dfrac{1}{x}$ is less than its minimum value.

(ii) Show that the minimum value of $\dfrac{(2x-1)(x-8)}{(x-1)(x-4)}$ is greater than its maximum value.

4. Show that $x^3 - 6x^2 + 12x - 3$ is neither a maximum nor a minimum when $x = 2$.

5. Show that the following functions possess neither a maximum nor a minimum :

(i) $x^3 - 3x^2 + 6x + 3$. (ii) $x^3 - 3x^2 + 9x - 1$.

(iii) $\sin(x + a)/\sin(x + b)$. (iv) $(ax + b)/(cx + d)$.

6. Show that $x^5 - 5x^4 + 5x^3 - 1$ is a maximum when $x = 1$, a minimum when $x = 3$; neither when $x = 0$.

7. Examine for maxima and minima of the following functions :

(i) $\sin x$. (ii) $\cos x$. (iii) x^5.

(iv) x^6. (v) $\frac{1}{5}x^5 - \frac{1}{4}x^4$. (vi) $e^x . \sin x$.

8. Test the following functions for maxima and minima at $x=0$:

(i) $\sin x - x + \dfrac{x^3}{3!} - \dfrac{x^5}{5!}$. (ii) $\cos x - 1 + \dfrac{x^2}{2!} - \dfrac{x^4}{4!}$.

9. Show that

(i) $\sqrt{3} \sin x + 3 \cos x$ is a maximum for $x = \frac{1}{6}\pi$.

(ii) $\sin x (1 + \cos x)$ is a maximum for $x = \frac{1}{3}\pi$.

[*C. P. 1942, '47*]

(iii) $\sin^3 x \cos x$ is a maximum when $x = \frac{1}{3}\pi$.

(iv) $x^2 + x \sin x + 4 \cos x$ is a maximum for $x = 0$.

(v) $\sec x + \log \cos^2 x$ is a maximum for $x = 0$ and a minimum for $x = \frac{1}{3}\pi$.

(vi) $\dfrac{2\theta - \sin 2\theta}{\theta^2}$ ($\theta > 0$) is maximum when $\theta = \frac{1}{2}\pi$.

10. If y is defined as a function of x by the equations

$$y = a(1 - \cos \theta),\ x = a\ (\theta - \sin \theta),$$

show that y is a maximum when $\theta = \pi$.

11. Show that

(i) the maximum value of $(1/x)^x$ is $e^{1/e}$.

(ii) the minimum value of $x/(\log x)$ is e.

(iii) the minimum value of $4e^{2x} + 9e^{-2x}$ is 12.

12. (i) Show that $4^x - 8x \log_e 2$ is a minimum when $x = 1$.

(ii) Show that $12 (\log x + 1) + x^2 - 10x + 3$ is a maximum when $x = 2$ and a minimum when $x = 3$.

(iii) Show that $x^2 \log (1/x)$ is a maximum for $x = 1/\sqrt{e}$.

13. If $f'(x)=(x-a)^{2m}(x-b)^{2n+1}$, when m and n are positive integers, show that $x=a$ gives neither a maximum nor a minimum value of $f(x)$, but $x=b$ gives a minimum.

14. Find the maxima and minima, if any, of

$$x^4/(x-1)(x-3)^3.$$

15. If $y=\dfrac{ax+b}{(x-1)(x-4)}$ has a turning value at $(2, -1)$, find a and b and show that the turning value is a maximum.

16. Prove that $\Sigma(x-a_1)^2$ is a minimum when x is the arithmetic mean of $a_1, a_2, a_3, \ldots a_n$.

17. (i) Given $x/2+y/3=1$, find the maximum value of xy and minimum value of x^2+y^2.

(ii) Given $xy=4$, find the maximum and minimum values of $4x+9y$.

18. (i) If $f(x)=1-\sqrt{(x^2)}$, when the square root is to be taken positive, show that $x=0$ gives a maximum for $f(x)$.

(ii) If $f(x)=a+(x-b)^{\frac{2}{3}}+(x-a)^{\frac{4}{5}}$, show that $f(x)$ is a minimum for $x=b$.

(iii) Show that $(x-a)^{\frac{1}{3}}(2x-a)^{\frac{2}{3}}$ is a maximum for $x=\frac{1}{2}a$, a minimum for $x=\frac{5}{8}a$ and neither for $x=a$. [$a>0$]

(iv) If $f(x)=|x|$, show that $f(0)$ is a minimum although $f'(0)$ does not exist.

19. Show that

(i) the largest rectangle with a given perimeter is a square.

(ii) the maximum rectangle inscribable in a circle is a square. [*C. P. 1936*]

20. Find the point on the parabola $2y = x^2$ which is nearest to the point (0, 3).

21. P is any point on the curve $y = f(x)$ and C is a fixed point not on the curve. If the length PC is either a maximum or a minimum, the line PC is perpendicular to the tangent at P.

22. Find the length of the perpendicular from the point (0, 2) upon the line $3x + 4y + 2 = 0$, showing that it is the shortest distance of the point from the line. Find also the foot of the perpendicular.

23. A cylindrical tin can, closed at the both ends, of a given capacity, has to be constructed. Show that the amount of the tin required will be a minimum when the height is equal to the diameter.

24. By the Post Office regulations, the combined length and girth of a parcel must not exceed 6 feet. Find the volume of the biggest cylindrical (right circular) packet that can be sent by the parcel post

25. A line drawn through the point P (1, 8) cuts the positive sides of the axes OX and OY at A and B. Find the intercepts of this line on the axes so that

(i) the area of the triangle OAB is a minimum ;

(ii) the length of the line AB is a minimum.

Find also in the above cases the area of the triangle and the length of the line respectively.

26. P is a point on an ellipse whose centre is C, and N is the foot of the perpendicular from C upon the tangent to the ellipse at P ; find the maximum value of PN.

[*C. P. 1945*]

27. The height of a particle projected with velocity u at an angle a with the horizontal, is $u \sin a.t - \frac{1}{2}gt^2$ at any time t. Find the greatest height attained and the time of reaching it.

28. The total waste per mile in an electric conductor is given by $W = C^2R + K^2/R$, where C is the current, R the resistance, and K a constant. What resistance will make the waste a minimum if the current C is kept constant ?

29. The force F exerted by a circular electric current of radius a on a magnet whose axis coincides with the axis of the coil is given by

$$F \propto x(a^2 + x^2)^{-\frac{5}{2}},$$

where x is the distance of the magnet from the centre of the circle. Show that F is greatest when $x = \frac{1}{2}a$.

***30.** Assuming that the intensity of light at a point on an illuminated surface varies directly as the sine of the angle at which the ray of light strikes the surface, and inversely as the square of the distance of the source from the point, find how high should a light be placed directly over the centre of a circular field of radius $30\sqrt{2}$ ft. in order to have a maximum illumination on the boundary.

31. (i) Find the altitude of the right cone of maximum volume that can be inscribed in a sphere of radius a.

(ii) Find the altitude of the right circular cylinder of maximum volume that can be inscribed in a given right circular cone of height h.

32. (i) For a given curved surface of a right circular cone when the volume is maximum, show that the semi-vertical angle is $\sin^{-1}(1/\sqrt{3})$.

(ii) For a given volume of a right cone, show that when the curved surface is minimum, the semi-vertical angle is $\sin^{-1}(1/\sqrt{3})$.

33. An open tank of a given volume consists of a square base with vertical sides. Show that the expense of lining the tank with lead will be least if the height of the tank is half the width.

34. If POP' and QOQ' be any two conjugate diameters of an ellipse, and from P and Q are drawn two perpendiculars to the major axis cutting it at M and N respectively, show that $PM+QN$ is a maximum when POP' and QOQ' are equi-conjugate diameters.

35. A window is in the form of a rectangle surmounted by a semi-circle. If the total perimeter be 25 ft., find the dimensions so that the greatest possible amount of light may be admitted.

36. A particle is moving in a straight line. Its distance x ft. from a fixed point O at any time t secs. is given by the relation

$$x=t^4-10t^3+24t^2+36t+12.$$

When is it moving most slowly ?

37. In enclosing a rectangular lawn that has one side along a neighbour's plot, a person has to pay for the fence for the three sides on his own ground and for half of that along the dividing line. What dimensions would give him the least cost if the lawn is to contain 4800 sq. ft. ?

38. A gardener having 120 ft. of fencing wishes to enclose a rectangular plot of land and also to erect a fence across the land parallel to two of the sides. What is the maximum area he can enclose ?

39. A shot is fired with a velocity u at a vertical wall whose distance from the point of projection is x. Find the greatest height above the level of the point of projection at which the bullet can hit the wall.

40. From the fixed point A on the circumference of a circle of radius c the perpendicular AY is let fall on the tangent at P. Show that the maximum area of ΔAPY is $\frac{3}{8}c^2\sqrt{3}$. [*C. P. 1930*]

41. The intensity of light varies inversely as the square of the distance from the source. If two lights are 150 ft. apart and one light is 8 times as strong as the other, where should an object be placed between the lights to have the least illumination ?

***42.** The boundary wall of a house is 27 ft. high, and is at a distance 8 ft. from the house. Show that a ladder, one end of which rests on the ground outside the wall, and which passes over the wall, must at least be $13\sqrt{13}$ ft. long in order to reach the house.

***43.** A man in a boat $\frac{1}{2}\sqrt{3}$ miles from the bank wishes to reach a village that is $5\frac{1}{2}$ miles distant along the bank from the point nearest to him. He can walk 4 miles per hour and row 2 miles per hour. Where should he land in order to reach the village in the least time ? Find also the time.

***44.** If for a steamer the consumption of coal varies as the cube of its speed, show that the most economical rate of steaming against a current will be a speed equal to $1\frac{1}{2}$ times that of the current.

***45.** For a train the cost of fuel varies as the square of its speed (in miles per hour), and the cost is Rs. 24 per hour when the speed is 12 m.p.h. If other expenses total Rs. 96 per hour, find the most economical speed and the cost for a journey of 100 miles.

***46.** Assuming Fermat's law, that a ray of light in passing from a point A to a point B in the same medium after meeting a reflecting surface takes the path for which the time is a minimum, prove the law of reflexion.

***47.** Assuming the law of refraction, if a ray of light passes through a prism in a plane perpendicular to its edge, prove that the deviation in its direction is minimum when the angle of incidence is equal to the angle of emergence.

ANSWERS

1. (i) $x=1$ (max.), $x=5$ (min.). (ii) $x=\frac{1}{2}$ (max.), $x=2$ (min.).
(iii) $x=4$ (max.), $x=16$ (min). (iv) $x=1$ (max.), $x=-1$ (min.).
(v) $x=1$ (min.), $x=2$ (max.), $x=3$ (min.).

2. Max. value $=1$, min. value $=25$ for (iii). Max. value $=3$, min. value $=\frac{1}{3}$ for (iv). Max. value $=-3$, and min. value $=-4$ in both cases for (v).

7. (i) $x=(2n+\frac{1}{2})\pi$ (max.) ; $x=(2n-\frac{1}{2})\pi$ (min.).

(ii) $x=2n\pi$ (max.), $x=(2n+1)\pi$ (min.).

(iii) Neither max. nor min. (iv) $x=0$ gives minimum.

(v) $x=0$ (max.), $x=1$ (min).

(vi) $x=2n\pi+\frac{3}{4}\pi$ (max.), $x=2n\pi-\frac{1}{4}\pi$ (min.).

8. (i) Neither max. nor min. (ii) Max. for $x=0$.

14. Min. for $x=0$, max. for $x=\frac{8}{5}$. **15.** $a=1, b=0$. **17.** (i) $\frac{3}{2}, \frac{36}{13}$.

(ii) Max. value $=-24$; min. value $=24$. **20.** $(\pm 2, 2)$.

22. 2 units ; $(-\frac{6}{5}, \frac{2}{5})$. **24.** $2\frac{6}{11}$ cu. ft. **25.** (i) 2, 16.

(ii) 5, 10. Area of the triangle in (i) $=16$ sq. units ; length of the line in (ii) $=5\sqrt{5}$ units. **26.** $a-b$. **27.** $(u^2 \sin^2 a)/2g$; $(u \sin a)/g$.

28. (K/C) units. **30.** 30 ft. **31.** (i) $\frac{4}{3}a$. (ii) $\frac{1}{3}h$.

35. Height of the rectangle = radius of the semi-circle.

36. At the end of 4 secs. **37.** 80 ft. $\times$ 60 ft. **38.** 600 sq. ft.

39. $(u^4-g^2x^2)/2u^2g$. **41.** 100 ft. from the stronger light.

43. $\frac{1}{2}$ mile from the point nearest to him ; $1\frac{3}{4}$ hours.

45. 24 m.p.h. ; Rs. 800.

CHAPTER VIII

INDETERMINATE FORMS

(*Evaluation of certain limits*)

8·1. The limit of $\phi(x)/\psi(x)$ as $x \to a$ is in general equal to the quotient of the limiting values of the numerator and denominator [See *Rule (iii) of Art. 2·7*], but when these two limits are both zero, that rule is no longer applicable, since the quotient takes the form 0/0 which is meaningless. We shall consider in the present chapter how to obtain the limiting values of the quotient in such cases, and also the limiting values in other cases of meaningless forms, apparently arising out of the indiscriminate use of the rules of Art. 2·7. The name 'indeterminate forms' as applied to these cases, is rather misleading and vague.

8·2. Form $\frac{0}{0}$ (*L' Hospital's theorem*).

If $\phi(x)$, $\psi(x)$, *as also their derivatives* $\phi'(x)$, $\psi'(x)$ *are continuous at* $x=a$, *and if* $\phi(a)=\psi(a)=0$ [*i.e.*, $\underset{x\to a}{Lt}\ \phi(x) = \underset{x\to a}{Lt}\ \psi(x)=0$], *then*

$$\underset{x\to a}{\mathrm{Lt}}\ \frac{\phi(x)}{\psi(x)} = \underset{x\to a}{\mathrm{Lt}}\ \frac{\phi'(x)}{\psi'(x)} = \frac{\phi'(a)}{\psi'(a)},$$

provided $\psi'(a) \neq 0$.

Since, $\phi(a)=0$ and $\psi(a)=0$, we have

$$\phi(x)=\phi(x)-\phi(a), \text{ and } \psi(x)=\psi(x)-\psi(a).$$

Now, by the Mean Value Theorem,

$$\phi(x)-\phi(a)=(x-a)\phi'\{a+\theta_1(x-a)\},\ 0<\theta_1<1$$
$$\psi(x)-\psi(a)=(x-a)\psi'\{a+\theta_2(x-a)\},\ 0<\theta_2<1.$$

$$\therefore \frac{\phi(x)}{\psi(x)} = \frac{\phi(x)-\phi(a)}{\psi(x)-\psi(a)} = \frac{\phi'\{a+\theta_1(x-a)\}}{\psi'\{a+\theta_2(x-a)\}}.$$

$$\therefore \underset{x\to a}{Lt} \frac{\phi(x)}{\psi(x)} = \frac{\phi'(a)}{\psi'(a)} = \underset{x\to a}{Lt} \frac{\phi'(x)}{\psi'(x)},$$

provided $\psi'(a) \neq 0$.

Generalization :

In case $\phi'(a)$ and $\psi'(a)$ are both zero, applying the above theorem again, we get

$$\underset{x\to a}{Lt} \frac{\phi(x)}{\psi'(x)} = \underset{x\to a}{Lt} \frac{\phi''(x)}{\psi''(x)} = \frac{\phi''(a)}{\psi''(a)},$$

provided $\phi''(x)$ and $\psi''(x)$ are continuous at $x=a$, and $\psi''(a) \neq 0$. If however $\phi''(a)=\psi''(a)=0$, then we again apply the above theorem, and obtain the limiting value as $\phi'''(a)/\psi'''(a)$, and so on.

[For *illustration*, see *Ex. 1, Art. 8 8.*]

Note 1. The above result can also be established by *Cauchy's Mean Value Theorem*. [See *Ex. 7(a), Art. 6·7.*]

Note 2. In the theorem of this article if x tends to ∞ instead of a, then the substitution $1/t$ for x would reduce it to the above form when t tends to zero.

8·3. Form $\frac{\infty}{\infty}$.

If $\underset{x\to a}{Lt}\, \phi(x) = \infty$ *and* $\underset{x\to a}{Lt}\, \psi(x) = \infty$, *then if* $\underset{x\to a}{Lt} \frac{\phi'(x)}{\psi'(x)}$ *exists, then* $\underset{x\to a}{Lt} \frac{\phi(x)}{\psi(x)}$ *will also exist, and its value is equal to the former limit.*

Let $\underset{x\to a}{Lt} \frac{\phi'(x)}{\psi'(x)} = l$. Then we can determine a positive number δ, such that in the interval $a-\delta < x < a+\delta$

$[x \neq a]$, $\dfrac{\phi'(x)}{\psi'(x)}$ is as near to l as we please. Also, since the limit exists, it follows that for x sufficiently close to a [but $\neq a$], $\phi'(x)$ and $\psi'(x)$ must both exist, and $\psi'(x) \neq 0$ there.

Now first consider the interval $a < x \leqslant a + \delta$, and let x_0 be any particular value therein, and take another value x, such that $a < x < x_0$.

Then by Cauchy's Mean Value Theorem [See § *6·7*, *Ex.* 7(*a*)],

$$\frac{\phi(x_0) - \phi(x)}{\psi(x_0) - \psi(x)} = \frac{\phi'(\xi)}{\psi'(\xi)};$$

where $x < \xi < x_0$ and so $a < \xi < a + \delta$.

Hence, $$\frac{\phi(x)\left\{\dfrac{\phi(x_0)}{\phi(x)} - 1\right\}}{\psi(x)\left\{\dfrac{\psi(x_0)}{\psi(x)} - 1\right\}} = \frac{\phi'(\xi)}{\psi'(\xi)},$$

or, $$\frac{\phi(x)}{\psi(x)} = \frac{\dfrac{\psi(x_0)}{\psi(x)} - 1}{\dfrac{\phi(x_0)}{\phi(x)} - 1} \cdot \frac{\phi'(\xi)}{\psi'(\xi)}. \qquad \cdots \quad \text{(i)}$$

Now keeping x_0 fixed, if we make $x \to a$, $\psi(x) \to \infty$ and $\phi(x) \to \infty$, and so $\left\{\dfrac{\psi(x_0)}{\psi(x)} - 1\right\} \Big/ \left\{\dfrac{\phi(x_0)}{\phi(x)} - 1\right\} \to \dfrac{0-1}{0-1}$ *i.e.*, $\to 1$.

Also $\dfrac{\phi'(\xi)}{\psi'(\xi)}$ is as near to l as we like, by a proper choice of δ.

Hence, from (i), $\dfrac{\phi(x)}{\psi(x)}$ is arbitrarily close to l, as $x \to a + 0$.

Thus $$\underset{x \to a+0}{Lt} \frac{\phi(x)}{\psi(x)} = l.$$

Similarly, considering the interval $a-\delta < x < a$, and proceeding exactly as before, we get $\underset{x\to a-0}{Lt} \dfrac{\phi(x)}{\psi(x)} = l$.

Hence, $\underset{x\to a}{Lt} \dfrac{\phi(x)}{\psi(x)} = \underset{x\to a}{Lt} \dfrac{\phi'(x)}{\psi'(x)}$.

We can also prove a modified form of the above theorem as follows :

If $\underset{x\to a}{Lt} \phi(x)$ *and* $\underset{x\to a}{Lt} \psi(x)$ *are both infinite, then*

$$\underset{x\to a}{\mathbf{Lt}} \frac{\phi(x)}{\psi(x)} \textbf{ (when it exists)} = \underset{x\to a}{\mathbf{Lt}} \frac{\phi'(x)}{\psi'(x)}.$$

$$\underset{x\to a}{Lt} \frac{\phi(x)}{\psi(x)} = \underset{x\to a}{Lt} \frac{1/\psi(x)}{1/\phi(x)} = \underset{x\to a}{Lt} \frac{f(x)}{g(x)} \text{ say,}$$

[where $f(x) = 1/\psi(x)$ and $g(x) = 1/\phi(x)$]

which being of the form 0/0, [See *Art. 8·2*, above]

$$= \underset{x\to a}{Lt} \frac{f'(x)}{g'(x)} = \underset{x\to a}{Lt} \frac{-\psi'(x)/\{\psi(x)\}^2}{-\phi'(x)/\{\phi(x)\}^2} = \underset{x\to a}{Lt} \left[\frac{\psi'(x)}{\phi'(x)}\left\{\frac{\phi(x)}{\psi(x)}\right\}^2\right].$$

$$\therefore\ \underset{x\to a}{Lt} \frac{\phi(x)}{\psi(x)} = \underset{x\to a}{Lt} \frac{\psi'(x)}{\phi'(x)}\cdot\left\{\underset{x\to a}{Lt} \frac{\phi(x)}{\psi(x)}\right\}^2. \qquad \cdots \quad (1)$$

Now, let $\underset{x\to a}{Lt} \phi(x)/\psi(x) = l$. $\qquad\cdots\qquad\cdots\quad (2)$

Three cases arise :

CASE I. l is neither zero, nor infinitely large.

Dividing both sides of (1) by l^2, we obtain

$$\frac{1}{l} = \underset{x\to a}{Lt} \frac{\psi'(x)}{\phi'(x)}; \quad \therefore\ l \text{ i.e., } \underset{x\to a}{Lt} \frac{\phi(x)}{\psi(x)} = \underset{x\to a}{Lt} \frac{\phi'(x)}{\psi'(x)}.$$

CASE II. $l=0$.

Adding 1 to each side of (2),

$$l+1=\underset{x\to a}{Lt}\ \frac{\phi(x)+\psi(x)}{\psi(x)}=\underset{x\to a}{Lt}\ \frac{\phi'(x)+\psi'(x)}{\psi'(x)}$$

[*by case (I)*]

$$=\underset{x\to a}{Lt}\ \frac{\phi'(x)}{\psi'(x)}+1.$$

$$\therefore\quad l,\ i.e.,\ \underset{x\to a}{Lt}\ \frac{\phi(x)}{\psi(x)}=\underset{x\to a}{Lt}\ \frac{\phi'(x)}{\psi'(x)}.$$

CASE III. When l is infinitely large,

$$\underset{x\to a}{Lt}\ \frac{1}{\{\phi(x)/\psi(x)\}}=\underset{x\to a}{Lt}\ \frac{\psi(x)}{\phi(x)}=\underset{x\to a}{Lt}\ \frac{\psi'(x)}{\phi'(x)}.$$

[*by case (II)*]

$$\therefore\qquad \underset{x\to a}{Lt}\ \frac{\phi(x)}{\psi(x)}=\underset{x\to a}{Lt}\ \frac{\phi'(x)}{\psi'(x)}.$$

Hence the theorem is proved in all cases.

For *illustration* see *Ex. 3, Art. 8·8.*

Note 1. The theorem is evidently true also when one or both the limits tends to $-\infty$.

Note 2. By substituting $x=1/t$, it can be shown that theorem is also true when x tends to ∞ instead of a.

8·4. Form $0\times\infty$.

Such forms arise when we want to find the limiting value of $\phi(x).\psi(x)$ as $x\to a$, where $\phi(x)\to 0$ and $\psi(x)\to\infty$ as $x\to a$.

We can write

$$\phi(x).\psi(x)=\frac{\phi(x)}{1/\psi(x)}\ \text{ or, }\ \frac{\psi(x)}{1/\phi(x)}$$

which being of the form $0/0$ and ∞/∞, as $x\to a$, can be evaluated by the methods of Arts. 8·2 and 8·3.

[See *Illustrative Example 4, Art. 8·8.*]

8·5. Form $\infty - \infty$.

Such forms arise when we want to find the limiting value of $\phi(x) - \psi(x)$ as $x \to a$, where $\phi(x) \to \infty$ and $\psi(x) \to \infty$ as $x \to a$.

We can write

$$\phi(x) - \psi(x) = \frac{1/\psi(x) - 1/\phi(x)}{1/\phi(x)\psi(x)}$$

which being of the form 0/0 can be evaluated by the method of Art. 8·2.

[See *Illustrative Example 5, Art. 8·8.*]

8·6. Forms 0^0, ∞^0, $1^{\pm\infty}$.

These forms occur when we want to evaluate the limits of functions of the form $\{\phi(x)\}^{\psi(x)}$ as $x \to a$,

when (i) both $\phi(x)$ and $\psi(x) \to 0$ as $x \to a$.

(ii) $\phi(x) \to \infty$ and $\psi(x) \to 0$ as $x \to a$.

(iii) $\phi(x) \to 1$ and $\psi(x) \to \pm \infty$ as $x \to a$.

If $\phi(x) > 0$, let $y = \{\phi(x)\}^{\psi(x)}$. $\therefore$ $\log y = \psi(x) \log \phi(x)$.

$\therefore$ *Lt* $\log y$ reduces to the form discussed in Art. 8·4, and hence can be evaluated.

Since, *Lt* $\log y = \log$ *Lt* y, the required limit *Lt* y can be obtained.

[See *Illustrative Example 6, Art. 8·8.*]

8·7. Use of power series.

In evaluating limits of certain expressions, it is sometimes found convenient to use the expansions of known functions in the expressions in power series in a finite form, and then to take the limit.

[See *Illustrative Example 7, Art. 8·8.*]

8·8. Illustrative Examples.

Ex. 1. *If $\phi(a), \phi'(a), \phi''(a), \ldots\ldots \phi^{n-1}(a)$ and $\psi(a), \psi'(a), \psi''(a), \ldots$ $\ldots\psi^{n-1}(a)$ are all zero, and $\psi^n(a) \neq 0$, then*

$$\underset{x\to a}{Lt}\ \frac{\phi(x)}{\psi(x)} = \underset{x\to a}{Lt}\ \frac{\phi^n(x)}{\psi^n(x)}.$$

Put $x = a+h$, so that when $x \to a$, $h \to 0$.

Now, by Taylor's theorem,

$$\phi(a+h) = \phi(a) + h\phi'(a) + \frac{h^2}{2!}\phi''(a) + \cdots\cdots$$

$$+ \frac{h^{n-1}}{(n-1)!}\phi^{n-1}(a) + \frac{h^n}{n!}\phi^n(a+\theta_1 h)$$

$$= \frac{h^n}{n!}\phi^n(a+\theta_1 h), \text{ where } 0 < \theta_1 < 1.$$

Similarly, $\psi(a+h) = \dfrac{h^n}{n!}\psi^n(a+\theta_2 h)$ where $0 < \theta_2 < 1$.

$$\therefore \quad \underset{x\to a}{Lt}\ \frac{\phi(x)}{\psi(x)} = \underset{h\to 0}{Lt}\ \frac{\phi(a+h)}{\psi(a+h)} = \underset{h\to 0}{Lt}\ \frac{h^n\phi^n(a+\theta_1 h)}{h^n\psi^n(a+\theta_2 h)}$$

$$= \underset{x\to a}{Lt}\ \frac{\phi^n(x)}{\psi^n(x)} = \frac{\phi^n(a)}{\psi^n(a)},$$

provided $\phi^n(x)$ and $\psi^n(x)$ are continuous at $x = a$.

Ex. 2. *Evaluate* $\underset{x\to 0}{Lt}\ \dfrac{e^x - e^{-x} - 2x}{x - \sin x}$.

The reqd. limit as it stands, being of the form 0/0, [See § *8·2*]

$$= \underset{x\to 0}{Lt}\ \frac{e^x + e^{-x} - 2}{1 - \cos x} \qquad \left[\text{form } \frac{0}{0}\right]$$

$$= \underset{x\to 0}{Lt}\ \frac{e^x - e^{-x}}{\sin x} \qquad \left[\text{form } \frac{0}{0}\right]$$

$$= \underset{x\to 0}{Lt}\ \frac{e^x + e^{-x}}{\cos x} = 2,$$

since, $\underset{x\to 0}{Lt}\ (e^x + e^{-x}) = 1 + 1 = 2$, and $\underset{x\to 0}{Lt}\ \cos x = 1$.

Ex. 3. *Evaluate* $\underset{x\to\infty}{Lt}\ \dfrac{x^4}{e^x}$.

The given limit, as it stands, being of the form $\dfrac{\infty}{\infty}$, can be written [*by* § *8·3*] as

$$=\underset{x\to\infty}{Lt}\ \frac{4x^3}{e^x}\ \left(\text{form}\ \frac{\infty}{\infty}\right)=\underset{x\to\infty}{Lt}\ \frac{12x^2}{e^x}\left(\text{form}\ \frac{\infty}{\infty}\right)$$

$$=\underset{x\to\infty}{Lt}\ \frac{24x}{e^x}\ \left(\text{form}\ \frac{\infty}{\infty}\right)=\underset{x\to\infty}{Lt}\ \frac{24}{e^x}=0.$$

Ex. 4. *Evaluate* $\underset{x\to\frac{1}{2}\pi}{Lt}\ (1-\sin x)\tan x$.

The given limit, as it stands, being of the form $0\times\infty$, can be written as

$$=\underset{x\to\frac{1}{2}\pi}{Lt}\ \frac{1-\sin x}{\cot x}\quad\left[\text{form}\ \frac{0}{0}\right]$$

$$=\underset{x\to\frac{1}{2}\pi}{Lt}\ \frac{-\cos x}{-\operatorname{cosec}^2 x}=0,$$

since, $\cos x=0$ and $\operatorname{cosec} x=1$ as $x\to\frac{1}{2}\pi$.

Ex. 5. *Evaluate* $\underset{x\to1}{Lt}\ \left\{\dfrac{1}{x^2-1}-\dfrac{2}{x^4-1}\right\}$.

The given limit, as it stands, being of the form $\infty-\infty$, can be written as

$$=\underset{x\to1}{Lt}\ \frac{x^2-1}{x^4-1}\qquad\left[\text{form}\ \frac{0}{0}\right]$$

$$=\underset{x\to1}{Lt}\ \frac{1}{x^2+1}=\frac{1}{2}.$$

Ex. 6. *Evaluate* $\underset{x\to0}{Lt}\ (\cos x)^{\cot^2 x}$. [*Patna, 1933*

The given limit as it stands is of the form 1^∞.

Let $\quad y=(\cos x)^{\cot^2 x}$.

$\therefore\quad \log y=\cot^2 x\log\cos x=\dfrac{\log\cos x}{\tan^2 x}$.

$$\text{Now} \quad \underset{x\to0}{Lt} \log y = \underset{x\to0}{Lt} \frac{\log \cos x}{\tan^2 x} \qquad \left[\text{form } \frac{0}{0}\right]$$

$$= \underset{x\to0}{Lt} \frac{-(\sin x/\cos x)}{2 \tan x \sec^2 x} = \underset{x\to0}{Lt} \left(-\tfrac{1}{2}\cos^2 x\right)$$

$$= -\tfrac{1}{2} \left(\because \underset{x\to0}{Lt} \cos^2 x = 1\right).$$

$$\text{Since} \quad \underset{x\to0}{Lt} \log y = \log \underset{x\to0}{Lt} y, \quad \therefore \log \underset{x\to0}{Lt} y = -\tfrac{1}{2}.$$

$$\therefore \quad \underset{x\to0}{Lt} y = e^{-\frac{1}{2}} \;\therefore\; \text{the reqd. limit} = e^{-\frac{1}{2}}.$$

Ex. 7. *Show that* $\underset{x\to0}{Lt} \dfrac{x - \sin x}{x^3} = \dfrac{1}{6}$. [*C. P. 1932*]

Writing down the expansion of $\sin x$ in a finite power series, we have

$$x - \sin x = x - \left\{x - \frac{x^3}{3!} + \frac{x^5}{5!} \sin\left(\frac{5\pi}{2} + \theta x\right)\right\}, \quad 0 < \theta < 1.$$

$$= \frac{x^3}{3!} - \frac{x^5}{5!} \sin\left(\frac{5\pi}{2} + \theta x\right)$$

$$= x^3 \left\{\frac{1}{3!} - \frac{x^2}{5!} \sin\left(\frac{5\pi}{2} + \theta x\right)\right\}.$$

$$\therefore \frac{x - \sin x}{x^3} = \frac{1}{3!} - \frac{x^2}{5!} \sin\left(\frac{5\pi}{2} + \theta x\right).$$

$$\therefore \underset{x\to0}{Lt} \frac{x - \sin x}{x^3} = \underset{x\to0}{Lt} \left\{\frac{1}{3!} - \frac{x^2}{5!} \sin\left(\frac{5\pi}{2} + \theta x\right)\right\} = \frac{1}{3!} = \frac{1}{6};$$

since, $\dfrac{x^2}{5!} \sin\left(\dfrac{5\pi}{2} + \theta x\right) \to 0$, as $x \to 0$, $\left|\sin\left(\dfrac{5\pi}{2} + \theta x\right)\right|$ being $\leqslant 1$.

Note. This being of the form 0/0 can also be obtained by the method of Art. 8·2.

Ex. 8. *Evaluate* $\underset{x\to0}{Lt} \dfrac{\sqrt{(a^2 + ax + x^2)} - \sqrt{(a^2 - ax + x^2)}}{\sqrt{(a + x)} - \sqrt{(a - x)}}$.

Rationalizing both the numerator and denominator, the required limit

$$= \underset{x\to0}{Lt} \frac{2ax\{\sqrt{(a+x)} + \sqrt{(a-x)}\}}{2x\{\sqrt{(a^2+ax+x^2)} + \sqrt{(a^2-ax+x^2)}\}}$$

$$= \underset{x\to0}{Lt} \frac{a\{\sqrt{(a+x)} + \sqrt{(a-x)}\}}{\{\sqrt{(a^2+ax+x^2)} + \sqrt{(a^2-ax+x^2)}\}}.$$

Now, the limit of the numerator $= a \cdot 2\sqrt{a}$ and that of the denominator $= 2a$.

$\therefore$ the reqd. limit $= \sqrt{a}$.

Note. An algebraical or trigonometrical transformation often enables us to obtain the limiting values without using calculus, as shown above, which case belongs to the form 0/0.

Ex. 9. *If* $\underset{x \to 0}{Lt} \dfrac{\sin 2x + a \sin x}{x^3}$ *be finite, find the value of 'a' and the limit.* [*C. P. 1931*]

The given limit being of the form 0/0

$$= \underset{x \to 0}{Lt} \frac{2 \cos 2x + a \cos x}{3x^2}. \qquad (\textit{by } \S\ 8{\cdot}2)$$

When $x \to 0$, the denominator $3x^2 = 0$; hence, in order that the limiting value of the expression may be finite, the numerator $(2 \cos 2x + a \cos x)$ should be zero as $x \to 0$. $\therefore\ 2 + a = 0$, *i.e.,* $a = -2$;

when $a = -2$, the given limit becomes

$$= \underset{x \to 0}{Lt} \frac{\sin 2x - 2 \sin x}{x^3} \qquad \left[\text{form } \frac{0}{0} \right]$$

$$= \underset{x \to 0}{Lt} \frac{2 \cos 2x - 2 \cos x}{3x^2} \qquad \left[\text{form } \frac{0}{0} \right]$$

$$= \underset{x \to 0}{Lt} \frac{-4 \sin 2x + 2 \sin x}{6x} \qquad \left[\text{form } \frac{0}{0} \right]$$

$$= \underset{x \to 0}{Lt} \frac{-8 \cos 2x + 2 \cos x}{6} = -\frac{6}{6}$$

$$= -1.$$

Ex 10. *Evaluate* $\underset{x \to 0}{Lt} \left(\dfrac{\tan x}{x} \right)^{\frac{1}{x}}$. [*C. P. 1947*]

Let $\quad u = \left(\dfrac{\tan x}{x} \right)^{\frac{1}{x}}$.

$\therefore \quad \log u = \dfrac{1}{x} \log \left(\dfrac{\tan x}{x} \right) = \log \left(\dfrac{\tan x}{x} \right) \Big/ x.$

Since $\underset{x \to 0}{Lt} \dfrac{\tan x}{x} = 1$, $\underset{x \to 0}{Lt} \log u$ is of the form $\dfrac{0}{0}$.

$$\therefore \quad \underset{x\to 0}{Lt}\,(\log u) = \underset{x\to 0}{Lt}\,\log\left(\frac{\tan x}{x}\right)\Big/ x$$

$$= \underset{x\to 0}{Lt}\left\{\frac{x}{\tan x}\cdot\frac{x\sec^2 x - \tan x}{x^2}\right\}\Big/ 1 \qquad (\textit{by § 8·2})$$

$$= \underset{x\to 0}{Lt}\,\frac{2x - \sin 2x}{x\sin 2x} = \underset{x\to 0}{Lt}\,\frac{2 - 2\cos 2x}{\sin 2x + 2x\cos 2x}$$

$$= \underset{x\to 0}{Lt}\,\frac{4\sin 2x}{4\cos 2x - 4x\sin 2x} = 0.$$

Since $\underset{x\to 0}{Lt}(\log u) = \log(\underset{x\to 0}{Lt}\, u)$, $\quad\therefore\quad \log(\underset{x\to 0}{Lt}\, u) = 0.$

$\therefore \quad \underset{x\to 0}{Lt}\, u = e^0 = 1$, *i.e.*, the reqd. limit $=1$.

Otherwise: Writing the finite form of the expansion of $\tan x$ by Maclaurin's theorem,

$$\tan x = x + \tfrac{1}{3}x^3 a \text{ where } a = \sec\theta x\,(1 + 2\tan^2\theta x),\ 0 < \theta < 1.$$

$$\therefore \quad \log u = \frac{1}{x}\log\frac{\tan x}{x} = \frac{1}{x}\log\frac{x + \frac{1}{3}x^3 a}{x} = \frac{1}{x}\log\left(1 + \tfrac{1}{3}x^2 a\right)$$

$$= \frac{1}{\frac{1}{3}x^2 a}\log\left(1 + \tfrac{1}{3}x^2 a\right)\tfrac{1}{3}xa = \frac{1}{v}\log(1+v)\cdot\tfrac{1}{3}xa,$$

where $v = \frac{1}{3}x^2 a$.

When $x\to 0$, $v\to 0$, also $\underset{v\to 0}{Lt}\,\frac{1}{v}\log(1+v) = 1$. [*Art. 2·9*]

Hence, $\underset{x\to 0}{Lt}\left(\log u\right) = \underset{v\to 0}{Lt}\,\frac{1}{v}\left(1+v\right)\cdot \underset{x\to 0}{Lt}\,(\tfrac{1}{3}xa)$.

$\therefore \quad \underset{x\to 0}{Lt}\,(\log u) = 0.$ Hence, etc.

Ex. 11. *Evaluate* $\underset{x\to 0}{Lt}\,\frac{(e^x - 1)\tan^2 x}{x^3}$.

$$\text{Given limit} = \underset{x\to 0}{Lt}\left[\frac{e^x - 1}{x}\cdot\left(\frac{\tan x}{x}\right)^2\right]$$

$$= \underset{x\to 0}{Lt}\,\frac{e^x - 1}{x}\times\left\{\underset{x\to 0}{Lt}\,\frac{\tan x}{x}\right\}^2.$$

Now, $\quad \underset{x\to 0}{Lt}\,\frac{e^x - 1}{x}\left(\text{being of the form } \frac{0}{0}\right) = \underset{x\to 0}{Lt}\,\frac{e^x}{1} = e^0 = 1.$

Also, $\quad \underset{x\to 0}{Lt}\,\frac{\tan x}{x} = 1. \quad \therefore \quad$ the reqd. limit $= 1\times 1^2 = 1.$

Note. Such forms are sometimes called *Compound Indeterminate forms.* In evaluating limits of such forms, the use of the theorems on limit (*Art. 2·8*) is of great help.

Examples VIII

Evaluate the following limits :

1. (i) $\underset{x\to 0}{Lt}\dfrac{x-\sin x\cos x}{x^3}$. (ii) $\underset{x\to 0}{Lt}\dfrac{\tan x-x}{x-\sin x}$.

(iii) $\underset{x\to 0}{Lt}\dfrac{e^{2x}-1}{\log(1+x)}$. (iv) $\underset{x\to a}{Lt}\dfrac{x^n-a^n}{x-a}$.

(v) $\underset{x\to 2}{Lt}\dfrac{x^3-2x^2+2x-4}{x^2-5x+6}$.

(vi) $\underset{x\to 0}{Lt}\dfrac{e^x+e^{-x}-2\cos x}{x\sin x}$. (vii) $\underset{x\to 0}{Lt}\dfrac{a^x-b^x}{x}$.

(viii) $\underset{x\to 0}{Lt}\dfrac{e^x+\sin x-1}{\log(1+x)}$. (ix) $\underset{x\to 0}{Lt}\dfrac{x-\sin^{-1}x}{\sin^3 x}$.

(x) $\underset{x\to 0}{Lt}\dfrac{e^x-e^{\sin x}}{x-\sin x}$. (xi) $\underset{x\to 0}{Lt}\dfrac{\tan nx-n\tan x}{n\sin x-\sin nx}$.

(xii) $\underset{x\to 2}{Lt}\dfrac{\sqrt{2x}-\sqrt{6-x}}{3x-2\sqrt{19-5x}}$. (xiii) $\underset{x\to 0}{Lt}\dfrac{2\sin x-\sin 2x}{\tan^3 x}$.

(xiv) $\underset{x\to 0}{Lt}\dfrac{\sin\log(1+x)}{\log(1+\sin x)}$.

(xv) $\underset{x\to 0}{Lt}\left[\left(\dfrac{\sin 2x+2\sin^2 x-2\sin x}{\cos x-\cos^2 x}\right)^2+\dfrac{1-\cos x}{\cos x\sin^2 x}\right]$.

(xvi) $\underset{x\to \frac{1}{2}\pi}{Lt}\left[\dfrac{(\frac{1}{2}\pi-x)\log\sin x}{e^{\cos x}-1+\log(1+x-\frac{1}{2}\pi)}+\dfrac{\cos x}{x-\frac{1}{2}\pi}\right]$.

2. (i) $\underset{x\to \frac{1}{2}\pi}{Lt}\dfrac{\tan 5x}{\tan x}$. (ii) $\underset{x\to 0}{Lt}\dfrac{\log(x^2)}{\log(\cot^2 x)}$.

(iii) $\underset{x\to\infty}{Lt}\dfrac{x^n}{e^x}$ (*n being positive*).

(iv) $\underset{x\to \frac{1}{2}}{Lt}\dfrac{\tan 3\pi x}{\sec \pi x}$. (v) $\underset{x\to 0+}{Lt}\dfrac{\log x-\cot\frac{1}{2}\pi x}{\cot \pi x}$.

(vi) $\underset{x\to 0}{Lt}\log_{\tan^2 x}(\tan^2 2x)$. (vii) $\underset{x\to\infty}{Lt}\dfrac{x^2+3x}{1-5x^2}$.

3. (i) $\underset{x\to 0}{Lt}\ x^2 \log (x^2)$. (ii) $\underset{x\to 1}{Lt}\ \operatorname{cosec} (\pi x) \log x$.

(iii) $\underset{x\to 0}{Lt}\ x \log \sin^2 x$. (iv) $\underset{x\to \frac{1}{2}\pi}{Lt}\ \sec x\,(x \sin x - \frac{1}{2}\pi)$.

(v) $\underset{x\to 0}{Lt}\ \sin x \,.\, \log x^2$. (vi) $\underset{x\to \frac{1}{2}\pi}{Lt}\ \sec 5x \cos 7x$.

(vii) $\underset{x\to 0+}{Lt}\ x^m (\log x)^n$, *m and n being positive.*

4. (i) $\underset{x\to \frac{1}{2}\pi}{Lt}\ (\sec x - \tan x)$. (ii) $\underset{x\to 0}{Lt}\ (x^{-1} - \cot x)$.

(iii) $\underset{x\to 0}{Lt}\left[\dfrac{1}{x^2} - \dfrac{1}{\sin^2 x}\right]$. (iv) $\underset{x\to 1}{Lt}\left[\dfrac{x}{x-1} - \dfrac{1}{\log x}\right]$.

(v) $\underset{x\to 2}{Lt}\left[\dfrac{4}{x^2-4} - \dfrac{1}{x-2}\right]$.

(vi) $\underset{x\to 0}{Lt}\left[\dfrac{1}{x} - \dfrac{1}{x^2}\log (1+x)\right]$.

(vii) $\underset{x\to \infty}{Lt}\ (x - \sqrt{x^2-9})$. (viii) $\underset{x\to \infty}{Lt}\ (\sqrt{x^2+2x} - x)$.

5. (i) $\underset{x\to 0}{Lt}\ x^{2x}$. (ii) $\underset{x\to 0}{Lt}\ x^{2 \sin x}$.

(iii) $\underset{x\to 0}{Lt}\ (\sin x)^{2 \tan x}$. (iv) $\underset{x\to 0}{Lt}\ (\cos x)^{\cot^2 x}$.

(v) $\underset{x\to \frac{1}{2}\pi}{Lt}\ (\sin x)^{\tan x}$. (vi) $\underset{x\to 1}{Lt}\ x^{\frac{1}{1-x}}$.

(vii) $\underset{x\to 0}{Lt}\ (\cot^2 x)^{\sin x}$. (viii) $\underset{x\to 0}{Lt}\ (1/x^2)^{\tan x}$.

(ix) $\underset{x\to 1-}{Lt}\ (1-x^2)^{1/\log (1-x)}$. (x) $\underset{x\to e}{Lt}\ (\log x)^{1/(1-\log x)}$.

(xi) $\underset{x\to \infty}{Lt}\left(1+\dfrac{1}{x^2}\right)^x$. (xii) $\underset{x\to 0}{Lt}\left(\dfrac{\sin x}{x}\right)^{\frac{1}{x}}$.

(xiii) $\underset{x\to 0}{Lt}\left(\dfrac{\tan x}{x}\right)^{1/x^2}$. (xiv) $\underset{x\to 0}{Lt}\left(\dfrac{\sin x}{x}\right)^{1/x^2}$.

6. $\underset{x\to\infty}{Lt} \dfrac{a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \cdots + a_n}{b_0x^m + b_1x^{m-1} + b_2x^{m-2} + \cdots + b_m}$

($a_0 \neq 0, b_0 \neq 0$), according as $n >=$ or $< m$ (n and m being positive integers).

7. $\underset{x\to\infty}{Lt} (a_0x^m + a_1x^{m-1} + \cdots + a_m)^{1/x}$, m being a positive integer ($a_0 \neq 0$).

8. $\underset{x\to\infty}{Lt}\ 2^x \sin \dfrac{a}{2^x}$ ($a \neq 0$). [*C. P. 1946*]

9. (i) $\underset{x\to\infty}{Lt} \dfrac{x + \cos x}{x+1}$. (ii) $\underset{x\to 1}{Lt} \dfrac{x^{\frac{3}{2}} - 1 + (x-1)^{\frac{3}{2}}}{(x^2-1)^{\frac{3}{2}} - x + 1}$.

10. If $\underset{x\to 0}{Lt} \dfrac{a \sin x - \sin 2x}{\tan^3 x}$ is finite, find the value of a, and the limit.

11. Adjust the constants a and b in order that

$$\underset{\theta\to 0}{Lt} \frac{\theta(1 + a \cos\theta) - b \sin\theta}{\theta^3} = 1.$$

12. Determine the values of a, b, c so that

(i) $\dfrac{ae^x - b \cos x + ce^{-x}}{x \sin x} \to 2$, as $x \to 0$.

(ii) $\dfrac{(a + b \cos x)\, x - c \sin x}{x^5} \to 1$ as $x \to 0$.

(iii) $\dfrac{a \sin x - bx + cx^2 + x^3}{2x^2 \log (1+x) - 2x^3 + x^4}$ may tend to a finite limit as $x \to 0$, and determine this limit.

Evaluate the following [*Ex. 13-19*] :

13. (i) $\underset{x\to 0}{Lt} \dfrac{xe^x - \log(1+x)}{x^2}$.

(ii) $\underset{x\to 0}{Lt} \left\{ \dfrac{1}{x} - \dfrac{1}{x^2} \log (1+x) \right\}$.

(iii) $\underset{x\to 0}{Lt} \dfrac{x \cos x - \log(1+x)}{x^2}$.

(iv) $\underset{x\to 0}{Lt} \dfrac{\tan x \,.\, \tan^{-1}x - x^2}{x^6}$.

14. $\underset{x\to 0}{Lt} \dfrac{e^x - e^{-x} + 2\sin x - 4x}{x^5}$.

15. $\underset{x\to 0}{Lt} \dfrac{\cos x - \log_e(1+x) + \sin x - 1}{e^x - (1+x)}$.

***16.** $\underset{x\to \frac{1}{2}\pi}{Lt} \left[\sqrt{\left\{\dfrac{2+\cos 2x - \sin x}{x \sin 2x + x\cos x}\right\}} - \left(\dfrac{\pi - 2x}{2\sin 2x}\right)^2\right]$.

17. $\underset{x\to a-0}{Lt} \left[\sqrt{(a^2 - x^2)} \,.\, \cot\left\{\dfrac{\pi}{2}\sqrt{\left(\dfrac{a-x}{a+x}\right)}\right\}\right]$.

18. $\underset{x\to 2}{Lt} \dfrac{x^2 - 4}{\sqrt{(x+2)} - \sqrt{(3x-2)}}$.

19. $\underset{x\to a+0}{Lt} \dfrac{\sqrt{x} - \sqrt{a} + \sqrt{(x-a)}}{\sqrt{(x^2 - a^2)}}$.

Show that [*Ex. 20-26*] :

20. $\underset{x\to \infty}{Lt} \; a^x \sin \dfrac{b}{a^x} = 0$ or b

according as $0 < a < 1$, or $a > 1$.

21. $\underset{x\to \infty}{Lt} \left\{x - x^2 \log\left(1 + \dfrac{1}{x}\right)\right\} = \frac{1}{2}$.

***22.** $\underset{x\to 0}{Lt} \dfrac{\log(1+x+x^2) + \log(1-x+x^2)}{\sec x - \cos x} = 1$.

***23.** $\underset{x\to 0}{Lt} \dfrac{e^x - 1}{x^4 \sin x}\left(\dfrac{3\sin x - \sin 3x}{\cos x - \cos 3x}\right)^4 = 1$.

***24.** $\underset{x\to 0}{Lt} \dfrac{\log_{\sin^2 x}(\cos x)}{\log_{\sin^2 \frac{1}{2}x}(\cos \frac{1}{2}x)} = 4$.

25. $\underset{x\to 0}{Lt} \dfrac{(1+x)^{\frac{1}{x}} - e}{x} = -\frac{1}{2}e$.

*26. $\left(\underset{x\to\infty}{Lt} \dfrac{a_1^{1/x} + a_2^{1/x} + \cdots + a_n^{1/x}}{n}\right)^{nx} = a_1 a_2 \ldots a_n.$

*27. Evaluate $\underset{a\to b}{Lt} \dfrac{a^x \sin bx - b^x \sin ax}{\tan bx - \tan ax}.$

*28. If $\phi(x) = x^2 \sin(1/x)$, and $\psi(x) = \tan x$, show that although $\underset{x\to 0}{Lt} \dfrac{\phi'(x)}{\psi'(x)}$ does not exist, $\underset{x\to 0}{Lt} \dfrac{\phi(x)}{\psi(x)}$ exists, and $=0$.

ANSWERS

1. (i) $\frac{2}{3}$. (ii) 2. (iii) 2. (iv) na^{n-1}. (v) -6. (vi) 2. (vii) $\log(a/b)$. (viii) 2. (ix) $-\frac{1}{6}$. (x) 1. (xi) 2. (xii) $\frac{9}{16}$. (xiii) 1. (xiv) 1. (xv) $16\frac{1}{2}$. (xvi) $\frac{1}{2}$.

2. (i) $\frac{1}{5}$. (ii) -1. (iii) 0. (iv) $\frac{1}{3}$. (v) -2. (vi) 1. (vii) $-\frac{1}{5}$.

3. (i) 0. (ii) $(-1/\pi)$. (iii) 0. (iv) -1. (v) 0. (vi) $-\frac{7}{6}$. (vii) 0.

4. (i) 0. (ii) 0. (iii) $-\frac{1}{3}$. (iv) $\frac{1}{2}$. (v) $-\frac{1}{4}$. (vi) $\frac{1}{2}$. (vii) 0. (viii) 1.

5. (i) 1. (ii) 1. (iii) 1. (iv) $e^{-\frac{1}{2}}$. (v) 1. (vi) $1/e$. (vii) 1. (viii) 1. (ix) e. (x) $1/e$. (xi) 1. (xii) 1. (xiii) $e^{\frac{1}{3}}$. (xiv) $e^{-\frac{1}{6}}$.

6. $+\infty$ or, $-\infty$ (corresponding to a_0/b_0 being positive or negative), a_0/b_0, 0 according as $n >$ = or $< m$.

7. 1. 8. a. 9. (i) 1. (ii) $-\frac{3}{2}$. 10. $a=2$; limit$=1$.

11. $a=-\frac{5}{2}$, $b=-\frac{3}{2}$. 12. (i) $a=1, b=2, c=1$. (ii) $a=120, b=60, c=180$. (iii) $a=6, b=6, c=0$; limit$=\frac{3}{40}$.

13. (i) $\frac{3}{2}$. (ii) $\frac{1}{2}$. (iii) $\frac{1}{2}$. (iv) $\frac{2}{9}$. 14. $\frac{1}{30}$. 15. 0.

16. $-\frac{1}{4}$. 17. $4a/\pi$. 18. -8. 19. $1/\sqrt{2a}$.

27. $b^{x-1}(b\cos bx - \sin bx)\cos^2 bx$.

CHAPTER IX

PARTIAL DIFFERENTIATION

(*Function of two or more variables*)

9·1. Definition.

If three variables u, x, y are so related that for every pair of values of x and y within the defined domain say, $a \leqslant x \leqslant b$ and $c \leqslant y \leqslant d$, u has a single definite value, u is said to be *a function of the two independent variables x and y*, and this is denoted by $u=f(x, y)$.

More generally (*i.e.*, without restricting to single-valued functions only), if the three variables u, x, y are so related that u is determined when x and y are known, *u is said to be a function of the two independent variables x and y.*

Illus: Since the area of a triangle is determined when its base and altitude are given, area of a triangle is a function of its base and altitude.

Similarly, the volume of a gas is a function of its pressure and temperature.

In a similar way, a function of three or more independent variables can be defined.

Thus, the volume of a parallelopiped is a function of three variables, its length, breadth and height.

Note 1. If to each pair of values of x and y, u has a single definite value, u is called a *single-valued function* (to which the definition refers, and with which we are mainly concerned in all mathematical investigations), and if to each set of values of x and y, u has more than one definite value, u is called a *multiple-valued function.* A multiple-valued function with proper limitations imposed on its value can in general be treated as defining two or more single-valued functions.

Note 2. *Geometrical representation of $z=f(x, y)$.*

When a single-valued function $z=f(x, y)$ is given, for each pair of values of x and y, there corresponds a point Q in the plane OXY, and if a perpendicular QP is then erected of length equal to the value of z obtained from the given relation, the points like P describe what is called a *surface* in three-dimensional space. Thus to a functional relation between three variables x, y, z, therefore corresponds a surface referred to axes OX, OY, OZ in space.

Note 3. Continuity.

The function $f(x, y)$ is said to be continuous at the point (a, b), if corresponding to a pre-assigned positive number ϵ, however small, there exists a positive number δ, such that

$$|f(x, y)-f(a, b)| < \epsilon,$$

whenever $0 \leqslant |x-a| \leqslant \delta$ and $0 \leqslant |y-b| \leqslant \delta$.

9·2. Partial Derivatives.

The result of differentiating $u=f(x, y)$, with respect to x, treating y as a constant, is called *the partial derivative of u with respect to x*, and is denoted by one of the symbols $\dfrac{\delta u}{\delta x}, \dfrac{\delta f}{\delta x}, f_x(x, y)$ [or briefly, f_x], u_x, etc.

Analytically, $\dfrac{\delta f}{\delta x} = \underset{\Delta x \to 0}{Lt} \dfrac{f(x+\Delta x, y)-f(x, y)}{\Delta x}$

when this limit exists.

The partial derivative of $u=f(x, y)$ with respect to y is similarly defined and is denoted by $\dfrac{\delta u}{\delta y}, \dfrac{\delta f}{\delta y}, f_y(x, y)$ [or simply f_y], u_y, etc.

Thus, $\dfrac{\delta f}{\delta y} = \underset{\Delta y \to 0}{Lt} \dfrac{f(x, y+\Delta y)-f(x, y)}{\Delta y}$,

provided this limit exists.

If $u=f(x, y, z)$, then the partial derivative of u with respect to x is the derivative of u with respect to x when both y and z are regarded as constants.

Thus, $\dfrac{\delta f}{\delta x}=\underset{\Delta x \to 0}{Lt} \dfrac{f(x+\Delta x, y, z)-f(x, y, z)}{\Delta x}$.

Similarly for $\dfrac{\delta f}{\delta y}, \dfrac{\delta f}{\delta z}$.

Illustrations :

Let $u=x^2+xy+y^2$; then

$$\frac{\delta u}{\delta x}=2x+y \; ; \; \frac{\delta u}{\delta y}=2y+x.$$

Let $u=yz+zx+xy$; then

$$\frac{\delta u}{\delta x}=y+z \; ; \; \frac{\delta u}{\delta y}=z+x \; ; \; \frac{\delta u}{\delta z}=x+y.$$

Note. The curl δ is generally used to denote the symbol of partial derivative, in order to distinguish it from the symbol d of ordinary derivative.

9·3. Successive Partial Derivatives.

Since each of the partial derivatives $\dfrac{\delta u}{\delta x}, \dfrac{\delta u}{\delta y}$ is in general a function of x and y, each may possess partial derivatives with respect to these two independent variables, and these are called the second order partial derivatives of u. The usual *notations* for these *second order partial derivatives are*

$$\frac{\delta}{\delta x}\left(\frac{\delta u}{\delta x}\right), \; i.e., \; \frac{\delta^2 u}{\delta x^2} \text{ or } f_{xx} \text{ etc.}$$

$$\frac{\delta}{\delta y}\left(\frac{\delta u}{\delta y}\right), \; i.e., \; \frac{\delta^2 u}{\delta y^2} \text{ or } f_{yy} \text{ etc.}$$

$$\frac{\delta}{\delta x}\left(\frac{\delta u}{\delta y}\right), \; i.e., \; \frac{\delta^2 u}{\delta x\, \delta y} \text{ or } f_{xy}.$$

$$\frac{\delta}{\delta y}\left(\frac{\delta u}{\delta x}\right), \; i.e., \; \frac{\delta^2 y}{\delta y\, \delta x} \text{ or } f_{yx}.$$

Although *for most of the functions that occur in applications we have,*

$$\frac{\delta^2 u}{\delta x\,\delta y}=\frac{\delta^2 u}{\delta y\,\delta x}$$

i.e., the partial derivative has the same value whether we differentiate partially first with respect to x and then with respect to y or the reverse, *it must not be supposed that the above relation holds good for all functions* ; because, the equality implies that the two limiting operations involved therein should be commutative, which may not be true always. Ex. 3, Art. 9·4 will elucidate the point. We can prove in particular that if the functions $\frac{\delta^2 u}{\delta y\,\delta x}$ and $\frac{\delta^2 u}{\delta x\,\delta y}$ both exist for a particular set of values of x, y, and one of them is continuous there, the equality will hold good.*

We have similar definitions and notations for partial derivatives of order higher than two.

If $z=f(x, y)$, partial derivatives of z are very often denoted by the following notations :

$$\frac{\delta z}{\delta x}=p,\ \frac{\delta z}{\delta y}=q,\ \frac{\delta^2 z}{\delta x^2}=r,\ \frac{\delta^2 z}{\delta x\,\delta y}=\frac{\delta^2 z}{\delta y\,\delta x}=s,\ \frac{\delta^2 z}{\delta y^2}=t.\ \dagger$$

Illustrations :

$$u=x^3+x^2y^2+y^3$$

$$\frac{\delta u}{\delta x}=3x^2+2xy^2\ ;\ \frac{\delta^2 u}{\delta x^2}=6x+2y^2\ ;\ \frac{\delta^2 u}{\delta y\,\delta x}=4xy.$$

$$\frac{\delta u}{\delta y}=3y^2+2x^2y\ ;\ \frac{\delta^2 u}{\delta y^2}=6y+2x^2\ ;\ \frac{\delta^2 u}{\delta x\,\delta y}=4xy.$$

* See *Appendix.*

† These notations were first introduced by Monge.

9·4. Illustrative Examples.

Ex. 1. *If $x=r\cos\theta$, $y=r\sin\theta$, so that $r=\sqrt{(x^2+y^2)}$, $\theta=\tan^{-1}(y/x)$, show that*

$$\frac{\delta x}{\delta r}\neq 1\Big/\frac{\delta r}{\delta x} \text{ and } \frac{\delta x}{\delta \theta}\neq 1\Big/\frac{\delta \theta}{\delta x}.$$

Here, $\dfrac{\delta x}{\delta r}=\cos\theta$; $\dfrac{\delta r}{\delta x}=\dfrac{x}{\sqrt{(x^2+y^2)}}=\dfrac{r\cos\theta}{r}=\cos\theta.$

$$\frac{\delta x}{\delta \theta}=-r\sin\theta\ ;\ \frac{\delta\theta}{\delta x}=-\frac{y}{x^2+y^2}=-\frac{r\sin\theta}{r^2}=-\frac{\sin\theta}{r}.$$

Hence, the required result follows.

Note. If y is a function of a *single variable* x, then we have seen that under certain circumstances (See § *4·7*), $\dfrac{dy}{dx}=1\Big/\dfrac{dx}{dy}$. *A similar property is not true*, as seen above, *when y is a function of more than one variable.*

Ex. 2. *If $u=f\left(\dfrac{y}{x}\right)$, show that $x\dfrac{\delta u}{\delta x}+y\dfrac{\delta u}{\delta y}=0.$*

$$u=f(z), \text{ say, where } z=y/x.$$

$$\frac{\delta u}{\delta x}=\frac{\delta u}{\delta z}\cdot\frac{\delta z}{\delta x}=f'(z)\cdot\frac{\delta z}{\delta x}=-\frac{y}{x^2}f'(z).$$

Similarly, $\dfrac{\delta u}{\delta y}=f'(z)\cdot\dfrac{\delta z}{\delta y}=\dfrac{1}{x}f'(z).$

$$\therefore\quad x\frac{\delta u}{\delta x}+y\frac{\delta u}{\delta y}=-\frac{y}{x}f'(z)+\frac{y}{x}f'(z)=0.$$

Ex. 3. *Show that*

$$\underset{x\to 0}{Lt}\ \underset{y\to 0}{Lt}\ \frac{x-y}{x+y}\neq \underset{y\to 0}{Lt}\ \underset{x\to 0}{Lt}\ \frac{x-y}{x+y}.$$

Left side $=\underset{x\to 0}{Lt}\ \dfrac{x}{x}=\underset{x\to 0}{Lt}\ 1=1.$

Right side $=\underset{y\to 0}{Lt}\ \dfrac{-y}{y}=\underset{y\to 0}{Lt}\ (-1)=-1.$

Hence, the result.

Examples IX(A)

1. Find f_x, f_y for the following functions $f(x, y)$;

(i) $ax^2+2hxy+by^2$. (ii) $\tan^{-1}(y/x)$.

(iii) $1/\sqrt{(x^2+y^2)}$. (iv) $\log(x^2+y^2)$.

(v) $x^2/a^2+y^2/b^2-1$.

2. Find $f_{xx}, f_{xy}, f_{yx}, f_{yy}$ for the following functions $f(x, y)$:

(i) $x^3+3x^2y+3xy^2+y^3$. (ii) $e^{x^2+xy+y^2}$.

(iii) $x\cos y+y\cos x$. (iv) $\log(x^2y+xy^2)$.

3. (i) If $V=x^2+y^2+z^2$, show that $xV_x+yV_y+zV_z=2V$.

(ii) If $u=x^2y+y^2z+z^2x$, show that

$$u_x+u_y+u_z=(x+y+z)^2.$$

(iii) If $u=f(xyz)$, show that

$$xu_x=yu_y=zu_z.$$

(iv) If $u=\frac{y}{z}+\frac{z}{x}+\frac{x}{y}$, prove that

$$x\frac{\delta u}{\delta x}+y\frac{\delta u}{\delta y}+z\frac{\delta u}{\delta z}=0.$$

4. (i) If $u=\sin^{-1}\frac{x}{y}+\tan^{-1}\frac{y}{x}$, show that $x\frac{\delta u}{\delta x}+y\frac{\delta u}{\delta y}=0$.

(ii) If $U=\frac{x+y}{1-xy}$, and $V=\frac{x(1-y^2)+y(1-x^2)}{(1+x^2)(1+y^2)}$,

prove that $U_xV_y=U_yV_x$.

5. (a) Show that $\frac{\delta^2u}{\delta x^2}+\frac{\delta^2u}{\delta y^2}=0$, if

(i) $u=\log(x^2+y^2)$. (ii) $u=\tan^{-1}(y/x)$.

(iii) $u=e^x(x\cos y-y\sin y)$.

(b) If $V=z\tan^{-1}\frac{y}{x}$, then $\frac{\delta^2V}{\delta x^2}+\frac{\delta^2V}{\delta y^2}+\frac{\delta^2V}{\delta z^2}=0$.

6. (i) If $f(x, y, z) = \begin{vmatrix} x^2 & y^2 & z^2 \\ x & y & z \\ 1 & 1 & 1 \end{vmatrix}$

show that $f_x + f_y + f_z = 0.$

*(ii) If $u = \begin{vmatrix} 1 & 1 & 1 & 1 \\ x & y & z & w \\ x^2 & y^2 & z^2 & w^2 \\ x^3 & y^3 & z^3 & w^3 \end{vmatrix}$

show that $u_x + u_y + u_z + u_w = 0.$

7. If $V = ax^2 + 2hxy + by^2$, show that

$$V_x^2 V_{yy} - 2V_x V_y V_{xy} + V_y^2 V_{xx} = 8(ab - h^2)V.$$

8. If $u = \log(x^3 + y^3 + z^3 - 3xyz)$, show that

(i) $\dfrac{\delta u}{\delta x} + \dfrac{\delta u}{\delta y} + \dfrac{\delta u}{\delta z} = \dfrac{3}{x + y + z}.$

(ii) $\dfrac{\delta^2 u}{\delta x^2} + \dfrac{\delta^2 u}{\delta y^2} + \dfrac{\delta^2 u}{\delta z^2} = -\dfrac{3}{(x + y + z)^2}.$ [*C. U. 1946*]

9. If $V = \sqrt{(x^2 + y^2 + z^2)}$, then $V_{xx} + V_{yy} + V_{zz} = 2/V.$

10. If $V = 1/\sqrt{(x^2 + y^2 + z^2)}$, then $V_{xx} + V_{yy} + V_{zz} = 0.$

11. If $u = e^{xyz}$, prove that

$$\frac{\delta^3 u}{\delta x\, \delta y\, \delta z} = (1 + 3xyz + x^2y^2z^2)\, e^{xyz}.$$ [*C. U. 1947*]

12. (i) If $V = (ax + by)^2 - (x^2 + y^2)$, where $a^2 + b^2 = 2$, show that $V_{xx} + V_{yy} = 0.$

(ii) If $u = 3(ax + by + cz)^2 - (x^2 + y^2 + z^2)$ and $a^2 + b^2 + c^2 = 1$, find the value of $\dfrac{\delta^2 u}{\delta x^2} + \dfrac{\delta^2 u}{\delta y^2} + \dfrac{\delta^2 u}{\delta z^2}$

[*C. U. 1934*]

(iii) If $u = ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy$, and $\sum \frac{\delta^2 u}{\delta x^2} = 0$, show that $a + b + c = 0$.

13. Show that if $u(x, y, z)$ satisfy the equation

$$\frac{\delta^2 u}{\delta x^2} + \frac{\delta^2 u}{\delta y^2} + \frac{\delta^2 u}{\delta z^2} = 0,$$

then (i) $\frac{\delta u}{\delta x}, \frac{\delta u}{\delta y}, \frac{\delta u}{\delta z}$ satisfy it,

and also (ii) $x\frac{\delta u}{\delta x} + y\frac{\delta u}{\delta y} + z\frac{\delta u}{\delta z}$ satisfies it.

14. If $u = \log(x^2 + y^2 + z^2)$, then

$$x\frac{\delta^2 u}{\delta y\,\delta z} = y\frac{\delta^2 u}{\delta z\,\delta x} - z\frac{\delta^2 u}{\delta x\,\delta y}.$$

15. If $u = \log r$ and $r^2 = x^2 + y^2 + z^2$, prove that

$$r^2\left(\frac{\delta^2 u}{\delta x^2} + \frac{\delta^2 u}{\delta y^2} + \frac{\delta^2 u}{\delta z^2}\right) = 1.$$

16. If $y = f(x + ct) + \phi(x - ct)$, show that

$$\frac{\delta^2 y}{\delta t^2} = c^2\frac{\delta^2 y}{\delta x^2}.$$

17. If $u = f(ax^2 + 2hxy + by^2)$, $v = \phi(ax^2 + 2hxy + by^2)$, show that $\frac{\delta}{\delta y}\left(u\frac{\delta v}{\delta x}\right) = \frac{\delta}{\delta x}\left(u\frac{\delta v}{\delta y}\right)$.

[*C. H. 1934*]

***18.** (i) If $U = x + y + z$, $V = x^2 + y^2 + z^2$,
$W = x^3 + y^3 + z^3 - 3xyz$.

(ii) If $U = x + y + z$, $V = x^2 + y^2 + z^2$, $W = yz + zx + xy$,

show that in each case $\begin{vmatrix} U_x & U_y & U_z \\ V_x & V_y & V_z \\ W_x & W_y & W_z \end{vmatrix} = 0.$

***19.** If α, β, γ be the roots of the cubic $x^3+px^2+qx+r=0$, show that

$$\begin{vmatrix} \dfrac{\delta p}{\delta \alpha} & \dfrac{\delta q}{\delta \alpha} & \dfrac{\delta r}{\delta \alpha} \\ \dfrac{\delta p}{\delta \beta} & \dfrac{\delta q}{\delta \beta} & \dfrac{\delta r}{\delta \beta} \\ \dfrac{\delta p}{\delta \gamma} & \dfrac{\delta q}{\delta \gamma} & \dfrac{\delta r}{\delta \gamma} \end{vmatrix}$$

vanishes when any two of the three roots are equal.

***20.** Show that

$$\frac{\delta^3}{\delta x\, \delta y\, \delta z}\begin{vmatrix} f_1(x) & f_2(x) & f_3(x) \\ \phi_1(y) & \phi_2(y) & \phi_3(y) \\ \psi_1(z) & \psi_2(z) & \psi_3(z) \end{vmatrix} = \begin{vmatrix} f_1'(x) & f_2'(x) & f_3'(x) \\ \phi_1'(y) & \phi_2'(y) & \phi_3'(y) \\ \psi_1'(z) & \psi_2'(z) & \psi_3'(z) \end{vmatrix}$$

where dashes denote differentiations w. r. t. the variables concerned.

ANSWERS

1. (i) $2(ax+hy)$; $2(hx+by)$. (ii) $-\dfrac{y}{x^2+y^2}, \dfrac{x}{x^2+y^2}$.

(iii) $-\dfrac{x}{(x^2+y^2)^{\frac{3}{2}}}, -\dfrac{y}{(x^2+y^2)^{\frac{3}{2}}}$. (iv) $\dfrac{2x}{x^2+y^2}, \dfrac{2y}{x^2+y^2}$.

(v) $\dfrac{2x}{a^2}, \dfrac{2y}{b^2}$. **2.** (i) $6(x+y), 6(x+y), 6(x+y), 6(x+y)$.

(ii) $z\{(2x+y)^2+2\}, z\{(2x+y)(x+2y)+1\}$,

$z\{(2x+y)(x+2y)+1\}, z\{(x+2y)^2+2\}$,

where $z=e^{x^2+xy+y^2}$.

(iii) $-y\cos x, -(\sin x+\sin y), -(\sin x+\sin y), -x\cos y$.

(iv) $-\left\{\dfrac{1}{x^2}+\dfrac{1}{(x+y)^2}\right\}, -\dfrac{1}{(x+y)^2}, -\dfrac{1}{(x+y)^2}, -\left\{\dfrac{1}{y^2}+\dfrac{1}{(x+y)^2}\right\}$.

12. (ii) 0.

9·5. Homogeneous Functions.

A function $f(x, y)$ is said to be *homogeneous* of degree n in the variables x and y if it can be expressed in the form $x^n \phi\left(\frac{y}{x}\right)$, or in the form $y^n \phi\left(\frac{x}{y}\right)$.

If V be a homogeneous function of degree n in x, y, z, then each of $\frac{\delta V}{\delta x}, \frac{\delta V}{\delta y}, \frac{\delta V}{\delta z}$ is a homogeneous function of degree $(n-1)$.

Since, $ax^2+2hxy+by^2=x^2\left\{a+2h\frac{y}{x}+b\left(\frac{y}{x}\right)^2\right\}=x^2\phi\left(\frac{y}{x}\right)$,

$ax^2+2hxy+by^2$ is a homogeneous function of degree 2 in x, y.

Similarly, y/x, $x \tan^{-1}(y/x)$, $x^2 \log(y/x)$ are homogeneous functions of degree 0, 1 and 2 respectively.

Note 1. An *alternative test* for a function $f(x, y)$ to be homogeneous of degree n, is that $f(tx, ty)=t^n f(x, y)$ for all values of t, where t is independent of x and y.

Note 2. The test that a *rational integral algebraic function* of x and y should be homogeneous of degree n is that the sum of the indices of x and y in every term must be n.

Note 3. Similarly, a function $f(x, y, z)$ is said to be homogeneous of degree n in the variables x, y, z, if it can be put in the form $x^n f\left(\frac{y}{x}, \frac{z}{x}\right)$, or if $f(tx, ty, tz)=t^n f(x, y, z)$; and so on, for any number of variables.

Thus, $f(x, y, z)=\sqrt{x}+\sqrt{y}+\sqrt{z}$ is a homogeneous function of degree $\frac{1}{2}$, since

$$f(tx, ty, tz)=\sqrt{tx}+\sqrt{ty}+\sqrt{tz}=t^{\frac{1}{2}} f(x, y, z).$$

9·6. Euler's Theorem on Homogeneous Functions.

If $f(x, y)$ be a homogeneous function of x and y of degree n, then

$$\mathbf{x\frac{\delta f}{\delta x}+y\frac{\delta f}{\delta y}=nf(x, y).}$$

Since $f(x, y)$ is a homogeneous function of degree n,

$$\text{let } f(x, y) = x^n \phi(y/x)$$

$$= x^n \phi(v), \text{ where } v = y/x.$$

$$\therefore \quad \frac{\delta f}{\delta x} = nx^{n-1} \phi(v) + x^n \phi'(v) \frac{\delta v}{\delta x}$$

$$= nx^{n-1} \phi(v) + x^n \phi'(v) \cdot \frac{-y}{x^2}.$$

$$\frac{\delta f}{\delta y} = x^n \phi'(v) \frac{\delta v}{\delta y} = x^n \phi'(v) \cdot \frac{1}{x}.$$

$$\therefore \quad x \frac{\delta f}{\delta x} + y \frac{\delta f}{\delta y} = nx^n \phi(v) = nf(x, y).$$

9·7. Differentiation of Implicit Functions.

Let the equation $f(x, y) = 0$ ... , ... (1)

define y as a differentiable function of x, and let f_x and f_y be continuous.

Then, we can find $\dfrac{dy}{dx}$ in terms of $\dfrac{\delta f}{dx}, \dfrac{\delta f}{dy}$ as follows :

We have, $f(x + \Delta x, y + \Delta y) = 0$.

$$\therefore \quad f(x + \Delta x, y + \Delta y) - f(x, y) = 0. \quad \cdots \quad (2)$$

Now, by the Mean Value Theorem, [See § *6·2*]

$$f(x + \Delta x, y + \Delta y) - f(x, y + \Delta y)$$

$$= \Delta x \frac{\delta}{\delta x} f(x + \theta_1 \Delta x, y + \Delta y). \quad [\, 0 < \theta_1 < 1 \,].$$

$$f(x, y + \Delta y) - f(x, y)$$

$$= \Delta y \frac{\delta}{\delta y} f(x, y + \theta_2 \Delta y) \qquad [\, 0 < \theta_2 < 1 \,].$$

Adding these two and using relation (2), and dividing by Δx, we get

$$\frac{\delta}{\delta x}f(x+\theta_1\Delta x,\ y+\Delta y)+\frac{\Delta y}{\Delta x}\cdot\frac{\delta}{\delta y}f(x,\ y+\theta_2\Delta y)=0. \quad \ldots \quad (3)$$

Since, y is a differentiable function of x, when $\Delta x\to 0$, $\Delta y\to 0$, and since, f_x and f_y are continuous, we get by making $\Delta x\to 0$ in (3),

$$\frac{\delta f}{\delta x}+\frac{dy}{dx}\,\frac{\delta f}{\delta y}=0.$$

$$\therefore \quad \frac{dy}{dx}=-\frac{\dfrac{\delta f}{\delta x}}{\dfrac{\delta f}{\delta y}} \text{ or } =-\frac{f_x}{f_y}\ (f_y\neq 0). \quad \cdots \quad (4)$$

9·8. Total Differential coefficient.

Let $u=f(x,y)$, where $x=\phi(t)$, $y=\psi(t)$.

Then, usually u is a function of t in this case.

To obtain the value of $\dfrac{du}{dt}$.

Let us suppose that f_x, f_y as also $\phi'(t)$, $\psi'(t)$ are continuous. When t changes to $t+\Delta t$, let x and y change to $x+\Delta x$, $y+\Delta y$.

Now, $u=f\{\phi(t),\ \psi(t)\}\equiv F(t)$ say.

$$\therefore \quad \frac{du}{dt}=\underset{\Delta t\to 0}{Lt}\ \frac{F(t+\Delta t)-F(t)}{\Delta t}$$

$$=\underset{\Delta t\to 0}{Lt}\ \frac{f\{\phi(t+\Delta t),\ \psi(t+\Delta t)\}-f\{\phi(t),\ \psi(t)\}}{\Delta t}$$

$$=\underset{\Delta t\to 0}{Lt}\ \frac{f(x+\Delta x,\ y+\Delta y)-f(x,\ y)}{\Delta t}$$

$$= \underset{\Delta t \to 0}{Lt} \left[\frac{f(x+\Delta x,\ y+\Delta y) - f(x,\ y+\Delta y)}{\Delta x} \cdot \frac{\Delta x}{\Delta t} + \frac{f(x,\ y+\Delta y) - f(x,\ y)}{\Delta y} \cdot \frac{\Delta y}{\Delta t} \right].$$

But by the Mean Value Theorem, [See § *6·2*]

$$f(x+\Delta x,\ y+\Delta y) - f(x,\ y+\Delta y) = \Delta x . f_x(x+\theta \Delta x,\ y+\Delta y)$$

and $$f(x,\ y+\Delta y) - f(x,\ y) = \Delta y . f_y(x,\ y+\theta' \Delta y),$$

where θ and θ' each lie between 0 and 1.

When $\Delta t \to 0$, $\Delta x \to 0$ and $\Delta y \to 0$ and $\Delta x / \Delta t \to \phi'(t)$, $\Delta y / \Delta t \to \psi'(t)$.

Also $$f_x(x+\theta\ \Delta x,\ y+\Delta y) \to f_x(x,\ y);$$
$$f_y(x,\ y+\theta'\ \Delta y) \to f_y(x,\ y),$$

$$\therefore \quad \frac{du}{dt} = f_x \frac{dx}{dt} + f_y \frac{dy}{dt}, \qquad \cdots \quad (1)$$

$$\text{i.e.,} \quad \frac{\mathbf{du}}{\mathbf{dt}} = \frac{\boldsymbol{\delta}\mathbf{u}}{\boldsymbol{\delta}\mathbf{x}} \frac{\mathbf{dx}}{\mathbf{dt}} + \frac{\boldsymbol{\delta}\mathbf{u}}{\boldsymbol{\delta}\mathbf{y}} \frac{\mathbf{dy}}{\mathbf{dt}}. \qquad \cdots \quad (2)$$

Note 1. As a particular case, if $u = f(x,\ y)$ where y is a function of x,

$$\frac{\mathbf{du}}{\mathbf{dx}} = \frac{\boldsymbol{\delta}\mathbf{u}}{\boldsymbol{\delta}\mathbf{x}} + \frac{\boldsymbol{\delta}\mathbf{u}}{\boldsymbol{\delta}\mathbf{y}} \frac{\mathbf{dy}}{\mathbf{dx}}.$$

$\frac{du}{dx}$ is called the *total differential coefficient of* u, to distinguish it from its partial differential coefficient.

Note 2. The above result can easily be extended to the case when u is a function of three or more variables.

Thus, if $u = f(x,\ y,\ z)$, where $x,\ y,\ z$ are all functions of t,

$$\frac{du}{dt} = \frac{\delta u}{\delta x} \frac{dx}{dt} + \frac{\delta u}{\delta y} \frac{dy}{dt} + \frac{\delta u}{\delta z} \frac{dz}{dt}.$$

9·9. Differentials.

We have already defined the differential of a function of a single independent variable; we now give the corresponding definition of the differential of a function of *two* independent variables x, y. Thus, if $u=f(x, y)$, we define du by the relation

$$du=f_x\,\Delta x+f_y\,\Delta y.$$

Putting $u=x$ and $u=y$ in turn, we obtain

$$dx=\Delta x, \quad dy=\Delta y \qquad \cdots \quad (1)$$

so that $$\mathbf{du=f_x\,dx+f_y\,dy.} \qquad \cdots \quad (2)$$

Multiplying both sides of the relation (1) of Art. 9·8 by dt, and noting that since, x, y are each function of t,

$$du=\frac{du}{dt}\,dt,\ dx=\frac{dx}{dt}\,dt,\ dy=\frac{dy}{dt}\,dt$$

we get, $$du=f_x\,dx+f_y\,dy$$

which is same in form as (2) above. But x, y here are not independent, but each is a function of t.

Hence, *the formula* $du=f_x\,dx+f_y\,dy$ *is true whether the variables* x *and* y *are independent or not.*

This remark is of great importance in applications.

Similarly, if $u=f(x, y, z)$,

$$\mathbf{du=f_x\,dx+f_y\,dy+f_z\,dz,}$$

whether x, y, z are independent or not.

Note. It should be noted that the relations (1) above are true only when x and y are independent variables. If x and y are not independent but functions of a third independent variable t, say $x=\phi(t)$, $y=\psi(t)$, then $dx=\phi'(t)\,dt$ and $dy=\psi'(t)\,dt$, where $dt=\Delta t$.

9·10. Exact (or Perfect) differential.

The expression

$$\phi(x, y)\,dx+\psi(x, y)\,dy \qquad \cdots \quad (1)$$

is called an exact (or perfect) differential if a function u of x, y exists such that its differential

$$du \text{ i.e., } \frac{\delta u}{\delta x}dx+\frac{\delta u}{\delta y}dy \qquad \cdots \quad (2)$$

is equal to (1) for all values of dx and dy.

Hence, comparing (1) and (2), we see that if (1) be an exact differential, it is necessary that

$$\frac{\delta u}{\delta x}=\phi(x, y) \text{ and } \frac{\delta u}{\delta y}=\psi(x, y)$$

Differentiating these relations with respect to y and x respectively, we have

$$\frac{\delta^2 u}{\delta y\,\delta x}=\frac{\delta \phi}{\delta y} \text{ and } \frac{\delta^2 u}{\delta x\delta y}=\frac{\delta \psi}{\delta x}$$

Since, in all ordinary cases, $\frac{\delta^2 u}{\delta y\delta x}=\frac{\delta^2 u}{\delta x\delta y}$, hence in all ordinary cases, in order that (1) *may be an exact differential*, it is necessary that,

$$\frac{\delta \phi}{\delta y}=\frac{\delta \psi}{\delta x}.$$

It can be easily shown that this condition is in general also sufficient.

9·11. Partial Derivatives of a Function of two Functions.

If $u=f(x_1, x_2)$,

where $x_1=\phi_1(x, y)$, $x_2=\phi_2(x, y)$, and x, y are independent variables then,

$$\frac{\delta u}{\delta x}=\frac{\delta u}{\delta x_1}\frac{\delta x_1}{\delta x}+\frac{\delta u}{\delta x_2}\frac{\delta x_2}{\delta x}$$

$$\frac{\delta u}{\delta y}=\frac{\delta u}{\delta x_1}\frac{\delta x_1}{\delta y}+\frac{\delta u}{\delta x_2}\frac{\delta x_2}{\delta y}.$$

We have

$$du=\frac{\delta u}{\delta x_1}dx_1+\frac{\delta u}{\delta x_2}dx_2 \qquad \cdots \quad (1)$$

$$dx_1=\frac{\delta x_1}{\delta x}dx+\frac{\delta x_1}{\delta y}dy,\ dx_2=\frac{\delta x_2}{\delta x}dx+\frac{\delta x_2}{\delta y}dy.$$

When values of x_1, x_2 in terms of x, y are substituted, u becomes a function of x, y ; hence

$$du=\frac{\delta u}{\delta x}dx+\frac{\delta u}{\delta y}dy. \qquad \cdots \quad (2)$$

Now, substituting the values of dx_1, dx_2 in (1), we get

$$du=\left(\frac{\delta u}{\delta x_1}\frac{\delta x_1}{\delta x}+\frac{\delta u}{\delta x_2}\frac{\delta x_2}{\delta x}\right)dx+\left(\frac{\delta u}{\delta x_1}\frac{\delta x_1}{\delta y}+\frac{\delta u}{\delta x_2}\frac{\delta x_2}{\delta y}\right)dy.$$

Comparing this with (2), since, dx, dy are independent, the required relations follow.

Note. The above result admits of easy generalization to the cases of more than two variables. Thus, if $u=f(x_1, x_2, x_3)$, where $x_1=\phi_1(x, y, z)$, $x_2=\phi_2(x, y, z)$, $x_3=\phi_3(x, y, z)$ and x, y, z are independent variables, then

$$\frac{\delta u}{\delta x}=\frac{\delta u}{\delta x_1}\frac{\delta x_1}{\delta x}+\frac{\delta u}{\delta x_2}\frac{\delta x_2}{\delta x}+\frac{\delta u}{\delta x_3}\frac{\delta x_3}{\delta x}$$

$$\frac{\partial u}{\partial y}=\frac{\partial u}{\partial x_1}\frac{\partial x_1}{\partial y}+\frac{\partial u}{\partial x_2}\frac{\partial x_2}{\partial y}+\frac{\partial u}{\partial x_3}\frac{\partial x_3}{\partial y}$$

$$\frac{\partial u}{\partial z}=\frac{\partial u}{\partial x_1}\frac{\partial x_1}{\partial z}+\frac{\partial u}{\partial x_2}\frac{\partial x_2}{\partial z}+\frac{\partial u}{\partial x_3}\frac{\partial x_3}{\partial z}.$$

9·12. Euler's Theorem on Homogeneous Functions *(generalization).*

If $f(x, y, z)$ be a homogeneous function in x, y, z of degree n, having continuous partial derivatives,

then $$x\frac{\delta f}{\delta x}+y\frac{\delta f}{\delta y}+z\frac{\delta f}{\delta z}=nf.$$

Proof : Since $f(x, y, z)$ is a homogeneous function,

$$\therefore \quad f(tx, ty, tz)=t^n f(x, y, z) \quad \cdots \quad \cdots \quad (1)$$

for all values of t.

Putting $tx=u$, $ty=v$, $tz=w$,
differentiating both sides of (1) with respect to t, we have,

$$\frac{\delta f}{\delta u}\cdot\frac{\delta u}{\delta t}+\frac{\delta f}{\delta v}\cdot\frac{\delta v}{\delta t}+\frac{\delta f}{\delta w}\cdot\frac{\delta w}{\delta t}=nt^{n-1}f(x, y, z),$$

$$\therefore \quad x\frac{\delta f}{\delta u}+y\frac{\delta f}{\delta v}+z\frac{\delta f}{\delta w}=nt^{n-1}f(x, y, z). \quad \cdots \quad (2)$$

Putting $t=1$ in (2),

$$x\frac{\delta f}{\delta x}+y\frac{\delta f}{\delta y}+z\frac{\delta f}{\delta z}=nf.$$

Note 1. The above method of proof is applicable to a function of any number of independent variables.

Note 2. The above result can also be established as in the case of two independent variables *i.e.*, by writing

$f(x, y, z)=x^n f\left(\frac{y}{x}, \frac{z}{x}\right)=x^n f(u, v)$, where $u=\frac{y}{x}$, $v=\frac{z}{x}$ and then obtaining $\frac{\delta f}{\delta x}, \frac{\delta f}{\delta y}, \frac{\delta f}{\delta z}$.

9·13. Converse of Euler's Theorem.

If $f(x, y, z)$ admits of continuous partial derivatives and satisfies the relation

$$x\frac{\partial f}{\partial x}+y\frac{\partial f}{\partial y}+z\frac{\partial f}{\partial z}=nf(x, y, z),$$

where n is a positive integer, prove that $f(x, y, z)$ is a homogeneous function of degree n. [*C. H. 1960*]

Proof: Put $\xi=\dfrac{x}{z}$, $\eta=\dfrac{y}{z}$, $\zeta=z$;

then $x=\xi\zeta$, $y=\eta\zeta$, $z=\zeta$. Suppose when expressed in terms of ξ, η, ζ,

$$f(x, y, z)=v(\xi, \eta, \zeta).$$

Then, $$x\frac{\partial f}{\partial x}=x\left(\frac{\partial v}{\partial \xi}\frac{\partial \xi}{\partial x}+\frac{\partial v}{\partial \eta}\frac{\partial \eta}{\partial x}+\frac{\partial v}{\partial \zeta}\frac{\partial \zeta}{\partial x}\right)$$

$$=x\left(\frac{\partial v}{\partial \xi}\cdot\frac{1}{z}+\frac{\partial v}{\partial \eta}\cdot 0+\frac{\partial v}{\partial \zeta}\cdot 0\right)$$

$$=\xi\frac{\partial v}{\partial \xi}.$$

Similarly $y\dfrac{\partial f}{\partial y}=\eta\dfrac{\partial v}{\partial \eta}$

and $$z\frac{\partial f}{\partial z}=z\left(\frac{\partial v}{\partial \xi}\frac{\partial \xi}{\partial z}+\frac{\partial v}{\partial \eta}\frac{\partial \eta}{\partial z}+\frac{\partial v}{\partial \zeta}\frac{\partial \zeta}{\partial z}\right)$$

$$=z\left(-\frac{\partial v}{\partial \xi}\cdot\frac{x}{z^2}-\frac{\partial v}{\partial \eta}\cdot\frac{y}{z^2}+\frac{\partial v}{\partial \zeta}\cdot 1\right)$$

$$=-\xi\frac{\partial v}{\partial \xi}-\eta\frac{\partial v}{\partial \eta}+\zeta\frac{\partial v}{\partial \zeta}.$$

Hence, the given relation reduces to

$$\zeta\frac{\partial v}{\partial \zeta}=nv, \quad \text{or,} \quad \frac{1}{v}\frac{\partial v}{\partial \zeta}=\frac{n}{\zeta}$$

whence, $\log v=n\log\zeta+$ a constant, where the constant is independent of ζ, but may depend on ξ and η; let this constant be denoted by $\log \phi(\xi, \eta)$.

Then, $\quad v = \zeta^n \phi(\xi, \eta)$,

i.e., $f(x, y, z) = z^n \phi\left(\frac{x}{z}, \frac{y}{z}\right)$

which, according to the definition of a homogeneous function, shows that $f(x, y, z)$ is a homogeneous function of degree n.

Note. If n be any rational number, the proof and the result remain unchanged.

9·14. Illustrative Examples.

Ex 1. *If* $u = \tan^{-1} \frac{x^3 + y^3}{x - y}$, *show that*

$$x \frac{\delta u}{\delta x} + y \frac{\delta u}{\delta y} = \sin 2u.$$

From the given relation, we get

$$\tan u = \frac{x^3 + y^3}{x - y} = \frac{x^3 \{1 + (y/x)^3\}}{x \{1 - (y/x)\}} = x^2 \phi\left(\frac{y}{x}\right).$$

$\therefore$ $\tan u$ is a homogeneous function of degree 2.

Let $v = \tan u$; $\therefore$ by Euler's Theorem,

$$x \frac{\delta v}{\delta x} + y \frac{\delta v}{\delta y} = 2v,$$

$$\therefore \; x \sec^2 u \frac{\delta u}{\delta x} + y \sec^2 u \frac{\delta u}{\delta y} = 2 \tan u.$$

$$\therefore \quad x \frac{\delta u}{\delta x} + y \frac{\delta u}{\delta y} = \frac{2 \tan u}{\sec^2 u} = 2 \sin u \cos u = \sin 2u.$$

Ex. 2. *If $\triangle$ be the area of a triangle ABC, show that*

$$d\triangle = R(\cos A \; da + \cos B \; db + \cos C \; dc),$$

where R is the circum-radius of the triangle.

From trigonometry, we have

$$\triangle^2 = \tfrac{1}{16}(2b^2c^2 + 2c^2a^2 + 2a^2b^2 - a^4 - b^4 - c^4).$$

Thus, $\triangle$ is a function of the three independent variables, a, b, c.
Hence, taking differential of both sides,

$$2\triangle d\triangle = \tfrac{1}{16}\{4a(b^2+c^2-a^2)\,da + 4b(c^2+a^2-b^2)\,db + 4c(a^2+b^2-c^2)\,dc\}$$

$$= \tfrac{1}{16}(4a\,.\,2bc\cos A\,da + 4b\,.\,2ca\cos B\,db + 4c\,.\,2ab\cos C\,dc)$$

$$= \tfrac{1}{2}abc(\cos A\,da + \cos B\,db + \cos C\,dc).$$

$$\therefore\ d\triangle = \frac{abc}{4\triangle}(\cos A\,da + \cos B\,db + \cos C\,dc)$$

$$= R(\cos A\,da + \cos B\,db + \cos C\,dc).$$

Ex. 8. *If $P\,dx + Q\,dy + R\,dz$ can be made a perfect differential of some function of x, y, z, on multiplication by a factor, prove that*

$$P\left(\frac{\partial Q}{\partial z} - \frac{\partial R}{\partial y}\right) + Q\left(\frac{\partial R}{\partial x} - \frac{\partial P}{\partial z}\right) + R\left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right) = 0.$$

[*C. H. 1949, 1954*]

Suppose u is a function of x, y, z and

$$\mu\,(P\,dx + Q\,dy + R\,dz) = du, \qquad \cdots \qquad \cdots \quad (1)$$

where μ is some function of x, y, z.

Also, $$du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy + \frac{\partial u}{\partial z}dz \qquad \cdots \qquad \cdots \quad (2)$$

since u is a function of x, y, z

Comparing (1) and (2),

$$\frac{\partial u}{\partial x} = \mu P \quad \cdots \quad (3)\ ; \quad \frac{\partial u}{\partial y} = \mu Q \quad \cdots \quad (4)\ ; \quad \frac{\partial u}{\partial z} = \mu R \qquad \cdots \quad (5)$$

$$\therefore \left.\begin{aligned}\frac{\partial^2 u}{\partial y \partial x} &= \mu\frac{\partial P}{\partial y} + P\frac{\partial \mu}{\partial y}\\ &= \mu\frac{\partial Q}{\partial x} + Q\frac{\partial \mu}{\partial x}\end{aligned}\right\} \cdots (6)$$

(on differentiating (3) with respect to y)
(on differentiating (4) with respect to x assuming $\frac{\partial^2 u}{\partial y \partial x} = \frac{\partial^2 u}{\partial x \partial y}$)

Similarly,

$$\frac{\partial^2 u}{\partial z \partial y} = \mu\frac{\partial Q}{\partial z} + Q\frac{\partial \mu}{\partial z} = \mu\frac{\partial R}{\partial y} + R\frac{\partial \mu}{\partial y} \qquad \cdots \qquad \cdots \quad (7)$$

$$\frac{\partial^2 u}{\partial x \partial z} = \mu\frac{\partial R}{\partial x} + R\frac{\partial \mu}{\partial x} = \mu\frac{\partial P}{\partial z} + P\frac{\partial \mu}{\partial z}. \qquad \cdots \qquad \cdots \quad (8)$$

From (6), (7), (8), we get on re-arranging

$$\mu\left(\frac{\partial P}{\partial y}-\frac{\partial Q}{\partial x}\right)=Q\frac{\partial \mu}{\partial x}-P\frac{\partial \mu}{\partial y} \quad \cdots \quad \cdots \quad (9)$$

$$\mu\left(\frac{\partial Q}{\partial z}-\frac{\partial R}{\partial y}\right)=R\frac{\partial \mu}{\partial y}-Q\frac{\partial \mu}{\partial z} \quad \cdots \quad \cdots \quad (10)$$

$$\mu\left(\frac{\partial R}{\partial x}-\frac{\partial P}{\partial z}\right)=P\frac{\partial \mu}{\partial z}-R\frac{\partial \mu}{\partial x}. \quad \cdots \quad \cdots \quad (11)$$

Multiplying (9) by R, (10) by P, (11) by Q and adding together, we get the required result.

Note. If $P\,dx+Q\,dy+R\,dz$ be *itself a perfect differential*, then we easily deduce the conditions that $\frac{\partial Q}{\partial z}-\frac{\partial R}{\partial y}=\frac{\partial R}{\partial x}-\frac{\partial P}{\partial z}=\frac{\partial P}{\partial y}-\frac{\partial Q}{\partial x}=0.$

Ex. 4. *If V be a function of x and y, prove that*

$$\frac{\partial^2 V}{\partial x^2}+\frac{\partial^2 V}{\partial y^2}=\frac{\partial^2 V}{\partial r^2}+\frac{1}{r}\frac{\partial V}{\partial r}+\frac{1}{r^2}\frac{\partial^2 V}{\partial \theta^2}$$

where $x=r\cos\theta$, $y=r\sin\theta$. [*C. H. 1953*]

Here $x=r\cos\theta$, $y=r\sin\theta$;

$\therefore \quad r=\sqrt{x^2+y^2}$, $\theta=\tan^{-1}\frac{y}{x}$.

Hence, $\quad \frac{\partial x}{\partial r}=\cos\theta$; $\frac{\partial r}{\partial x}=\frac{x}{\sqrt{x^2+y^2}}=\frac{r\cos\theta}{r}=\cos\theta.$

$$\frac{\partial y}{\partial r}=\sin\theta\ ;\ \frac{\partial r}{\partial y}=\frac{y}{\sqrt{x^2+y^2}}=\frac{r\sin\theta}{r}=\sin\theta.$$

$$\frac{\partial \theta}{\partial x}=\frac{1}{1+\frac{y^2}{x^2}}\left(-\frac{y}{x^2}\right)=-\frac{x^2}{x^2+y^2}\cdot\frac{y}{x^2}=-\frac{r\sin\theta}{r^2}=-\frac{\sin\theta}{r}.$$

Similarly, $\frac{\partial \theta}{\partial y}=\frac{\cos\theta}{r}$.

Since V is a function of (x, y), and x and y are functions of r, θ, so V is a function of (r, θ).

Hence, $\frac{\partial V}{\partial x}=\frac{\partial V}{\partial r}\cdot\frac{\partial r}{\partial x}+\frac{\partial V}{\partial \theta}\cdot\frac{\partial \theta}{\partial x}$

$$=\cos\theta\cdot\frac{\partial V}{\partial r}-\frac{\sin\theta}{r}\frac{\partial V}{\partial \theta} \quad \cdots \quad \cdots \quad (1)$$

$$\frac{\partial V}{\partial y}=\frac{\partial V}{\partial r}\cdot\frac{\partial r}{\partial y}+\frac{\partial V}{\partial \theta}\cdot\frac{\partial \theta}{\partial y}$$

$$=\sin\theta.\frac{\partial V}{\partial r}+\frac{\cos\theta}{r}\frac{\partial V}{\partial \theta}. \qquad \dots \quad (2)$$

Thus we have the following *equivalence of Cartesian and polar operators* :

$$\frac{\partial}{\partial x}\equiv\left(\cos\theta\,\frac{\partial}{\partial r}-\frac{\sin\theta}{r}\,\frac{\partial}{\partial \theta}\right)$$

$$\frac{\partial}{\partial y}\equiv\left(\sin\theta\,\frac{\partial}{\partial r}+\frac{\cos\theta}{r}\,\frac{\partial}{\partial \theta}\right);$$

$$\therefore\ \frac{\partial}{\partial x}\left(\frac{\partial V}{\partial x}\right)=\left(\cos\theta\,\frac{\partial}{\partial r}-\frac{\sin\theta}{r}\,\frac{\partial}{\partial \theta}\right)\left(\cos\theta\,\frac{\partial V}{\partial r}-\frac{\sin\theta}{r}\,\frac{\partial V}{\partial \theta}\right)$$

$$=\cos\theta\left[\cos\theta\,\frac{\partial^2 V}{\partial r^2}-\frac{\sin\theta}{r}\,\frac{\partial^2 V}{\partial r\,\partial \theta}+\frac{\partial V}{\partial \theta}\,\frac{\sin\theta}{r^2}\right]$$

$$-\frac{\sin\theta}{r}\left[\cos\theta\,\frac{\partial^2 V}{\partial \theta\,\partial r}-\sin\theta\,\frac{\partial V}{\partial r}-\frac{1}{r}\,\frac{\partial V}{\partial \theta}\cos\theta-\frac{1}{r}\sin\theta\,\frac{\partial^2 V}{\partial \theta^2}\right]$$

$$\therefore\quad \frac{\partial^2 V}{\partial x^2}=\cos^2\theta\,\frac{\partial^2 V}{\partial r^2}-\frac{2\sin\theta\cos\theta}{r}\,\frac{\partial^2 V}{\partial r\,\partial \theta}+\frac{\sin^2\theta}{r^2}\,\frac{\partial^2 V}{\partial \theta^2}+\frac{\sin^2\theta}{r}\,\frac{\partial V}{\partial r}$$
$$+\frac{2\sin\theta\cos\theta}{r^2}\,\frac{\partial V}{\partial \theta}.$$

Similarly,

$$\frac{\partial^2 V}{\partial y^2}=\sin^2\theta\,\frac{\partial^2 V}{\partial r^2}+\frac{2\sin\theta\cos\theta}{r}\,\frac{\partial^2 V}{\partial r\,\partial \theta}+\frac{\cos^2\theta}{r^2}\,\frac{\partial^2 V}{\partial \theta^2}+\frac{\cos^2\theta}{r}\,\frac{\partial V}{\partial r}$$
$$-\frac{2\sin\theta\cos\theta}{r^2}\,\frac{\partial V}{\partial \theta}.$$

$$\therefore\ \frac{\partial^2 V}{\partial x^2}+\frac{\partial^2 V}{\partial y^2}=\frac{\partial^2 V}{\partial r^2}+\frac{1}{r^2}\,\frac{\partial^2 V}{\partial \theta^2}+\frac{1}{r}\,\frac{\partial V}{\partial r}.$$

Examples IX(B)

1. Verify Euler's theorem for the following functions :

(i) $u=ax^2+2hxy+by^2$. (ii) $u=x^3+y^3+3x^2y+3xy^2$.

(iii) $u=\dfrac{x-y}{x+y}$. (iv) $u=\sin\dfrac{x^2+y^2}{xy}$.

(v) $u=(x^{\frac{1}{4}}+y^{\frac{1}{4}})/(x^{\frac{1}{5}}+y^{\frac{1}{5}})$.

2. Find $\dfrac{dy}{dx}$ in the following cases ;

(i) $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$. (ii) $x^y + y^x = a^b$. [C. H. 1944]

(iii) $(\cos x)^y = (\sin y)^x$. (iv) $e^x + e^y = 2xy$.

(v) $y^x + x^y = (x+y)^{x+y}$.

3. (i) If $u = \phi(H_n)$, where H_n is a homogeneous function of nth degree in x, y, z, show that $x\dfrac{\delta u}{\delta x} + y\dfrac{\delta u}{\delta y} + z\dfrac{\delta u}{\delta z} = n\dfrac{F(u)}{F'(u)}$, where $F(u) = H_n$.

(ii) If $u = \cos^{-1}\{(x+y)/(\sqrt{x} + \sqrt{y})\}$, show that

$$x\frac{\delta u}{\delta x} + y\frac{\delta u}{\delta y} + \tfrac{1}{2}\cot u = 0.$$

4. If $V = \sin^{-1}(x^2 + y^2)/(x+y)$, then

$$xV_x + yV_y = \tan V.$$

5. If $u = x\phi(y/x) + \psi(y/x)$, show that

(i) $x\dfrac{\delta u}{\delta x} + y\dfrac{\delta u}{\delta y} = x\phi(y/x)$.

(ii) $x^2u_{xx} + 2xyu_{xy} + y^2u_{yy} = 0$.

6. (i) If $v = f(u)$, u being a homogeneous function of degree n in x, y, show that

$$x\frac{\delta v}{\delta x} + y\frac{\delta v}{\delta y} = nu\frac{dv}{du}. \qquad [\text{ C. P. 1948 }]$$

(ii) If V be a homogeneous function in x, y, z of degree n, prove that $\dfrac{\delta V}{\delta x}, \dfrac{\delta V}{\delta y}$ and $\dfrac{\delta V}{\delta z}$ are each a homogeneous function in x, y, z of degree $(n-1)$.

*(iii) If V be a homogeneous function of the n^{th} degree in x, y, z and if $V=f(X, Y, Z)$, where X, Y, Z are respectively $\frac{\delta V}{\delta x}, \frac{\delta V}{\delta y}, \frac{\delta V}{\delta z}$, show that $X\frac{\delta V}{\delta X}+Y\frac{\delta V}{\delta Y}+Z\frac{\delta V}{\delta Z}=\frac{n}{n-1}V$.

7. (i) If H be a homogeneous function of degree n in x and y and if $u=(x^2+y^2)^{-\frac{1}{2}n}$, then

$$\frac{\delta}{\delta x}\left(H\frac{\delta u}{\delta x}\right)+\frac{\delta}{\delta y}\left(H\frac{\delta u}{\delta y}\right)=0.$$

*(ii) If H be a homogeneous function of x, y, z of degree n and if $u=(x^2+y^2+z^2)^{-\frac{1}{2}(n+1)}$, then

$$\frac{\delta}{\delta x}\left(H\frac{\delta u}{\delta x}\right)+\frac{\delta}{\delta y}\left(H\frac{\delta u}{\delta y}\right)+\frac{\delta}{dz}\left(H\frac{\delta u}{\delta z}\right)=0.$$

8. If $x=r\cos\theta$, $y=r\sin\theta$, then

(i) $dx^2+dy^2=dr^2+r^2\,d\theta^2$.

(ii) $x\,dy-y\,dx=r^2\,d\theta$.

9. If $\phi(x, y)=0$, $\psi(x, z)=0$, then

$$\frac{\delta\psi}{\delta x}\cdot\frac{\delta\phi}{\delta y}\cdot\frac{dy}{\delta z}=\frac{\delta\phi}{\delta x}\cdot\frac{\delta\psi}{\delta z}.$$

10. Express Δ, the area of $\triangle ABC$, as a function of a, b, C and hence show that

$$\frac{d\Delta}{\Delta}=\frac{da}{a}+\frac{db}{b}+\cot C\,dC.$$

11. If $x^2+y^2+z^2-2xyz=1$, show that

$$\frac{dx}{\sqrt{1-x^2}}+\frac{dy}{\sqrt{1-y^2}}+\frac{dz}{\sqrt{1-z^2}}=0.$$

12. (i) If $ax^2+by^2+cz^2=1$, and $lx+my+nz=0$, then

$$\frac{dx}{bny-cmz}=\frac{dy}{clz-anx}=\frac{dz}{amx-bly}.$$

(ii) If $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$, and $\frac{x^2}{a^2+\lambda}+\frac{y^2}{b^2+\lambda}+\frac{z^2}{c^2+\lambda}=1$,

prove that $\frac{x(b^2-c^2)}{dx}+\frac{y(c^2-a^2)}{dy}+\frac{z(a^2-b^2)}{dz}=0.$

*13. The radius of a right circular cone is measured as 5 inches with a possible error of ·01 inches, and altitude as 8 inches with a possible error of ·024 inches. Find the possible relative error and percentage error in the volume as calculated from these measurements.

*14. The side a of a triangle ABC is calculated from b, c, A. If there be small errors db, dc, dA in the measured values of b, c, A, show that the error in the calculated value of a is given by

$$da=\cos B.dc+\cos C.db+b\sin C.dA.$$

*15. If $f(p, t, v)=0$, show that

$$\left(\frac{dp}{dt}\right)_{v\text{ const}}\times\left(\frac{dt}{dv}\right)_{p\text{ const}}\times\left(\frac{dv}{dp}\right)_{t\text{ const}}=-1.$$

*16. If $u=F(y-z, z-x, x-y)$, prove that

$$\frac{\delta u}{\delta x}+\frac{\delta u}{\delta y}+\frac{\delta u}{\delta z}=0.$$

*17. If $u=F(x^2+y^2+z^2)f(xy+yz+zx)$, prove that

$$(y-z)\frac{\delta u}{\delta x}+(z-x)\frac{\delta u}{\delta y}+(x-y)\frac{\delta u}{\delta z}=0.$$

*18. If $u=f(x^2+2yz, y^2+2zx)$, prove that

$$(y^2-zx)\frac{\delta u}{\delta x}+(x^2-yz)\frac{\delta u}{\delta y}+(z^2-xy)\frac{\delta u}{\delta z}=0.$$

[*C. H. 1947*]

***19.** If $F(v^2 - x^2,\ v^2 - y^2,\ v^2 - z^2) = 0$,

where v is a function of x, y, z, show that

$$\frac{1}{x}\frac{\delta v}{\delta x} + \frac{1}{y}\frac{\delta v}{\delta y} + \frac{1}{z}\frac{\delta v}{\delta z} = \frac{1}{v}.$$

***20.** If u be a homogeneous function of x and y of n dimensions, prove that

$$\left(x\frac{\delta}{\delta x} + y\frac{\delta}{\delta y}\right)^2 u = n(n-1)u,$$

where $\left(x\frac{\delta}{\delta x} + y\frac{\delta}{\delta y}\right)^2 u = x^2\frac{\delta^2 u}{\delta x^2} + 2xy\frac{\delta^2 u}{\delta x\,\delta y} + y^2\frac{\delta^2 u}{\delta y^2}.$

[*C. U. 1946*]

***21.** If $u = x\phi(x+y) + y\psi(x+y)$, show that

$$\frac{\delta^2 u}{\delta x^2} - 2\frac{\delta^2 u}{\delta x\,\delta y} + \frac{\delta^2 u}{\delta y^2} = 0.$$

***22.** If V be a function of r alone, where $r^2 = x^2 + y^2 + z^2$, show that $\frac{\delta^2 V}{\delta x^2} + \frac{\delta^2 V}{\delta y^2} + \frac{\delta^2 V}{\delta z^2} = \frac{d^2 V}{dr^2} + \frac{2}{r}\frac{dV}{dr}.$

ANSWERS

2. (i) $-\frac{y^{\frac{1}{3}}}{x^{\frac{1}{3}}}$. (ii) $-\frac{yx^{y-1} + y^x \log y}{xy^{x-1} + x^y \log x}$. (iii) $\frac{y \tan x + \log \sin y}{\log \cos x - x \cot y}$.

(iv) $\frac{e^x - 2y}{2x - e^y}$. (v) $-\frac{y^x \log y + yx^{y-1} - (x+y)^{x+y}\{\log(x+y)+1\}}{x^y \log x + xy^{x-1} - (x+y)^{x+y}\{\log(x+y)+1\}}$.

13. ·007 (relative error) ; ·7 (percentage error).

CHAPTER X

TANGENT AND NORMAL

10·1. We shall now consider certain properties of curves represented by continuous functions. If the equation of the curve is given in the explicit form $y=f(x)$, we shall assume that $f(x)$ has a derivative at every point, except, in some cases, at isolated points. If the equation of the curve is given in the implicit form $f(x, y)=0$, we shall assume that the function $f(x, y)$ possesses continuous partial derivatives f_x and f_y which are not simultaneously zero. When the equation of the curve is given in the parametric form $x=\phi(t)$, $y=\psi(t)$, we shall assume that $\phi'(t)$ and $\psi'(t)$ are not simultaneously zero.

10·2. Equation of the tangent.

Def. *The tangent at P to a given curve is defined as the limiting position of the secant PQ,* (when such a limit exists) *as the point Q approaches P along the curve* (whether Q is taken on one side or the other of the point P).

(i) *Let the equation of the curve be* $y=f(x)$ and let the given point P on the curve be (x, y) and any other neighbouring point Q on the curve be $(x+\Delta x, y+\Delta y)$.

The equation of the secant PQ is (X, Y denoting current co-ordinates),

$$Y-y=\frac{y+\Delta y-y}{x+\Delta x-x}(X-x)=\frac{\Delta y}{\Delta x}(X-x)$$

$\therefore$ the equation of the tangent at P is

$$Y-y=\underset{\Delta x\to 0}{Lt}\frac{\Delta y}{\Delta x}(X-x)=\frac{dy}{dx}(X-x),$$

provided dy/dx is finite.

Thus, *the tangent to the curve* $y=f(x)$ at (x, y) (not parallel to y-axis) is

$$\mathbf{Y-y=\frac{dy}{dx}(X-x).} \qquad \cdots \quad (1)$$

(ii) *When the equation of the curve is* $f(x, y)=0$,

since, $\dfrac{dy}{dx}=-\dfrac{f_x}{f_y}$, $(f_y \neq 0)$

the equation of the tangent to the curve at (x, y) is

$$\mathbf{(X-x)\,f_x+(Y-y)\,f_y=0.} \qquad \cdots \quad (2)$$

(iii) *When the equation of the curve is* $x=\phi(t)$, $y=\psi(t)$,

since, $\dfrac{dy}{dx}=\dfrac{dy}{dt}\Big/\dfrac{dx}{dt}=\dfrac{\psi'(t)}{\phi'(t)}$, $\phi'(t) \neq 0$,

the equation of the tangent at the point 't' is

$$\left.\begin{aligned} Y-\psi(t)&=\frac{\psi'(t)}{\phi'(t)}\{X-\phi(t)\} \\ \textit{i.e.,}\ \psi'(t)\,X-\phi'(t)\,Y&=\phi(t)\psi'(t)-\psi(t)\phi'(t) \end{aligned}\right\} \cdots \quad (3)$$

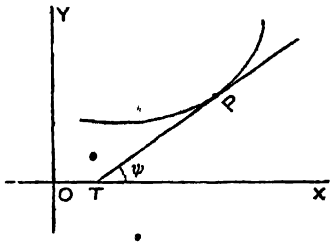
Note 1. When left-hand and right-hand derivatives at (x, y) are infinite, with equal or opposite signs, the tangent at (x, y) can be conveniently obtained by using the *alternative form of the equation of the tangent* $X-x=(Y-y)(dx/dy)$ which can be easily established as before. [See *Ex. 32, Examples X(A)*]

Note 2. In the notation of Co-ordinate Geometry, the equation of the tangent to the curve $y=f(x)$ at (x_1, y_1) can be written as

$$y-y_1=f'(x_1)(x-x_1).$$

In the application of Differential Calculus to the theory of plane curves, for the sake of convenience, the current co-ordinates in the equation of the tangent and normal are usually denoted by (X, Y), while those of any particular point are denoted by (x, y). The current co-ordinates in the equation of the curve are however, as usual, denoted by (x, y).

10·3. Geometrical meaning of $\frac{dy}{dx}$.



The equation (1) of the tangent can be written as

$$Y = \frac{dy}{dx} \cdot X + \left(y - x\frac{dy}{dx} \right)$$

which being of the form $y = mx + c$, the standard equation of a straight line, we conclude that

dx is the 'm' of the tangent at (x, y).

If ψ be the angle which the positive direction of the tangent at P makes with the positive direction of x-axis, then

$$\tan \psi = m = \frac{dy}{dx}.$$

Hence, *the derivative* $\frac{dy}{dx}$ *at* (x, y) *is equal to the trigonometrical tangent of the angle which the tangent to the curve at* (x, y) *makes with the positive direction of the x-axis.*

[See *Art. 4·14*]

Note 1. It is customary to denote by ψ, the angle which the tangent at any point on a curve makes with the x-axis.

Note 2. *The positive direction of the tangent* is the direction of the arc-length s increasing. Henceforth, this direction will be spoken of as the direction of the tangent or simply as the tangent.

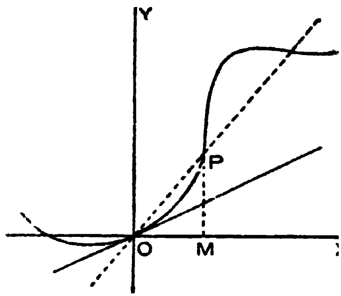
Note 3. $\tan\psi$, *i.e.*, $\frac{dy}{dx}$ is also called the **gradient** of the curve at the point $P(x, y)$.

Note 4. *The tangent at* (x, y) *is parallel to the x-axis,* if $\psi = 0$, *i.e.*, if $\tan\psi = 0$, *i.e.*, if $\frac{dy}{dx} = 0$.

The tangent at (x, y) *is perpendicular to the x-axis* (*i.e.*, parallel to the y-axis), if $\psi = \frac{1}{2}\pi$, *i.e.*, if $\cot\psi = 0$, *i.e.*, if $1\Big/\frac{dy}{dx}$, or, $\frac{dx}{dy} = 0$.

10·4. Tangent at the origin.

If a curve passing through the origin be given by a rational integral algebraic equation, the equation of the tangent (or tangents) at the origin is obtained by equating to zero the terms of the lowest degree in the equation.



Let the equation of a curve of the nth degree passing through the origin be

$$a_1 x + b_1 y + a_2 x^2 + b_2 xy + c_2 y^2 + \cdots\cdots + a_n x^n + \cdots\cdots + k_n y^n = 0. \quad \cdots \quad (1)$$

Let $P(x, y)$ be a point on the curve near the origin O.

The equation of the secant OP is $Y = \frac{y}{x} X$.

$\therefore$ the equation of the tangent at O is

$$Y = \underset{\substack{x \to 0 \\ y \to 0}}{Lt} \frac{y}{x} \cdot X = mX \text{ (say)}. \qquad \cdots \quad (2)$$

Thus, the 'm' of the tangent at the origin is $\underset{\substack{x \to 0 \\ y \to 0}}{Lt} \frac{y}{x}$.

CASE I. Let us suppose that m is finite i.e., the y-axis is not the tangent at the origin.

(i) Let us suppose $b_1 \neq 0$.

Dividing (1) by x, we get

$$a_1 + b_1 \frac{y}{x} + a_2 x + b_2 y + c_2 y . \frac{y}{x} + \cdots\cdots = 0.$$

Now, let $x \to 0$, $y \to 0$ then $Lt\,(y/x) = m$.

$\therefore$ $a_1 + b_1 m = 0$, the other terms vanishing.

$\therefore$ $m = -a_1/b_1$. $\qquad \cdots \qquad \cdots \quad (3)$

From (2) and (3), the equation of the tangent at the origin is $\qquad a_1 X + b_1 Y = 0$,

or, taking x and y as current co-ordinates,

$$a_1 x + b_1 y = 0.$$

(ii) If $b_1 = 0$, then from (3), it follows that $a_1 = 0$; now in this case, let us suppose that b_2 and c_2 are not both zero. Then, the equation of the curve (1) can be written as

$$a_2 x^2 + b_2 xy + c_2 y^2 + a_3 x^3 + \cdots\cdots = 0. \qquad \cdots \quad (4)$$

Dividing by x^2, $a_2 + b_2 \frac{y}{x} + c_2 \left(\frac{y}{x}\right)^2 + a_3 x + \cdots\cdots = 0.$

When $x \to 0$, $y \to 0$, we have

$$a_2 + b_2 m + c_2 m^2 = 0, \text{ the other terms vanishing.} \quad \cdots (5)$$

From (5), it is clear that there are two values of m and hence there are two tangents at the origin and their equation which is obtained by eliminating m between (2) and (5) is

$$a_2 X^2 + b_2 XY + c_2 Y^2 = 0,$$

or, taking x and y as current co-ordinates

$$a_2 x^2 + b_2 xy + c_2 y^2 = 0.$$

If $a_1 = b_1 = a_2 = b_2 = c_2 = 0$, it can be shown similarly that the rule holds good then also ; and so on.

CASE II. When the tangent at the origin is the y-axis, then $Lt\,(x/y)$ as x and y both $\to 0$, being the tangent of the inclination of the tangent at the origin to the y-axis, is zero. Hence, dividing throughout the equation of the curve by y, and assuming $a_1 \neq 0$, and making x and y both approach zero, we find $b_1 = 0$. Hence, the equation of the curve now being

$$a_1 x + a_2 x^2 + b_2 xy + c_2 y^2 + \cdots\cdots = 0$$

we see that the theorem is still true in this case.

Illus. : If the equation of a curve be $x^2 - y^2 + x^3 + 3x^2 y - y^3 = 0$, the tangents at the origin are given by $x^2 - y^2 = 0$, *i.e.*, $x + y = 0$ and $x - y = 0$.

10·5. Equation of the normal.

Def. *The normal at any point of a curve is the straight line through that point drawn perpendicular to the tangent at that point.*

Let any line (not parallel to the co-ordinate axes) through the point (x, y) be

$$Y - y = m(X - x).$$

This will be perpendicular to the tangent (not parallel to the co-ordinate axes) to the curve $y = f(x)$ at (x, y).

i.e., to $Y-y=\dfrac{dy}{dx}(X-x)$, if $m\cdot\dfrac{dy}{dx}=-1$, *i.e.*, if $m=-1\Big/\dfrac{dy}{dx}$.

Substituting this value of m in the above equation, we see that *the normal to the curve* $y=f(x)$ *at* (x, y) (when not parallel to the co-ordinate axes) is

$$\frac{\mathbf{dy}}{\mathbf{dx}}(\mathbf{Y}-\mathbf{y})+(\mathbf{X}-\mathbf{x})=\mathbf{0}. \qquad \cdots \quad (1)$$

Similarly, if the equation of the curve is $f(x, y)=0$, the equation of the normal at (x, y) is

$$\frac{\mathbf{X}-\mathbf{x}}{\mathbf{f_x}}=\frac{\mathbf{Y}-\mathbf{y}}{\mathbf{f_y}} \qquad \cdots \qquad \cdots \quad (2)$$

and if the equation of the curve is $x=\phi(t)$, $y=\psi(t)$, the equation of the normal at the point 't' is

$$\phi'(t)\,X+\psi'(t)\,Y=\phi(t)\,\phi'(t)+\psi(t)\,\psi'(t). \qquad \cdots \quad (3)$$

Note 1. When the tangents are parallel to OX and OY, the normals are $X=x$ and $Y=y$ respectively.

Note 2. The *positive direction of the normal* makes an angle $+\frac{1}{2}\pi$ with the tangent, or $\frac{1}{2}\pi+\psi$ with the x-axis.

10·6. Angle of intersection of two curves.

The *angle of intersection of two curves* is the angle between the tangents to the two curves at their common point of intersection.

Suppose the two curves $f(x, y)=0$, $\phi(x, y)=0$ intersect at the point (x, y).

The tangents to the curves at (x, y) are

$$Xf_x+Yf_y-(xf_x+yf_y)=0, \qquad [\text{ by § } \mathit{10{\cdot}2(2)}\ ]$$
$$X\phi_x+Y\phi_y-(x\phi_x+y\phi_y)=0.$$

The angle α at which these lines cut is given by

$$\tan\alpha=\frac{f_x\phi_y \sim \phi_x f_y}{f_x\phi_x+f_y\phi_y}.$$

Hence, *if the curves touch at* (x, y), $\alpha=0$, *i.e.*, $\tan\alpha=0$,

$$i.e., \quad f_x\phi_y=\phi_x f_y, \quad i.e., \; f_x/\phi_x=f_y/\phi_y,$$

and *if they cut orthogonally at* (x, y), $\alpha=\frac{1}{2}\pi$, *i.e.*, $\cot\alpha=0$,

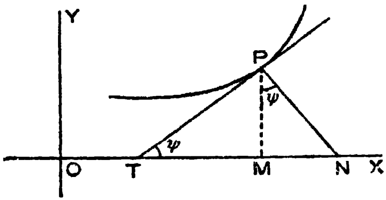
$$i.e., \quad f_x\phi_x+f_y\phi_y=0.$$

Note. If the equations of the curves are given in the forms $y=f(x)$, $y=\phi(x)$, the angle of their intersection is given by $\tan^{-1}\dfrac{f'(x)\sim\phi'(x)}{1+f'(x)\phi'(x)}$.

Hence, the curves cut orthogonally if $f'(x)\phi'(x)=-1$.

10·7. Cartesian Subtangent and Subnormal.

Let the tangent and normal at any point $P(x, y)$ on a curve meet the x-axis in T and N respectively and let PM be drawn perpendicular to OX.



Then, TM is called the *subtangent*, and MN, the *subnormal* at P.

In the right-angled Δ^s PTM, PNM,

since, $\quad \angle NPM=\angle PTM=\psi$, and $PM=y$,

$$\textit{Subtangent}=TM=y\cot\psi=\mathbf{y}\bigg/\left(\frac{\mathbf{dy}}{\mathbf{dx}}\right).$$

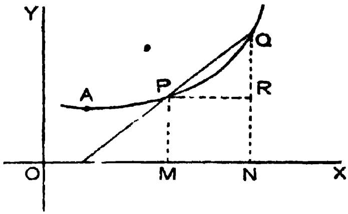
$$\textit{Subnormal}=MN=y\tan\psi=\mathbf{y}\,\frac{\mathbf{dy}}{\mathbf{dx}}.$$

Note. PT and PN are often called as the *length of the tangent* and the *length of the normal* (or sometimes simply, tangent and normal) respectively. Thus, from $\triangle^s$ PTM and PNM,

$$PT = y \operatorname{cosec} \psi = y\sqrt{1+\cot^2\psi} = y\sqrt{1+(1/y_1)^2} = (y\sqrt{1+y_1^2})/y_1.$$
$$PN = y \sec \psi = y\sqrt{1+\tan^2\psi} = y\sqrt{1+y_1^2}.$$

10·8. Derivative of arc-length (Cartesian).

Let $P(x, y)$ be the given point, and $Q(x+\Delta x, y+\Delta y)$ be any point near P on the curve.



Let s denote the length of the arc AP measured from a fixed point A on the curve, and let $s+\Delta s$ denote the arc AQ, so that arc $PQ = \Delta s$. Here, s is obviously a function of x, and hence of y. We shall *assume* the fundamental limit

$$\underset{P \to Q}{Lt} \frac{\text{chord } PQ}{\text{arc } PQ} = 1.*$$

From the figure, $(\text{chord } PQ)^2 = PR^2 + QR^2 = (\Delta x)^2 + (\Delta y)^2.$

$$\therefore \quad \left(\frac{\text{chord } PQ}{\Delta s}\right)^2 \cdot \left(\frac{\Delta s}{\Delta x}\right)^2 = 1 + \left(\frac{\Delta y}{\Delta x}\right)^2.$$

Now, let $Q \to P$ as a limiting position ; then $\Delta x \to 0$ and we have

$$\left(\frac{ds}{dx}\right)^2 = 1 + \left(\frac{dy}{dx}\right)^2, \quad \cdots \quad \cdots \quad (1)$$

$$\text{or,} \quad \frac{\mathbf{ds}}{\mathbf{dx}} = \sqrt{1+\left(\frac{\mathbf{dy}}{\mathbf{dx}}\right)^2} \quad \cdots \quad \cdots \quad (2)$$

---

* For *proof* see *Appendix.*

Since, $\dfrac{ds}{dy}=\dfrac{ds}{dx}\cdot\dfrac{dx}{dy}$, we get on multiplying both sides of (2) by $\dfrac{dx}{dy}$,

$$\frac{\mathbf{ds}}{\mathbf{dy}}=\sqrt{1+\left(\frac{\mathbf{dx}}{\mathbf{dy}}\right)^2}. \quad \cdots \quad \cdots \quad (3)$$

Cor. Multiplying both sides of (1), (2) and (3) by dx^2, dx, dy, we get the corresponding differential form

$$\mathbf{ds^2=dx^2+dy^2}$$

$$ds=\sqrt{1+\left(\frac{dy}{dx}\right)^2}\cdot dx\ ;\ ds=\sqrt{1+\left(\frac{dx}{dy}\right)^2}\cdot dy.$$

10·9. Values of sin ψ, cos ψ.

From $\triangle PQR$ [See *Fig*, § *10·8*] sin $QPR=\dfrac{RQ}{PQ}=\dfrac{\Delta y}{\Delta s}\cdot\dfrac{\Delta s}{PQ}$.

In the limiting position when $Q \to P$, the secant PQ becomes the tangent at P, $\angle QPR \to \psi$ and $\Delta s \to 0$ and $\Delta s/PQ=(\text{arc } PQ)/(\text{chord } PQ) \to 1$.

$$\therefore \quad \sin\psi=\underset{\Delta s\to 0}{Lt}\frac{\Delta y}{\Delta s}=\frac{\mathbf{dy}}{\mathbf{ds}}. \quad \cdots \quad \cdots \quad (1)$$

Similarly, $\cos\psi=\underset{\Delta s\to 0}{Lt}\dfrac{\Delta x}{\Delta s}=\dfrac{\mathbf{dx}}{\mathbf{ds}}$. $\quad\cdots \quad \cdots \quad (2)$

Since $\tan\psi=\dfrac{dy}{dx}$ and $\cot\psi=\dfrac{dx}{dy}$, we get from (2) and (3) of Art. 10·8, $\quad \dfrac{ds}{dx}=\sec\psi,\ \dfrac{ds}{dy}=\operatorname{cosec}\psi,$

whence also cos ψ, sin ψ are obtained.

$$\therefore \quad \left(\frac{\mathbf{dx}}{\mathbf{ds}}\right)^2+\left(\frac{\mathbf{dy}}{\mathbf{ds}}\right)^2=\mathbf{1}. \quad \cdots \quad \cdots \quad (3)$$

Cor. If $x=\phi(t)$, $y=\psi(t)$, $\dfrac{dx}{dt}=\dfrac{dx}{ds}\cdot\dfrac{ds}{dt}$; $\dfrac{dy}{dt}=\dfrac{dy}{ds}\cdot\dfrac{ds}{dt}$.

$$\therefore \quad \left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2=\left\{\left(\frac{dx}{ds}\right)^2+\left(\frac{dy}{ds}\right)^2\right\}\left(\frac{ds}{dt}\right)^2.$$

$$\therefore \quad \left(\frac{\mathbf{dx}}{\mathbf{dt}}\right)^2+\left(\frac{\mathbf{dy}}{\mathbf{dt}}\right)^2=\left(\frac{\mathbf{ds}}{\mathbf{dt}}\right)^2. \qquad \cdots \quad (4)$$

Note. Relations (2) and (3) of Art. 10·8 can also be deduced from the values of $\sin\psi$, $\cos\psi$, $\tan\psi$.

10·10. Illustrative Examples.

Ex. 1. *Find the equation of the tangent at* (x, y) *to the curve* $(x/a)^{\frac{2}{3}}+(y/b)^{\frac{2}{3}}=1$.

Here the equation of the curve is $f(x, y)\equiv(x/a)^{\frac{2}{3}}+(y/b)^{\frac{2}{3}}-1=0$.

The equation of the tangent is

$$(X-x)\,f_x+(Y-y)\,f_y=0,$$

i.e., $(X-x)\cdot\frac{2}{3}x^{-\frac{1}{3}}/a^{\frac{2}{3}}+(Y-y)\cdot\frac{2}{3}y^{-\frac{1}{3}}/b^{\frac{2}{3}}=0,$

i.e., $Xx^{-\frac{1}{3}}/a^{\frac{2}{3}}+Yy^{-\frac{1}{3}}/b^{\frac{2}{3}}=(x/a)^{\frac{2}{3}}+(y/b)^{\frac{2}{3}},$

i.e., $Xx^{-\frac{1}{3}}/a^{\frac{2}{3}}+Yy^{-\frac{1}{3}}/b^{\frac{2}{3}}=1.$

Note. The equation of the tangent should be simplified as much as possible as in the above example.

Ex. 2. *Find the angle of intersection of the curves* $x^2-y^2=a^2$ *and* $x^2+y^2=a^2\sqrt{2}$. [*Patna, 1940*]

Adding and subtracting the equations of the two curves, we find their common points of intersection given by $2x^2=a^2(\sqrt{2}+1)$, *i.e.*, $x=\pm a\sqrt{(\sqrt{2}+1)}/\sqrt{2}$ and $2y^2=a^2(\sqrt{2}-1)$, *i.e.*, $y=\pm a\sqrt{(\sqrt{2}-1)}/\sqrt{2}$.

Since the equations of the curves can be written as

$$f(x, y)\equiv x^2-y^2-a^2=0 \text{ and } \phi(x, y)\equiv x^2+y^2-a^2\sqrt{2}=0,$$

hence if α be the angle of intersection of the curves at (x, y), we have by Art. 10·6,

$$\tan\alpha=\frac{2x\,.\,2y-(2x)(-2y)}{2x\,.\,2x+(-2y)(2y)}=\frac{\pm 2xy}{x^2-y^2}=1,$$

on substituting the values of x and y found above. Hence, $\alpha=\frac{1}{4}\pi$.

$\therefore$ the curves intersect at an angle of 45°.

Ex. 3. *Find the condition that the conics*

$$ax^2+by^2=1 \text{ and } a_1x^2+b_1y^2=1$$

shall cut orthogonally.

The equations of the conics are

$$f(x, y) \equiv ax^2+by^2-1=0, \qquad \cdots \qquad \cdots \quad (1)$$

$$\phi(x, y) \equiv a_1x^2+b_1y^2-1=0. \qquad \cdots \qquad \cdots \quad (2)$$

Now, the condition that they should cut orthogonally at (x, y) is by § 10·6,

$$f_x\phi_x+f_y\phi_y=0,$$

$$i.e., \; 2ax.2a_1x+2by.2b_1y=0,$$

$$i.e., \; aa_1x^2+bb_1y^2=0. \qquad \cdots \qquad \cdots \quad (3)$$

Since the point (x, y) is common to both (1) and (2), the required condition is obtained by eliminating (x, y) from (1), (2) and (3).

Subtracting (2) from (1), $(a-a_1)\,x^2+(b-b_1)\,y^2=0. \qquad \cdots \quad (4)$

Comparing (3) and (4), we get

$$\frac{a-a_1}{aa_1}=\frac{b-b_1}{bb_1}, \text{ or, } \frac{1}{a_1}-\frac{1}{a}=\frac{1}{b_1}-\frac{1}{b}$$

which is the required condition.

Ex. 4. *If $x \cos \alpha+y \sin \alpha=p$ touch the curve*

$$\frac{x^m}{a^m}+\frac{y^m}{b^m}=1,$$

show that $(a \cos \alpha)^{\frac{m}{m-1}}+(b \sin \alpha)^{\frac{m}{m-1}}=p^{\frac{m}{m-1}}$. [C. P. 1939]

The equation of the tangent to the given curve at (x, y) by formula (2) of Art. 10·2 is

$$(X-x)\cdot\frac{mx^{m-1}}{a^m}+(Y-y)\cdot\frac{my^{m-1}}{b^m}=0,$$

$$i.e., \; Xx^{m-1}/a^m+Yy^{m-1}/b^m=x^m/a^m+y^m/b^m=1. \qquad \cdots \quad (1)$$

If $X \cos \alpha+Y \sin \alpha=p \cdots (2)$ touch the given curve, equations (1) and (2) must be identical.

Hence, $$\frac{x^{m-1}/a^m}{\cos\alpha}=\frac{y^{m-1}/b^m}{\sin\alpha}=\frac{1}{p},$$

i.e., $$\frac{x^{m-1}/a^{m-1}}{a\cos\alpha}=\frac{y^{m-1}/b^{m-1}}{b\sin\alpha}=\frac{1}{p}.$$

$$\therefore\quad \left(\frac{x}{a}\right)^{m-1}=\frac{a\cos\alpha}{p},\ \left(\frac{y}{b}\right)^{m-1}=\frac{b\sin\alpha}{p}.$$

$$\therefore\quad \left(\frac{a\cos\alpha}{p}\right)^{\frac{m}{m-1}}+\left(\frac{b\sin\alpha}{p}\right)^{\frac{m}{m-1}}=\left(\frac{x}{a}\right)^m+\left(\frac{y}{b}\right)^m=1,$$

i.e., $$(a\cos\alpha)^{\frac{m}{m-1}}+(b\sin\alpha)^{\frac{m}{m-1}}=p^{\frac{m}{m-1}}.$$

Ex. 5. *If x_1, y_1 be the parts of the axes of x and y intercepted by the tangent at any point (x, y) to the curve $(x/a)^{\frac{2}{3}}+(y/b)^{\frac{2}{3}}=1$, show that $x_1^2/a^2+y_1^2/b^2=1$.* [*C. P. 1941*]

The equation of the tangent at (x, y) to the given curve is, as in Ex. 1,

$$Xx^{-\frac{1}{3}}/a^{\frac{2}{3}}+Yy^{-\frac{1}{3}}/b^{\frac{2}{3}}=1.$$

Where it meets the x-axis, $Y=0$; hence $X=a^{\frac{2}{3}}x^{\frac{1}{3}}$, *i.e.*, $x_1=a^{\frac{2}{3}}x^{\frac{1}{3}}$; and where it meets the y-axis, $X=0$, hence $Y=b^{\frac{2}{3}}y^{\frac{1}{3}}$, *i.e.*, $y_1=b^{\frac{2}{3}}y^{\frac{1}{3}}$.

$$\therefore\ x_1^2/a^2+y_1^2/b^2=a^{\frac{4}{3}}x^{\frac{2}{3}}/a^2+b^{\frac{4}{3}}y^{\frac{2}{3}}/b^2=(x/a)^{\frac{2}{3}}+(y/b)^{\frac{2}{3}}=1.$$

Examples X(A)

1. Find the equation of the tangent at the point (x, y) on each of the following curves :

(i) $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1.$ (ii) $\dfrac{x^m}{a^m}+\dfrac{y^m}{b^m}=1.$

(iii) $x^{\frac{2}{3}}+y^{\frac{2}{3}}=a^{\frac{2}{3}}.$ (iv) $x^3-3axy+y^3=0.$

(v) $(x^2+y^2)^2=a^2(x^2-y^2).$

2. (i) Find the equation of the tangent at the point θ on each of the following curves :

(a) $x = a \cos \theta,\ y = b \sin \theta.$

(b) $x = a \cos^3\theta,\ y = b \sin^3\theta.$

(c) $x = a\ (\theta + \sin \theta),\ y = a\ (1 - \cos \theta).$

(ii) Find the equation of the normal at 't' on the curve

$$x = a\ (2 \cos t + \cos 2t),\ y = a\ (2 \sin t - \sin 2t).$$

3. (i) Find the tangent at the point $(1, \ -1)$ to the curve

$$x^3 + xy^2 - 3x^2 + 4x + 5y + 2 = 0.$$

(ii) Show that the tangent at (a, b) to the curve $(x/a)^3 + (y/b)^3 = 2$ is $x/a + y/b = 2.$

[*C. P. 1943*]

(iii) Show that the normal at the point $\theta = \frac{1}{4}\pi$ on the curve $x = 3 \cos \theta - \cos^3\theta,\ \ y = 3 \sin \theta - \sin^3\theta$ passes through the origin.

4. (i) Find the tangent and the normal to the curve

$$y(x-2)(x-3) - x + 7 = 0$$

at the point where it cuts the x-axis.

(ii) Show that of the tangents at the points where the curve $y = (x-1)(x-2)(x-3)$ is met by the x-axis, two are parallel, and the third makes an angle of $135°$ with the x-axis.

(iii) Find the tangent to the curve $xy^2 = 4(4-x)$ at the point where it is cut by the line $y = x$.

5. (i) Find where the tangent is parallel to the x-axis for the curves :

(a) $y = x^3 - 3x^2 - 9x + 15.$ (b) $ax^2 + 2hxy + by^2 = 1.$

(ii) Find where the tangent is perpendicular to the x-axis for the curves :

(a) $y^2 = x^2\ (a - x).$ (b) $ax^2 + 2hxy + by^2 = 1.$

(c) $y = (x-3)^2(x-2).$ [*C. P. 1935*]

(iii) Show that the tangents to the curve

$$3x^2 + 4xy + 5y^2 - 4 = 0$$

at the points in which it is intersected by the lines

$$3x + 2y = 0 \text{ and } 2x + 5y = 0$$

are parallel to the axes of co-ordinates.

(iv) Find at what points on the curve

$$y = 2x^3 - 15x^2 + 34x - 20$$

the tangents are parallel to $y + 2x = 0$.

(v) Find the points on the curve $y = x^2 + 3x + 4$, the tangents at which pass through the origin.

6. Show that the tangent to the curve $x^3 + y^3 = 3axy$ at the point other than the origin, where it meets the parabola $y^2 = ax$, is parallel to the y-axis.

7. Prove that all points of the curve

$$y^2 = 4a\,\{x + a \sin (x/a)\}$$

at which the tangent is parallel to the x-axis lie on a parabola.

8. Tangents are drawn from the origin to the curve $y = \sin x$. Prove that their points of contact lie on

$$x^2 y^2 = x^2 - y^2.$$

9. (i) Show that the curve $(x/a)^n + (y/b)^n = 2$ touches the straight line $x/a + y/b = 2$ at the point (a, b), whatever be the value of n.

(ii) Prove that $x/a + y/b = 1$ touches the curve

$$x/a + \log (y/b) = 0.$$

10. (i) If $lx + my = 1$ touches the curve $(ax)^n + (by)^n = 1$, show that

$$(l/a)^{\frac{n}{n-1}} + (m/b)^{\frac{n}{n-1}} = 1.$$

(ii) If $lx + my = 1$ is normal to the parabola $y^2 = 4ax$, then $al^3 + 2alm^2 = m^2$.

11. Prove that the condition that $x \cos \alpha + y \sin \alpha = p$ should touch $x^m y^n = a^{m+n}$

is $\quad p^{m+n} m^m n^n = (m+n)^{m+n} a^{m+n} \sin^n \alpha \cos^m \alpha.$

12. Find the angles of intersection of the following curves :

(i) $x^2 - y^2 = 2a^2$ and $x^2 + y^2 = 4a^2$.

(ii) $x^2 = 4y$ and $y(x^2 + 4) = 8$.

(iii) $y = x^3$ and $6y = 7 - x^2$.

13. (i) Prove that the curves $\dfrac{x^2}{a} + \dfrac{y^2}{b} = 1$ and $\dfrac{x^2}{a'} + \dfrac{y^2}{b'} = 1$ will cut orthogonally if $a - b = a' - b'$.

(ii) Find the condition that the curves $ax^3 + by^3 = 1$ and $a'x^3 + b'y^3 = 1$ should cut orthogonally.

(iii) Show that the curves $x^3 - 3xy^2 = -2$ and $3x^2 y - y^3 = 2$ cut orthogonally.

14. (i) Prove that the sum of the intercepts of the tangent to the curve $\sqrt{x} + \sqrt{y} = \sqrt{a}$ upon the co-ordinate axes is constant.

(ii) Find the abscissa of the point on the curves (*a*) $ay^2 = x^3$, (*b*) $\sqrt{xy} = a + x$, the normal at which cuts off equal intercepts from the co-ordinate axes.

15. Show that the portion of the tangent at any point on the following curves intercepted between the axes is of constant length.

(i) $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$. [*C. P. 1940*]

(ii) $x = a \cos^3\theta,\ y = a \sin^3\theta$.

***16.** If the tangent at (x_1, y_1) to the curve $x^3 + y^3 = a^3$ meets the curve again in (x_2, y_2), show that

$$x_2/x_1 + y_2/y_1 = -1.$$

17. (i) Show that at any point of the parabola $y^2 = 4ax$, the subnormal is constant and the subtangent varies as the abscissa of the point of contact.

(ii) Show that at any point of the hyperbola $xy = c^2$, the subtangent varies as the abscissa and the subnormal varies as the cube of the ordinate of the point of contact.

18. Prove that the subtangent is of constant length in the curve $\log y = x \log^e a$.

19. Show that for the curve $by^2 = (x+a)^3$, the square of the subtangent varies as the subnormal.

20. Show that at any point of the curve $x^{m+n} = k^{m-n} y^{2n}$, the mth power of the subtangent varies as the nth power of the subnormal.

21. In the curve $x^m y^n = a^{m+n}$, show that the subtangent at any point varies as the abscissa of the point.

22. Show that in any curve, the rectangle contained by the subtangent and subnormal is equal to the square on the corresponding ordinate.

23. Find the lengths of the subtangent, subnormal, tangent and normal of the curves

(i) $x = a(\theta + \sin\theta)$, $y = a(1 - \cos\theta)$ at 'θ'.

(ii) $x = a(\cos t + t \sin t)$, $y = a(\sin t - t \cos t)$ at 't'.

24. Find the value of n so that the subnormal at any point on the curve $xy^n = a^{n+1}$ may be constant.

25. Show that in any curve

$$\frac{\text{Subnormal}}{\text{Subtangent}} = \left(\frac{\text{length of normal}}{\text{length of tangent}}\right)^2.$$

26. Show that the length of the tangent at any point on the following curves is constant :

(i) $x = \sqrt{a^2 - y^2} + \dfrac{a}{2} \log \dfrac{a - \sqrt{a^2 - y^2}}{a + \sqrt{a^2 - y^2}}$.

(ii) $x = a(\cos t + \log \tan \frac{1}{2}t)$, $y = a \sin t$.

(iii) $s = a \log (a/y)$.

27. (i) If p_1 and p_2 be the perpendiculars from the origin on the tangent and normal respectively at any point (x, y) on a curve, then

$$p_1 = x \sin \psi - y \cos \psi$$
$$p_2 = x \cos \psi + y \sin \psi$$

where, as usual, $\tan \psi = dy/dx$.

(ii) If in the above case the curve be $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$, show that $4p_1{}^2 + p_2{}^2 = a^2$.

28. In the curve $x^m y^n = a^{m+n}$, show that the portion of the tangent intercepted between the axes is divided at its point of contact into segments which are in a constant ratio.

29. (i) In the catenary $y = c \cosh (x/c)$ show that the length of the perpendicular from the foot of the ordinate on the tangent is of constant length. [*C. P. 1943*]

(ii) Show that for the catenary $y = c \cosh (x/c)$, the length of the normal at any point is y^2/c.

***30.** Prove that the equation of the tangent to the curve $x = af(t)/\psi(t)$, $y = a\phi(t) / \psi(t)$ may be written in the form

$$\begin{vmatrix} x & y & a \\ f(t) & \phi(t) & \psi(t) \\ f'(t) & \phi'(t) & \psi'(t) \end{vmatrix} = 0.$$

***31.** Find the equation of the tangent at the origin on the curve

$$y = x^2 \sin (1/x) \text{ for } x \neq 0, \; = 0 \text{ for } x = 0.$$

***32.** Show that for the curve $y = x^{\frac{2}{3}}$, the tangent at the origin is $x = 0$, although dy/dx does not exist there.

33. If α and β be the intercepts on the axes of x and y cut off by the tangent to the curve $(x/a)^n + (y/b)^n = 1$, then

$$(a/\alpha)^{\frac{n}{n-1}} + (b/\beta)^{\frac{n}{n-1}} = 1.$$

34. Find $\dfrac{ds}{dx}$ for the following curves :

(i) $y^2 = 4ax$. (ii) $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$.

(iii) $y = \frac{1}{2}a(e^{\frac{x}{a}} + e^{-\frac{x}{a}})$.

(iv) $x = a(1 - \cos\theta)$, $y = a(\theta + \sin\theta)$.

35. If for the ellipse $x^2/a^2 + y^2/b^2 = 1$, $x = a \sin\phi$, show that

$$\frac{ds}{d\phi} = a\sqrt{1 - e^2 \sin^2\phi}.$$

***36.** Two curves are defined as follows :

(i) $x = t^3$,

$y = t^3 \sin(1/t)$ for $t \neq 0$, $= 0$ for $t = 0$;

(ii) $x = 2t + t^2 \sin(1/t)$, $y = t^2 \sin(1/t)$, for $t \neq 0$,

$x = 0$, $y = 0$ for $t = 0$;

show that for the first curve, although dx/dt, dy/dt are *continuous* for $t = 0$, the *curve has no tangent* at the point ; and for the second curve, although dx/dt, dy/dt *are not continuous* for $t = 0$, *the curve has a tangent* at the point.

ANSWERS

1. (i) $\dfrac{Xx}{a^2} + \dfrac{Yy}{b^2} = 1$. (ii) $\dfrac{Xx^{m-1}}{a^m} + \dfrac{Yy^{m-1}}{b^m} = 1$.

(iii) $Xx^{-\frac{1}{3}} + Yy^{-\frac{1}{3}} = a^{\frac{2}{3}}$. (iv) $X(x^2 - ay) + Y(y^2 - ax) = axy$.

(v) $X\{2x(x^2 + y^2) - a^2x\} + Y\{2y(x^2 + y^2) + a^2y\} = a^2(x^2 - y^2)$.

2. (i) (*a*) $\dfrac{X}{a}\cos\theta + \dfrac{Y}{b}\sin\theta = 1$.

(*b*) $bX\sin\theta + aY\cos\theta = ab\sin\theta\cos\theta$.

(*c*) $X\sin\frac{1}{2}\theta - Y\cos\frac{1}{2}\theta = a\theta\sin\frac{1}{2}\theta$.

(ii) $X\cos\frac{1}{2}t - Y\sin\frac{1}{2}t = 3a\cos\frac{3}{2}t$.

3. (i) $2x+3y+1=0$.

4. (i) Tangent $x-20y-7=0$; normal $20x+y-140=0$.

(iii) $x+y-4=0$. **5.** (i) (*a*) $(3,-12)$, $(-1, 20)$.

(*b*) where $ax+hy=0$ intersects the curve.

(ii) (*a*) $(0, 0)$, $(a, 0)$. (*b*) where $hx+by=0$ intersects the curve.

(*c*) No such point exists.

(iv) $(2, 4)$; $(3, 1)$. (v) $(2, 14)$; $(-2, 2)$.

12. (i) $\frac{1}{3}\pi$. (ii) $\tan^{-1} 3$. (iii) $\frac{1}{2}\pi$.

13. (ii) $aa'(b-b')^{\frac{4}{3}}+bb'(a-a')^{\frac{4}{3}}=0$.

14. (ii) (*a*) $\frac{4}{9}a$. (*b*) $\pm a/\sqrt{2}$.

23. (i) $a \sin\theta$, $2a \sin^2 \frac{1}{2}\theta \tan \frac{1}{2}\theta$, $2a \sin \frac{1}{2}\theta$, $2a \sin \frac{1}{2}\theta \tan \frac{1}{2}\theta$.

(ii) $y \cot t$, $y \tan t$, $y \operatorname{cosec} t$, $y \sec t$. **24.** -2. **31.** $y=0$.

34. (i) $\sqrt{\dfrac{a+x}{x}}$. (ii) $\left(\dfrac{a}{x}\right)^{\frac{1}{3}}$. (iii) $\dfrac{y}{a}$. (iv) $\sqrt{\left(\dfrac{2a}{x}\right)}$.

10·11. Angle between Radius Vector and Tangent.

If ϕ be the angle between the tangent and radius vector at any point on the curve $r=f(\theta)$, then

$$\tan\Phi=\frac{r\,d\theta}{dr},\quad \sin\Phi=\frac{r\,d\theta}{ds},\quad \cos\Phi=\frac{dr}{ds}.$$

Let $P(r, \theta)$ be the given point on the curve $r=f(\theta)$, and $Q(r+\Delta r, \theta+\Delta\theta)$ be a point on the curve in the neighbourhood of P. Let QP be the secant through Q, P. Draw PN perpendicular on OQ.

Then, $\angle PON=\Delta\theta$, $PN=r \sin \Delta\theta$, $ON=r \cos \Delta\theta$.

Let ϕ be the angle which the tangent PT at P makes with the radius vector OP, *i.e.*, $\angle OPT=\phi$.

From the right-angled $\triangle PQN$,

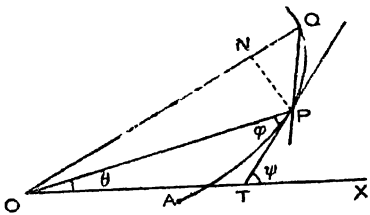
$$\tan PQN = \frac{PN}{NQ} = \frac{PN}{OQ - ON} = \frac{r \sin \Delta\theta}{r + \Delta r - r \cos \Delta\theta}$$

$$= \frac{r \sin \Delta\theta}{r(1 - \cos \Delta\theta) + \Delta r} = \frac{r \sin \Delta\theta}{2r \sin^2 \frac{1}{2}\Delta\theta + \Delta r}$$

$$= \frac{r \,.\, (\sin \Delta\theta / \Delta\theta)}{\frac{1}{2} r \Delta\theta \cdot (\sin \frac{1}{2}\Delta\theta / \frac{1}{2}\Delta\theta)^2 + (\Delta r / \Delta\theta)}.$$

[on dividing both numerator and denominator by $\Delta\theta$]

Now let $Q \to P$, then $\Delta\theta \to 0$, and the secant QP becomes the tangent PT, and $\angle PQN \to \angle OPT$, *i.e.*, ϕ,



$$\therefore \quad \tan \phi = \underset{\Delta\theta \to 0}{Lt} \frac{r \,.\, (\sin \Delta\theta / \Delta\theta)}{\frac{1}{2} r \Delta\theta \, (\sin \frac{1}{2}\Delta\theta / \frac{1}{2}\Delta\theta)^2 + (\Delta r / \Delta\theta)}$$

$$= r \Big/ \frac{dr}{d\theta} \; (i.e., \; = r/r') = r \, \frac{d\theta}{dr}.$$

[since $Lt\ (\sin \Delta\theta / \Delta\theta)$ and $Lt\ (\sin \frac{1}{2}\Delta\theta / \frac{1}{2}\Delta\theta)$ are each equal to 1]

Now let s denote the length of the arc AP measured from a fixed point A on the curve, and let $s + \Delta s$ denote the arc AQ, so that the arc $PQ = \Delta s$. Here s is obviously a function of θ, and hence of r.

$$\sin PQN = \frac{PN}{PQ} = \frac{r \sin \Delta\theta}{\Delta\theta} . \frac{\Delta\theta}{\Delta s} . \frac{\Delta s}{PQ}.$$

Now, let $Q \to P$; then $\Delta\theta \to 0$, $\Delta s \to 0$ and then $\Delta s/PQ = (\text{arc } PQ)/(\text{chord } PQ) \to 1$.

$$\therefore \quad \sin\phi = \underset{\Delta s \to 0}{Lt}\ r\frac{\Delta\theta}{\Delta s} = r\frac{d\theta}{ds}.$$

Again,

$$\cos PQN = \frac{QN}{PQ} = \frac{OQ - ON}{PQ} = \frac{(r+\Delta r) - r\cos\Delta\theta}{PQ}$$

$$= \frac{r(1-\cos\Delta\theta) + \Delta r}{PQ} = \frac{r \,.\, 2\sin^2\frac{1}{2}\Delta\theta + \Delta r}{PQ}$$

$$= \tfrac{1}{2}r\,\Delta\theta\cdot\left(\frac{\sin\frac{1}{2}\Delta\theta}{\frac{1}{2}\Delta\theta}\right)^2 . \frac{\Delta\theta}{\Delta s}.\frac{\Delta s}{PQ} + \frac{\Delta r}{\Delta s}.\frac{\Delta s}{PQ}.$$

Now, let $Q \to P$. Then as before,

$$\cos\phi = \underset{\Delta s \to 0}{Lt}\ \frac{\Delta r}{\Delta s} = \frac{dr}{ds}.$$

Otherwise :

$$\cos\phi = \cot\phi \,.\, \sin\phi = \frac{dr}{r\,d\theta}.\frac{r\,d\theta}{ds} = \frac{dr}{d\theta}.\frac{d\theta}{ds} = \frac{dr}{ds}.$$

Cor. 1. From $\triangle OPT$, $\angle PTX = \angle POT + \angle OPT$.

$$\therefore \quad \Psi = \theta + \Phi.$$

Cor. 2. $\left(\dfrac{\mathbf{dr}}{\mathbf{ds}}\right)^2 + \mathbf{r}^2\left(\dfrac{\mathbf{d\theta}}{\mathbf{ds}}\right)^2 = \cos^2\phi + \sin^2\phi = 1.$

10·12. Derivative of arc-length (*Polar*).

With the notation and the figure of the previous article, we have

$$PQ^2 = PN^2 + QN^2 = (r\sin\Delta\theta)^2 + (r + \Delta r - r\cos\Delta\theta)^2$$
$$= (r\sin\Delta\theta)^2 + (r \,.\, 2\sin^2\tfrac{1}{2}\Delta\theta + \Delta r)^2.$$

Dividing both sides by $\Delta\theta^2$, we get

$$\left(\frac{PQ}{\Delta s}\cdot\frac{\Delta s}{\Delta\theta}\right)^2 = r^2\left(\frac{\sin\Delta\theta}{\Delta\theta}\right)^2 + \left\{\tfrac{1}{2}r\,\Delta\theta\cdot\left(\frac{\sin\frac{1}{2}\Delta\theta}{\frac{1}{2}\Delta\theta}\right)^2 + \left(\frac{\Delta r}{\Delta\theta}\right)\right\}^2.$$

$\therefore$ in the limiting position when $Q \to P$ and $\Delta\theta \to 0$,

$$\left(\frac{ds}{d\theta}\right)^2 = r^2 + \left(\frac{dr}{d\theta}\right)^2 \quad \cdots \quad (1)$$

i.e., $$\frac{ds}{d\theta} = \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}. \quad \cdots \quad (2)$$

Multiplying both sides of (2) by $\frac{d\theta}{dr}$, we get

$$\frac{ds}{dr} = \sqrt{1 + \left(r\frac{d\theta}{dr}\right)^2}. \quad \cdots \quad (3)$$

Cor. Multiplying (1), (2) and (3) by $d\theta^2$, $d\theta$, dr, we get the corresponding differential forms

$$\mathbf{ds^2 = dr^2 + r^2\ d\theta^2.}$$

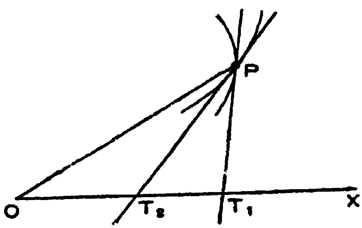
$$ds = \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\ d\theta$$

$$ds = \sqrt{1 + \left(\frac{r\ d\theta}{dr}\right)^2}\ dr.$$

Note. Relations (2) and (3) can also be deduced from the values of $\sin\phi$, $\cos\phi$, $\tan\phi$.

10·13. Angle of intersection of two curves (*Polar*).

Suppose two curves $r = f(\theta)$, $r = \phi(\theta)$ intersect at the



point P, and let PT_1, PT_2 be the tangents at P to the two curves, and let $\angle OPT_1 = \phi_1$, $\angle OPT_2 = \phi_2$.

Then if α be the angle between the two curves,

$$\alpha=\phi_1-\phi_2.$$

$$\therefore \quad \tan\alpha=\frac{\tan\phi_1-\tan\phi_2}{1+\tan\phi_1\tan\phi_2}.$$

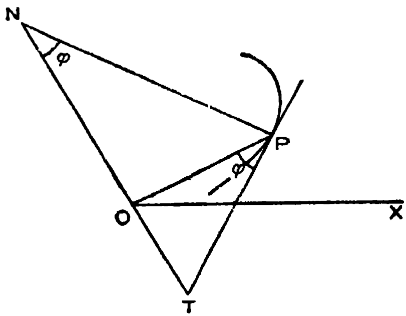
Since, $\tan\phi_1=r/r'=f(\theta)/f'(\theta)$

and $\tan\phi_2=r/r'=\phi(\theta)/\phi'(\theta)$, we get

$$\tan\alpha=\frac{f(\theta)\phi'(\theta)-f'(\theta)\phi(\theta)}{f'(\theta)\phi'(\theta)+f(\theta)\phi(\theta)}.$$

10·14. Polar Subtangent and Subnormal.

Let P be any point on the curve $r=f(\theta)$, and let the



tangent PT and normal PN at P meet the line drawn through the pole O perpendicular to the radius vector OP, in T and N respectively.

Then OT is called the *polar subtangent*

and ON is called the *polar subnormal.*

Since, $\angle OPT=\phi$, $OT=OP\tan\phi=r.r\dfrac{d\theta}{dr}$.

$\therefore$ **polar subtangent** $=\mathbf{r^2\dfrac{d\theta}{dr}}$.

Again, $ON = OP \tan OPN = r \cot \phi = r \cdot \dfrac{dr}{r\,d\theta}$.

$\therefore$ **polar subnormal** $= \dfrac{\mathbf{dr}}{\mathbf{d\theta}}$.

Note. If $u = \dfrac{1}{r}$, $\dfrac{du}{d\theta} = -\dfrac{1}{r^2}\dfrac{dr}{d\theta}$,

$\therefore$ *polar subtangent* $= r^2 \dfrac{d\theta}{dr} = -\dfrac{d\theta}{du}$.

10·15. Perpendicular from pole on Tangent.

Let p be the length of the perpendicular ON from the pole O on the tangent PT at any point P.

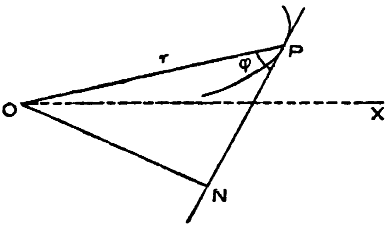
Then from $\triangle OPN$, $ON = OP \sin \phi$.

$$\therefore \quad \mathbf{p = r \sin \Phi}. \qquad \cdots \qquad \cdots \quad (1)$$

Again, $\dfrac{1}{p^2} = \dfrac{1}{r^2}\operatorname{cosec}^2\phi = \dfrac{1}{r^2}(1 + \cot^2\phi)$

$$= \frac{1}{r^2}\left\{1 + \frac{1}{r^2}\left(\frac{dr}{d\theta}\right)^2\right\}.$$

$$\therefore \quad \mathbf{\frac{1}{p^2} = \frac{1}{r^2} + \frac{1}{r^4}\left(\frac{dr}{d\theta}\right)^2}. \qquad \cdots \qquad \cdots \quad (2)$$



The symbol u is generally used to denote $1/r$, the reciprocal of the radius vector.

$$\therefore \quad \frac{du}{d\theta} = -\frac{1}{r^2}\frac{dr}{d\theta}.$$

Hence, the relation (2) becomes

$$\mathbf{\frac{1}{p^2} = u^2 + \left(\frac{du}{d\theta}\right)^2}. \qquad \cdots \qquad \cdots \quad (3)$$

10·16. The (p, r) or Pedal equation of a curve.

The relation between the perpendicular (p) on the tangent at any point P on a curve and the radius vector (r) of the point of contact P from some given point O, is called the *(p, r) or pedal equation of the curve* with regard to O. Such equations are found very useful in the application of the principles of Statics and Dynamics.

(*i*) *Pedal equation deduced from Cartesian equation.*

Let us take the origin at the point with regard to which the pedal equation is to be obtained, and let $f(x, y)=0$ be the equation of the curve.

The tangent at (x, y) is

$$Xf_x + Yf_y - (xf_x + yf_y) = 0.$$

If p be the perpendicular from the origin on it,

$$p^2 = \frac{(xf_x + yf_y)^2}{f_x{}^2 + f_y{}^2}. \quad \cdots \quad \cdots \quad (1)$$

Also, $$r^2 = x^2 + y^2, \quad \cdots \quad \cdots \quad (2)$$

and $$f(x, y) = 0. \quad \cdots \quad \cdots \quad (3)$$

If x and y be eliminated between (1), (2) and (3), the required pedal equation is obtained.

(*ii*) *Pedal equation deduced from polar equation.*

Let us take the pole at the point with regard to which the pedal equation is to be obtained, and let $f(r, \theta)=0$ be the equation of the curve.

Let p be the perpendicular from the origin on the tangent at (r, θ); then

$$f(r, \theta) = 0, \quad \cdots \quad \cdots \quad (1)$$

$$\tan\phi = \frac{r\,d\theta}{dr}. \quad \cdots \quad (2)$$

$$p = r\sin\phi. \quad \cdots \quad \cdots \quad (3)$$

If θ and ϕ be eliminated between (1), (2) and (3), the required pedal equation is obtained.

Note 1. When in any case nothing is mentioned about the given point with regard to which the pedal equation is to be obtained, the given point is to be taken as the origin in the Cartesian system and the pole in the Polar system.

Note 2. In some elementary cases, pedal equations can be easily obtained from geometrical properties. [See *Ex. 6, Art. 10·17*]

10·17. Illustrative Examples.

Ex. 1. *Obtain the values of $\sin\phi$, $\cos\phi$, $\tan\phi$ and arc-differential in polar co-ordinates by transformation from Cartesian system.*

Since, $x = r\cos\theta,\ y = r\sin\theta,$

$\therefore\quad dx = \cos\theta\, dr - r\sin\theta\, d\theta,$

and $\quad dy = \sin\theta\, dr + r\cos\theta\, d\theta.$

$\therefore\quad dx^2 + dy^2 = dr^2 + r^2\, d\theta^2,$

i.e., $\quad ds^2 = dr^2 + r^2\, d\theta^2. \qquad \cdots \qquad \cdots \quad (1)$

Also, $\quad x\, dy - y\, dx = r^2\, d\theta. \qquad \cdots \qquad \cdots \quad (2)$

Again, since $x^2 + y^2 = r^2$, $\therefore\ x\, dx + y\, dy = r\, dr. \qquad \cdots \quad (3)$

Now, $\quad \psi = \theta + \phi. \quad \therefore\ \phi = \psi - \theta.$

$\therefore\quad \cos\phi = \cos\psi\cos\theta + \sin\psi\sin\theta$

$$= \frac{dx}{ds}\cdot\frac{x}{r} + \frac{dy}{ds}\cdot\frac{y}{r} = \frac{x\, dx + y\, dy}{r\, ds} = \frac{dr}{ds}. \qquad [\ by\ (3)\ ]$$

Again, $\sin\phi = \sin\psi\cos\theta - \cos\psi\sin\theta$

$$= \frac{dy}{ds}\cdot\frac{x}{r} - \frac{dx}{ds}\cdot\frac{y}{r} = \frac{x\, dy - y\, dx}{r\, ds} = \frac{r\, d\theta}{ds}. \qquad [\ by\ (2)\ ]$$

$$\tan\phi = \sin\phi \div \cos\phi = \frac{r\, d\theta}{ds} \div \frac{dr}{ds} = \frac{r\, d\theta}{dr}.$$

Ex. 2. *Find the angle of intersection of the curves*

$$r = \sin\theta + \cos\theta \text{ and } r = 2\sin\theta.$$

Here, $\tan\phi_1 = \dfrac{r}{r'} = \dfrac{\cos\theta + \sin\theta}{\cos\theta - \sin\theta} = \dfrac{1 + \tan\theta}{1 - \tan\theta} = \tan\left(\tfrac{1}{4}\pi + \theta\right).$

$\tan\phi_2 = \dfrac{r}{r'} = \dfrac{2\sin\theta}{2\cos\theta} = \tan\theta.$

$\therefore\ \phi_1 = \tfrac{1}{4}\pi + \theta,\ \phi_2 = \theta.$

$\therefore$ angle of intersection $= \phi_1 - \phi_2 = \tfrac{1}{4}\pi.$

Ex. 3. *Prove that the curves*

$$r^n = a^n \cos n\theta \text{ and } r^n = b^n \sin n\theta$$

cut orthogonally.

Taking logarithm of the 1st equation

$$n \log r = n \log a + \log \cos n\theta.$$

Differentiating with respect to θ,

$$n \cdot \frac{1}{r}\frac{dr}{d\theta} = -\frac{n \sin n\theta}{\cos n\theta}.$$

$$\therefore \cot \phi_1 = -\tan n\theta = \cot (\tfrac{1}{2}\pi + n\theta).$$

Similarly, from the 2nd equation, we get

$$\cot \phi_2 = \cot n\theta.$$

$$\therefore \phi_1 = \tfrac{1}{2}\pi + n\theta\ ;\ \phi_2 = n\theta.$$

$\therefore$ angle of intersection $\phi_1 - \phi_2 = \frac{1}{2}\pi$.

Ex. 4. *Find the pedal equation of the parabola $y^2 = 4ax$ with regard to its vertex.*

Differentiating the given equation, $yy_1 = 2a$. $\therefore$ $y_1 = 2a/y$.

$\therefore$ the tangent at (x, y) is

$$Y - y = (2a/y)(X - x),$$

i.e., $2aX - yY + 2ax = 0$, $\quad (\because y^2 = 4ax)$

$$\therefore \quad p^2 = \frac{4a^2x^2}{4a^2 + y^2} = \frac{4a^2x^2}{4a^2 + 4ax} = \frac{ax^2}{x + a} \quad \cdots \quad \cdots \quad (1)$$

and $r^2 = x^2 + y^2 = x^2 + 4ax$. $\quad \cdots \quad \cdots \quad (2)$

From (1) and (2),

$$ax^2 - p^2x - ap^2 = 0 \quad \cdots \quad \cdots \quad (3)$$

$$x^2 + 4ax - r^2 = 0. \quad \cdots \quad \cdots \quad (4)$$

By eliminating x between (3) and (4), the required relation between p and r will be obtained.

By cross-multiplication,

$$\frac{x^2}{p^2r^2 + 4a^2p^2} = \frac{x}{ar^2 - ap^2} = \frac{1}{4a^2 + p^2}.$$

$$\therefore \quad (p^2r^2 + 4a^2p^2)(4a^2 + p^2) = (ar^2 - ap^2)^2$$

is the required pedal equation.

Ex. 5. *Find the pedal equation of $r^m = a^m \cos m\theta$.*

Taking logarithm of the given equation

$$m \log r = m \log a + \log \cos m\theta.$$

Differentiating with respect to θ,

$$m.\frac{1}{r}\frac{dr}{d\theta} = -\frac{m \sin m\theta}{\cos m\theta}.$$

$$\therefore \quad \cot \phi = -\tan m\theta = \cot(\tfrac{1}{2}\pi + m\theta).$$

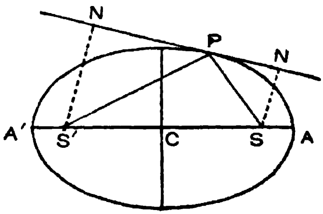
$$\therefore \quad \phi = \tfrac{1}{2}\pi + m\theta.$$

Again, $p = r \sin \phi = r \sin(\tfrac{1}{2}\pi + m\theta) = r \cos m\theta$

$= r \cdot \dfrac{r^m}{a^m}$ from the equation of the curve.

$\therefore \quad r^{m+1} = a^m p$, is the required pedal equation.

Ex. 6. *Find geometrically the pedal equation of the ellipse with respect to a focus.*



SN, $S'N'$ are drawn perpendiculars on the tangent at any point P on the ellipse. $SP = r$, $S'P = r'$, $SN = p$, $S'N' = p'$. We know from Co-ordinate geometry that,

$$r + r' = 2a \text{ and } pp' = b^2.$$

Since, $\angle SPN = \angle S'PN'$, $\therefore$ Δ' SPN, $S'PN'$ are similar.

$$\therefore \quad \frac{r}{p} = \frac{r'}{p'} = \sqrt{\frac{rr'}{pp'}} = \sqrt{\frac{r(2a-r)}{b^2}}.$$

$$\therefore \quad \frac{r^2}{p^2} = \frac{r(2a-r)}{b^2}, \text{ or, } \frac{b^2}{p^2} = \frac{2a}{r} - 1,$$

which is the required pedal equation.

Noting that the semi-latus rectum l of the ellipse $= b^2/a$, the above equation may be written as $\dfrac{l}{p^2} = \dfrac{2}{r} - \dfrac{1}{a}$.

Ex. 7. *Find the geometrical meaning of* $\frac{dp}{d\psi}$, *and hence deduce*

$$\mathbf{r^2 = p^2 + \left(\frac{dp}{d\psi}\right)^2}.$$

We have $p = r \sin \phi$.

Differentiating with respect to ψ,

$$\frac{dp}{d\psi} = r \cos \phi \frac{d\phi}{d\psi} + \sin \phi \frac{dr}{d\psi}$$

$$= r \cos \phi \frac{d\phi}{d\psi} + \cos \phi . r \frac{d\theta}{dr} \cdot \frac{dr}{d\psi} \qquad \left[\because \frac{\sin \phi}{\cos \phi} = \frac{r\, d\theta}{dr}\right]$$

$$= r \cos \phi \frac{d}{d\psi} (\theta + \phi)$$

$$= \mathbf{r \cos \phi} \quad (\because \theta + \phi = \psi)$$

$= PN$ (See *fig.*, § *10·15*)

= projection of the radius vector on the tangent.

From $\triangle OPN$, $OP^2 = ON^2 + PN^2$.

$$\therefore \quad r^2 = p^2 + \left(\frac{dp}{d\psi}\right)^2.$$

Examples X(B)

1. Find $\frac{ds}{d\theta}$ for the following curves :

(i) $r = a(1 + \cos \theta)$. (ii) $r = ae^{\theta \cot \alpha}$.

(iii) $r^2 = a^2 \cos 2\theta$. (iv) $r^n = a^n \cos n\theta$.

2. Find $\frac{ds}{dr}$ for the curves : (i) $r = a\theta$. (ii) $r = a/\theta$.

3. Show that in the equiangular spiral $r = ae^{\theta \cot \alpha}$, the tangent is inclined at a constant angle to the radius vector.

4. Show that for $\log r = a\theta + b$, $p \propto r$.

5. Find ϕ in the terms of θ for the following curves :

(i) Cardioide $r = a(1 - \cos\theta)$.

(ii) Parabola $r = 2a/(1 - \cos\theta)$.

(iii) Hyperbola $r^2 \cos 2\theta = a^2$.

(iv) Lemniscate $r^2 = a^2 \cos 2\theta$.

6. Find the angle of intersection of the following curves :

(i) $r = a \sin 2\theta$, $r = a \cos 2\theta$.

(ii) $r = 6 \cos\theta$, $r = 2(1 + \cos\theta)$.

(iii) $r^2 = 16 \sin 2\theta$, $r^2 \sin 2\theta = 4$.

7. Show that the following curves cut orthogonally :

(i) $r = a(1 + \cos\theta)$, $r = b(1 - \cos\theta)$.

(ii) $r = a/(1 + \cos\theta)$, $r = b/(1 - \cos\theta)$.

8. Show that the curves

$$r^n = a^n \sec(n\theta + \alpha),\ r^n = b^n \sec(n\theta + \beta)$$

intersect at an angle which is independent of a and b.

9. Prove that

$$\tan\phi = \left(x\frac{dy}{dx} - y\right)\Big/\left(x + y\frac{dy}{dx}\right),$$

where ϕ is the angle which the tangent to a curve makes with the radius vector drawn from the origin. [*C. P. 1931*]

[Use $\phi = \psi - \theta$; $\tan\theta = y/x$]

10. Show that for the curve $r\theta = a$, the polar subtangent is constant and for the curve $r = a\theta$, the polar subnormal is constant.

11. Show that for the curve $r=e^{\theta}$, the polar subtangent is equal to the polar subnormal.

12. Find the polar subtangents of

(i) $r=ae^{\theta \cot \alpha}$. (ii) $r=a(1-\cos\theta)$.

(iii) $r=2a/(1-\cos\theta)$. (iv) $r=l/(1+e\cos\theta)$.

13. Show that the locus of the extremity of the polar subtangent of the curve $u+f(\theta)=0$ is $u=f'(\frac{1}{2}\pi+\theta)$.

14. Prove that the locus of the extremity of the polar subnormal of the curve $r=f(\theta)$ is $r=f'(\theta-\frac{1}{2}\pi)$.

Hence deduce that the locus of the extremity of the polar subnormal of the equiangular spiral $r=ae^{\theta \cot \alpha}$ is another equiangular spiral.

15. Show that the pedal equation of the ellipse

$$x^2/a^2+y^2/b^2=1$$

with regard to the centre is $a^2b^2/p^2=a^2+b^2-r^2$.

16. (i) Show that the pedal equation of the astroid $x^{\frac{2}{3}}+y^{\frac{2}{3}}=a^{\frac{2}{3}}$ is $r^2+3p^2=a^2$.

(ii) Show that the pedal equation of the parabola $y^2=4a(x+a)$ is $p^2=ar$. [*C. P. 1931*]

17. Show geometrically that the pedal equation of a circle with regard to a point on the circumference is $pd=r^2$, where d is the diameter of the circle.

18. Show that the pedal equation of

(i) the cardioide $r=a(1+\cos\theta)$ is $r^3=2ap^2$.

(ii) the parabola $r=2a/(1-\cos\theta)$ is $p^2=ar$.

(iii) the hyperbola $r^2\cos 2\theta=a^2$ is $pr=a^2$.

(iv) the lemniscate $r^2 = a^2 \cos 2\theta$ is $r^3 = a^2 p$.

(v) the equiangular spiral $r = ae^{\theta \cot \alpha}$ is $p = r \sin \alpha$.

(vi) the class of curves $r^n = a^n \sin n\theta$ is $r^{n+1} = a^n p$.

(vii) the reciprocal spiral $r\theta = a$ is $p^2 (a^2 + r^2) = a^2 r^2$.

[*C. P. 1938*]

ANSWERS

1. (i) $2a \cos \frac{1}{2}\theta$. (ii) $r \operatorname{cosec} \alpha$. (iii) a^2/r. (iv) $a \sec^{\frac{n-1}{n}} n\theta$.

2. (i) $\sqrt{r^2 + a^2}/a$. (ii) $-\sqrt{r^2 + a^2}/r$.

5. (i) $\frac{1}{2}\theta$. (ii) $\pi - \frac{1}{2}\theta$. (iii) $\frac{1}{2}\pi - 2\theta$. (iv) $\frac{1}{2}\pi + 2\theta$.

6. (i) $\tan^{-1} \frac{4}{3}$. (ii) $\frac{1}{6}\pi$. (iii) $\frac{2}{3}\pi$. **12.** (i) $r \tan \alpha$.

(ii) $(2a \sin^3 \frac{1}{2}\theta)/(\cos \frac{1}{2}\theta)$. (iii) $2a \operatorname{cosec} \theta$. (iv) $l/(e \sin \theta)$.

P as the curve has, and this explains the nomenclature of the above circle. Since the curve and the circle of curvature at any point $P(x, y)$ have the same tangent and the same curvature, hence x, y, *y', y'' have the same values at P for the circle of curvature and the curve.* [See *Art. 11·2*]

11·2. Formulæ for radius of Curvature.

(A) For the **Cartesian equation $y = f(x)$.**

We know $\dfrac{dy}{dx} = \tan\psi$.

$\therefore$ differentiating with respect to x,

$$\frac{d^2y}{dx^2} = \sec^2\psi\frac{d\psi}{dx} = \sec^2\psi\frac{d\psi}{ds}\cdot\frac{ds}{dx}$$

$$= \sec^3\psi\frac{d\psi}{ds}. \qquad \left[\because \frac{dx}{ds} = \cos\psi\right]$$

$$\therefore \quad \rho = \frac{ds}{d\psi} = \sec^3\psi\Big/\frac{d^2y}{dx^2}.$$

Since $\sec\psi = (1+\tan^2\psi)^{\frac{1}{2}} = \left\{1 + \left(\dfrac{dy}{dx}\right)^2\right\}^{\frac{1}{2}}$,

$$\therefore \qquad \rho = \frac{\left\{1 + \left(\dfrac{dy}{dx}\right)^2\right\}^{\frac{3}{2}}}{\dfrac{d^2y}{dx^2}} = \frac{(1+y_1^2)^{\frac{3}{2}}}{y_2} \qquad \cdots \quad \text{(i)}$$

where $y_2 \neq 0$.

Note 1. Making the convention of attaching positive sign to $(1+y_1^2)^{\frac{3}{2}}$, ρ is positive or negative according as y_2 is positive or negative.

Note 2. *The above formula fails* when at any point y_1 becomes infinite, *i.e.*, when the tangent at the point is parallel to the y-axis (For *illustration*, see *Ex. 4*, *§ 11·5*). In such cases the following formula, *for the equation of the curve as* $x=\phi(y)$, would be found useful.

$$\frac{dx}{dy}=\cot\psi\ ; \quad \therefore \text{ differentiating with respect to } y,$$

$$\frac{d^2x}{dy^2}=-\operatorname{cosec}^2\psi\,\frac{d\psi}{dy}=-\operatorname{cosec}^2\psi\cdot\frac{d\psi}{ds}\cdot\frac{ds}{dy}$$

$$=-\operatorname{cosec}^3\psi\cdot\frac{1}{\rho}\cdot \qquad \left[\because\ \frac{dy}{ds}=\sin\psi\right]$$

$$\therefore\quad \rho=-\operatorname{cosec}^3\psi\Big/\frac{d^2x}{dy^2}.$$

Since $\operatorname{cosec}^2\psi=1+\cot^2\psi=1+\left(\frac{dx}{dy}\right)^2$,

$\therefore$ considering the magnitude only of the radius of curvature

$$\rho=\frac{\left\{1+\left(\frac{dx}{dy}\right)^2\right\}^{\frac{3}{2}}}{\frac{d^2x}{dy^2}}=\frac{(1+x_1{}^2)^{\frac{3}{2}}}{x_2} \qquad \cdots \quad \text{(i a)}$$

where $x_2 \neq 0$.

(B) For the **Parametric equation** $x=\phi(t),\ y=\psi(t)$.

Here, $$\frac{dy}{dx}=\frac{dy}{dt}\Big/\frac{dx}{dt}=\frac{y'}{x'} \quad (x'\neq 0)$$

where dashes denote differentiations with respect to t.

$$\therefore\quad \frac{d^2y}{dx^2}=\frac{d}{dx}\left(\frac{y'}{x'}\right)=\frac{d}{dt}\left(\frac{y'}{x'}\right)\cdot\frac{dt}{dx}=\frac{x'y''-y'x''}{x'^3}.$$

Then substituting the values of $\frac{dy}{dx}, \frac{d^2y}{dx^2}$ in the formula (i) above, we get

$$\rho=\frac{(x'^2+y'^2)^{\frac{3}{2}}}{x'y''-y'x''} \qquad \cdots \quad \text{(ii)}$$

where dashes denote differentiations with respect to t, and where $(x'\,y''-y'\,x'')\neq 0$.

(F) For the Tangential polar equation $p=f(\psi)$.

When the tangential polar equation *i.e.*, the relation between p and ψ of a curve is given,

$$\frac{dp}{d\psi}=\frac{dp}{dr}\cdot\frac{dr}{ds}\cdot\frac{ds}{d\psi}=\frac{dp}{dr}\cos\phi.\rho=\frac{dp}{dr}\cdot\cos\phi.r\frac{dr}{dp}$$

$$=r\cos\phi.$$

$$\therefore\quad p^2+\left(\frac{dp}{d\psi}\right)^2=r^2\sin^2\phi+r^2\cos^2\phi=r^2.$$

Differentiating with respect to p,

$$2p+2\frac{dp}{d\psi}\cdot\frac{d^2p}{d\psi^2}\cdot\frac{d\psi}{dp}=2r\frac{dr}{dp}.$$

$$\therefore\quad \boldsymbol{\rho=p+\frac{d^2p}{d\psi^2}}. \qquad \cdots \quad \text{(vi)}$$

Alternative Method :

If p be the length of the perpendicular from the origin on the tangent at (x, y) *viz.*, $Y-y_1X+xy_1-y=0$,

$$\text{then } p=\frac{xy_1-y}{\sqrt{1+y_1^2}}=\frac{x\tan\psi-y}{\sqrt{1+\tan^2\psi}}.$$

$$\therefore\quad p=x\sin\psi-y\cos\psi.$$

$$\therefore\ \frac{dp}{d\psi}=\frac{dx}{d\psi}\sin\psi+x\cos\psi-\frac{dy}{d\psi}\cos\psi+y\sin\psi$$

$$=x\cos\psi+y\sin\psi.$$

$$\left[\text{Since } \frac{dx}{d\psi}=\frac{dx}{ds}\cdot\frac{ds}{d\psi}=\rho\cos\psi\ ;\ \frac{dy}{d\psi}=\frac{dy}{ds}\cdot\frac{ds}{d\psi}=\rho\sin\psi.\right]$$

$$\text{Similarly, } \frac{d^2p}{d\psi^2}=\frac{dx}{d\psi}\cos\psi-x\sin\psi+\frac{dy}{d\psi}\sin\psi+y\cos\psi$$

$$=\rho\cos^2\psi-x\sin\psi+\rho\sin^2\psi+y\cos\psi$$

$$=\rho-(x\sin\psi-y\cos\psi)=\rho-p.$$

Hence, the result follows.

11·3. Curvature at the origin.

(i) *Method of substitution.*

Radius of curvature at the origin can be found by substituting $x=0$, $y=0$ in the value of ρ obtained from Art. 11·2, or by directly substituting the values of $(y_1)_0$ and $(y_2)_0$ in the formula.

(ii) *Method of Expansion.*

In some cases the above method fails, or becomes laborious. In such cases, the values of $(y_1)_0$ and $(y_2)_0$ can be easily obtained in the following way by assuming the equation of the curve to be $y=f(x)$, and writing for y in the given equation its expansion by Maclaurin's theorem *viz.*, $xf'(0)+\dfrac{x^2}{2!}f''(0)+\cdots$ [$f(0)$ being zero here, since the curve passes through the origin] *i.e.*, $px+qx^2/2!+\cdots$, where p, q stand for $f'(0)$, $f''(0)$, *i.e.*, $(y_1)_0$, $(y_2)_0$, and then equating coefficients of like powers of x in the identity obtained.

This is illustrated in Example 9 of Art. 11·5.

(iii) Newton's Formula.

If the curve passes through the origin, and the axis of x is the tangent at the origin, we have

$$x=0,\ y=0,\ (y_1)_0,\ i.e.,\ p=0.$$

$\therefore$ by Maclaurin's Theorem,

$$y=qx^2/2!+\cdots\cdots$$

Dividing by $x^2|2!$ and taking limits as $x\to 0$, we get

$$Lt\ (2y/x^2)=q.$$

It should be noted here that as $x\to 0$, y also $\to 0$.

But from formula of Art. 11·2, at the origin $\rho = \dfrac{(1+p^2)^{\frac{3}{2}}}{q} = \dfrac{1}{q}$.

$$\therefore \quad \rho = \mathop{\mathrm{Lt}}_{\substack{x\to 0\\ y\to 0}} \frac{x^2}{2y}. \qquad \cdots \quad (1)$$

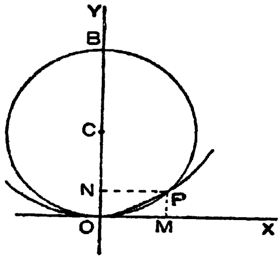
Similarly, *if a curve passes through the origin, and the axis of y is the tangent there,* we have at the origin

$$\rho = \mathop{\mathrm{Lt}}_{\substack{x\to 0\\ y\to 0}} \frac{y^2}{2x}. \qquad \cdots \quad (2)$$

Geometrically :

Let the x-axis be the tangent at the origin.

Draw a circle touching the curve at O, and passing



through a point P (x, y) near O, on the curve. Now when $P \to O$ along the curve, the limiting position of the circle is the circle of curvature. *

Let OB be the diameter of the circle, and draw PN perp. to it, and PM perp. to OX. Let r be the radius of the circle.

Then, $\quad ON.NB = PN^2$, *i.e.*, $ON\,(OB - ON) = PN^2$,

$$\therefore \quad OB = \frac{PN^2}{ON} + ON = \frac{PN^2 + ON^2}{ON} = \frac{OP^2}{ON},$$

i.e., $$2r = \frac{OP^2}{PM} = \frac{x^2 + y^2}{y} = \frac{x^2}{y} + y.$$

* See *Appendix.*

In the limit when $P \to O$, $x \to 0$, $y \to 0$, $r \to \rho$, and hence we get as before

$$\rho = \frac{1}{2} Lt \frac{x^2}{y}.$$

Similarly, *when y-axis is the tangent at the origin,*

we obtain $\rho = \frac{1}{2} Lt \frac{y^2}{x}$.

Analytically :

The equation of the circle passing through the origin and having the x-axis as the tangent at the origin is

$$x^2 + y^2 - 2fy = 0. \qquad \cdots \quad (1)$$

If r be the radius of the circle, $r = f$.

Since (i) passes through the point (x, y) on the curve,

$\therefore \quad x^2 + y^2 - 2fy = 0$ whence $f = (x^2 + y^2)/2y$.

$\therefore \quad \rho = Lt\, r = Lt\, f = Lt \frac{x^2 + y^2}{2y} = Lt \frac{x^2}{2y}$.

General Case :

If $ax + by = 0$ be the tangent at the origin, then, proceeding as above, we get

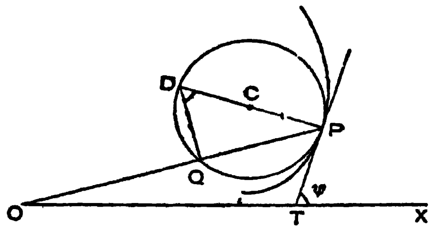
$$OB = \frac{OP^2}{PM} = \frac{x^2 + y^2}{(ax + by)/\sqrt{(a^2 + b^2)}},$$

$$\therefore \quad \rho = \tfrac{1}{2}\sqrt{a^2 + b^2} \cdot \underset{\substack{x \to 0 \\ y \to 0}}{Lt} \frac{x^2 + y^2}{ax + by}. \qquad \cdots \quad (3)$$

Note. It should be noted that as $x \to 0$, $y \to 0$, $y/x \to \left(-\frac{a}{b}\right)$, the '$m$' of the tangent line $ax + by = 0$. Here, it is supposed that $a \neq 0$, $b \neq 0$.

11·4. Chord of curvature through the origin (pole).

Let PQ be a chord passing through the origin O, of



the circle of curvature at P on the given curve, and let C be the centre of curvature and PT be tangent at P.

Join PC, produce it to D ; join DQ.

Then $\angle PQD =$ a rt. $\angle$, being in a semi-circle.

$$\angle OPT = \phi \quad \text{and} \quad \angle PTX = \psi.$$

From $\triangle PQD$, **chord PQ** $= PD \cos DPQ$

$$= 2\rho \cos (\tfrac{1}{2}\pi - \phi)$$

$$= \mathbf{2\rho \sin \phi}$$

$$= 2.r \frac{dr}{dp} \cdot \frac{p}{r}$$

$$= \mathbf{2p \frac{dr}{dp}}.$$

Note 1. From above it is clear that the *chord of curvature through the origin* can be easily obtained when the pedal equation of the curve is given.

Note 2. If the chord PQ, without passing through the origin, makes an angle a with the tangent PT, *i.e.*, $\angle QPT = a$, then obviously $\angle PDQ = a$, and hence

$$PQ = 2\rho \sin a.$$

Hence, *the chord of curvature parallel to the x-axis is* $2\rho \sin \psi$

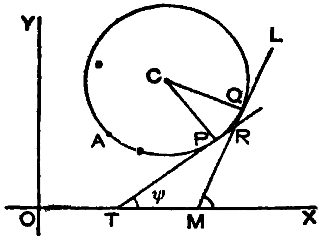
$$(\because \text{ here } \angle PDQ = \psi)$$

and *the chord of curvature parallel to the y-axis is* $2\rho \cos \psi$.

$$(\because \text{ here } \angle PDQ = \tfrac{1}{2}\pi - \psi)$$

11·5. Illustrative Examples.

Ex. 1. *Show that the circle is a curve of uniform curvature and its radius of curvature at every point is constant, being equal to the radius of the circle.*



Let C be the centre of a circle of radius a. Let P be the given point, Q a point near it, and let PT, QM be tangents at P, Q and let $\angle PTX = \psi$, $\angle QMX = \psi + \Delta\psi$; join CP, CQ.

$\therefore \quad \angle PCQ = \angle PRM = \Delta\psi.$

$\therefore \quad \dfrac{\Delta\psi}{\Delta s} = \dfrac{\text{angle } PCQ}{\Delta s} = \dfrac{\Delta s/a}{\Delta s} = \dfrac{1}{a}$, since $\angle PCQ$ is measured in radians.

$\therefore$ as in Art. 11·1,

$$\text{curvature} = \underset{\Delta s\to 0}{Lt} \frac{\Delta\psi}{\Delta s} = \underset{\Delta s\to 0}{Lt} \frac{1}{a} = \frac{1}{a} \text{ (constant) and hence } \rho = a.$$

Ex. 2. *Find the radius of curvature at the point (s, ψ) of the curve $s = a \sec\psi \tan\psi + a \log(\sec\psi + \tan\psi)$.*

Here,
$$\rho = \frac{ds}{d\psi} = a\,(\sec\psi . \sec^2\psi + \tan^2\psi \sec\psi)$$
$$+ a \cdot \frac{1}{\sec\psi + \tan\psi} \cdot \sec\psi\,(\sec\psi + \tan\psi)$$
$$= a\sec\psi\,(\sec^2\psi + \sec^2\psi - 1) + a\sec\psi = 2a\sec^3\psi.$$

Ex. 3. *Find the radius of curvature at the point (x, y) on the curve $y = a \log\sec(x/a)$.*

Here, $\quad y_1 = a \cdot \dfrac{1}{\sec(x/a)} \cdot \sec(x/a)\tan(x/a) \cdot \dfrac{1}{a} = \tan(x/a).$

$\therefore\ y_2 = (1/a)\sec^2(x/a)$. Also, $1+y_1{}^2 = 1+\tan^2(x/a) = \sec^2(x/a)$.

$$\therefore\ \rho = \frac{(1+y_1{}^2)^{\frac{3}{2}}}{y_2} = \frac{\{\sec^2(x/a)\}^{\frac{3}{2}}}{(1/a)\sec^2(x/a)} = a\sec(x/a).$$

Ex. 4. *Find the radius of curvature of the parabola $y^2 = 4x$ at the vertex* (0, 0).

The tangent at the vertex being the y-axis, $\dfrac{dy}{dx}$ at the vertex (0, 0) is infinite. Hence, formula (i) of Art. 11·2 being not applicable, let us apply formula (i*a*). [See *Note 2, Art. 11·2*]

Here, $\dfrac{dx}{dy} = \frac{1}{2}y$; $\dfrac{d^2x}{dy^2} = \frac{1}{2}$.

$\therefore$ at the vertex, $x_1 = 0$, $x_2 = \frac{1}{2}$.

$$\therefore \text{ at the vertex, } \rho = \frac{(1+x_1{}^2)^{\frac{3}{2}}}{x_2} = \frac{1}{\frac{1}{2}} = 2.$$

Ex. 5. *Find the radius of curvature at the point 'θ' on the cycloid $x = a\ (\theta + \sin\theta)$, $y = a\ (1-\cos\theta)$.* [*C. P. 1944*]

Here, $x' = a\,(1+\cos\theta)$, $y' = a\sin\theta$,

$x'' = -a\sin\theta$, $y'' = a\cos\theta$.

$\therefore$ by formula (ii) of Art. 11·2,

$$\rho = \frac{\{a^2(1+\cos\theta)^2 + a^2\sin^2\theta\}^{\frac{3}{2}}}{a^2\cos\theta\,(1+\cos\theta) + a^2\sin^2\theta} = a\cdot\frac{8\cos^3\frac{1}{2}\theta}{2\cos^2\frac{1}{2}\theta}$$
$$= 4a\cos\tfrac{1}{2}\theta.$$

Note. ρ can also be obtained by using formula (i) of Art. 11·2 by first obtaining the values of y_1 and y_2 in terms of θ.

Ex. 6. *Find the radius of curvature at the point (r, θ) on the cardioide $r = a\,(1-\cos\theta)$, and show that it varies as $\sqrt{r}$.*

Here, $r_1 = a\sin\theta$, $r_2 = a\cos\theta$,

$\therefore$ by applying formula (iv) of Art. 11·2,

$$\rho = \frac{\{a^2(1-\cos\theta)^2 + a^2\sin^2\theta\}^{\frac{3}{2}}}{a^2(1-\cos\theta)^2 + 2a^2\sin^2\theta - a^2\cos\theta\,(1-\cos\theta)}$$
$$= \frac{a\,(2-2\cos\theta)^{\frac{3}{2}}}{3(1-\cos\theta)} = \frac{2^{\frac{3}{2}}a}{3}(1-\cos\theta)^{\frac{1}{2}} = \frac{2^{\frac{3}{2}}a}{3}\left(2\sin^2\frac{\theta}{2}\right)^{\frac{1}{2}}$$
$$= \tfrac{4}{3}a\sin\tfrac{1}{2}\theta. \qquad \cdots \qquad \cdots \qquad \cdots \quad (1)$$

Since, $r = a(1-\cos\theta) = a.2\sin^2\frac{1}{2}\theta$, $\therefore$ $\sin\frac{1}{2}\theta = \sqrt{(r/2a)}$.

Hence, from (1), $\rho = \frac{2}{3}\sqrt{2a}\,.\sqrt{r}$. $\therefore$ $\rho \propto \sqrt{r}$.

Note. In cases where it is easier to transform a polar equation into a pedal one, to find the radius of curvature it is convenient to transform the polar equation into the pedal form first, and then use formula (v) of Art. 11·2.

Ex. 7. *Find the radius of curvature at the point (p, r) of the curve $r^{m+1} = a^m p$.*

We have $p = \dfrac{r^{m+1}}{a^m}$. $\therefore$ $\dfrac{dp}{dr} = \dfrac{(m+1)\,r^m}{a^m}$. $\therefore$ $\dfrac{dr}{dp} = \dfrac{a^m}{(m+1)\,r^m}$.

$$\therefore \quad \rho = r\frac{dr}{dp} = r\cdot\frac{a^m}{(m+1)\,r^m} = \frac{a^m}{(m+1)\,r^{m-1}}.$$

Ex. 8. *Find the radius of curvature at the origin for the curve $x^3 + y^3 - 2x^2 + 6y = 0$.*

Here, $y=0$, *i.e.*, the x-axis is the tangent at the origin,

$\therefore$ at the origin $Lt\,\dfrac{x^2}{y} = 2\rho$.

Dividing the equation of the curve by y, we have

$$x\cdot\frac{x^2}{y} + y^2 - 2\,\frac{x^2}{y} + 6 = 0.$$

Now, taking limits as $x \to 0$, and $y \to 0$, we have

$$-2\,.\,2\rho + 6 = 0, \quad \text{or,} \quad \rho = \tfrac{3}{2}.$$

Ex. 9. *Find the radius of curvature at the origin of the conic*

$$y - x = x^2 + 2xy + y^2. \qquad [\,C.\,P.\ 1948\,]$$

First Method: Differentiating the equation successively with respect to x, $y_1 - 1 = 2(x + xy_1 + y + yy_1)$,

and $\qquad y_2 = 2(1 + xy_2 + 2y_1 + yy_2 + y_1^2)$.

$\therefore$ at the origin, *i.e.*, when $x=0$, $y=0$, $y_1 = 1$ and $y_2 = 8$.

$\therefore$ at the origin, $\rho = \dfrac{(1+y_1^2)^{\frac{3}{2}}}{y_2} = \dfrac{(1+1)^{\frac{3}{2}}}{8} = \dfrac{\sqrt{8}}{8} = \dfrac{\sqrt{2}}{4} = {\cdot}35$ nearly.

Second Method :

Putting $y = px + \frac{qx^2}{2!} + \cdots\cdots$ on both sides of the equation, we have

$$(p-1)\,x + \frac{qx^2}{2!} + \text{higher powers of } x = (1+2p+p^2)\,x^2$$
$$+ \text{higher powers of } x.$$

Equating coefficients of x and x^2 on both sides,

$$p-1=0, \text{ i.e., } p=1,$$

and $\frac{1}{2}q = 1+2p+p^2$. $\therefore\ q=8$.

Since here p and q are the values of y_1, y_2 at the origin, using the formula $\rho = \frac{(1+y_1{}^2)^{\frac{3}{2}}}{y_2}$, we get ρ at the origin.

Third Method (Newtonian Method) :

Since $y-x=0$ is the tangent at the origin here, by the formula for the Newtonian method at the origin,

$$\rho = \tfrac{1}{2}\sqrt{2}\,.\,Lt\,\frac{x^2+y^2}{y-x} = \tfrac{1}{2}\sqrt{2}\,.\,Lt\,\frac{x^2+y^2}{x^2+2xy+y^2}$$

(from the equation of the curve)

$$= \tfrac{1}{2}\sqrt{2}\,.\,Lt\,\frac{1+(y/x)^2}{1+2\,(y/x)+(y/x)^2}$$

(on dividing numerator and denominator by x^2)

$$= \tfrac{1}{2}\sqrt{2}\,.\,\frac{1+1}{1+2+1} = \tfrac{1}{4}\sqrt{2}.$$

Since $Lt\,(y/x)$ being the value of 'm' of the tangent at the origin *viz*, $y-x=0$, is equal to 1.

Ex. 10. *Show that the chord of curvature through the pole of the curve* $r^m = a^m \cos m\theta$ *is* $\frac{2r}{m+1}$.

Taking logarithm of the given equation

$$m \log r = m \log a + \log \cos m\theta.$$

Differentiating with respect to θ, we have

$$\frac{1}{r}\,\frac{dr}{d\theta} = \frac{-\sin m\theta}{\cos m\theta} = -\tan m\theta.$$

$\therefore \quad \cot \phi = \cot (\frac{1}{2}\pi + m\theta)$, *i.e.*, $\phi = \frac{1}{2}\pi + m\theta$.

$\therefore \quad p = r \sin \phi = r \cos m\theta = r.r^m/a^m = r^{m+1}/a^m$.

$\therefore \quad \dfrac{dp}{dr} = \dfrac{(m+1)\,r^m}{a^m}$.

$\therefore \quad$ chord of curvature $= 2\rho \sin \phi = 2r \dfrac{dr}{dp} \cdot \dfrac{p}{r} = 2 \dfrac{dr}{dp} \cdot p$

$$= 2 \cdot \frac{a^m}{(m+1)\,r^m} \cdot \frac{r^{m+1}}{a^m} = \frac{2r}{m+1}.$$

Ex. 11. *For any curve prove that*

$$\frac{1}{\rho^2} = \left(\frac{d^2x}{ds^2}\right)^2 + \left(\frac{d^2y}{ds^2}\right)^2.$$

We have $\dfrac{dx}{ds} = \cos \psi$. $\therefore \dfrac{d^2x}{ds^2} = -\sin \psi \dfrac{d\psi}{ds} = -\sin \psi \cdot \dfrac{1}{\rho}$, $\quad \cdots \quad (1)$

and $\dfrac{dy}{ds} = \sin \psi$. $\therefore \dfrac{d^2y}{ds^2} = \cos \psi \dfrac{d\psi}{ds} = \cos \psi \dfrac{1}{\rho}$. $\quad \cdots \quad (2)$

Now, squaring and adding (1) and (2), the required relation follows.

Ex. 12. *For any curve prove that*

$$\rho = \frac{r}{\sin \phi \left(1 + \dfrac{d\phi}{d\theta}\right)}, \text{ where } \tan \phi = r \frac{d\theta}{dr}.$$

$$\sin \phi \left(1 + \frac{d\phi}{d\theta}\right) = \sin \phi + \frac{d\phi}{d\theta} \sin \phi = r \frac{d\theta}{ds} + \frac{d\phi}{d\theta} \cdot r \frac{d\theta}{ds}$$

$$= r \left(\frac{d\theta}{ds} + \frac{d\phi}{d\theta} \cdot \frac{d\theta}{ds}\right) = r \left(\frac{d\theta}{ds} + \frac{d\phi}{ds}\right)$$

$$= r \frac{d}{ps} (\theta + \phi) = r \frac{d\psi}{ds} \; (\because \; \theta + \phi = \psi).$$

$\therefore \quad$ right side $= \dfrac{r}{r \dfrac{d\psi}{ds}} = \dfrac{ds}{d\psi} = \rho$.

Examples XI(A)

1. Find the radius of curvature at any point (s, ψ) on the following curves:

(i) $s = a\psi$. (ii) $s = 4a \sin \psi$.

(iii) $s = c \tan \psi$. (iv) $s = 8a \sin^2 \frac{1}{6}\psi$.

(v) $s = a(e^{m\psi} - 1)$. (vi) $s = c \log \sec \psi$.

(vii) $s = m(\sec^3\psi - 1)$. (viii) $s = a \log \tan (\frac{1}{4}\pi + \frac{1}{2}\psi)$.

2. Find the radius of curvature at any point (x, y) for the curves (i) to (viii), and at the points indicated for the curves (ix) to (xiv) :

(i) $y^2 = 4ax$. (ii) $e^{y/a} = \sec (x/a)$.

(iii) $y = \log \sin x$. (iv) $ay^2 = x^3$. (v) $xy = c^2$.

(vi) $y = \frac{1}{2}a. \left(e^{\frac{x}{a}} + e^{-\frac{x}{a}}\right)$. (vii) $x^2/a^2 + y^2/b^2 = 1$.

(viii) $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$. [*C. P. 1943*].

(ix) $y = x^3 - 2x^2 + 7x$ at the origin.

(x) $y = 4 \sin x - \sin 2x$ at $x = \frac{1}{2}\pi$.

(xi) $9x^2 + 4y^2 = 36x$ at (2, 3).

(xii) $y = e^{-x^2}$ at (0, 1).

(xiii) $\sqrt{x} + \sqrt{y} = \sqrt{a}$ at the point where $y = x$ cuts it.

(xiv) $y = xe^{-x}$ at its maximum point.

3. Find the radius of curvature at any point of the curves (i) to (vi), and at the points indicated for the curves (vii) and (viii) :

(i) $x = a \cos \theta$, $y = a \sin \theta$. (ii) $x = at^2$, $y = 2at$.

(iii) $x = a \cos \phi$, $y = b \sin \phi$.

(iv) $x = a \sec \phi$, $y = b \tan \phi$.

(v) $x = a(\cos t + t \sin t)$, $y = a(\sin t - t \cos t)$.

(vi) $x = a \sin 2\theta \,(1 + \cos 2\theta)$, $y = a \cos 2\theta \,(1 - \cos 2\theta)$.

(vii) $x = a \cos^3\theta$, $y = a \sin^3\theta$ at $\theta = \frac{1}{4}\pi$.

(viii) $x = a(\theta + \sin \theta)$, $y = a(1 - \cos \theta)$ at $\theta = 0$.

4. Find the radius of curvature at any point (r, θ) for the curves (i) to (xi), and at the points indicated for the curves (xii) to (xvi) :

(i) $r=a\theta$. (ii) $r=a\cos\theta$. (iii) $r=a\sec^2\frac{1}{2}\theta$.

(iv) $r=a(1-\cos\theta)$. (v) $r^2=a^2\cos 2\theta$.

(vi) $r=ae^{\theta\cot\alpha}$. (vii) $r^3=a^3\cos 3\theta$.

(viii) $r=a+b\cos\theta$. (ix) $r^m=a^m\cos m\theta$.

(x) $r^2\cos 2\theta=a^2$. (xi) $r=a\sec 2\theta$.

(xii) $r=2a\cos\theta-a$ at $\theta=0$.

(xiii) $r=a\sin n\theta$ at the origin.

(xiv) $r=l/(1+e\cos\theta)$ at $\theta=\pi$, $[\, e<1 \,]$.

(xv) $r^2=a^2\cos 2\theta$ at $\theta=0$.

(xvi) $r=a(\theta+\sin\theta)$ at $\theta=0$.

5. Find the radius of curvature at any point (p, r) on the following curves whose pedal equations are

(i) $p=r\sin\alpha$. (ii) $r^2=2ap$. (iii) $p^2=ar$.

(iv) $pr=a^2$. (v) $r^3=2ap^2$. (vi) $r^3=a^2p$.

(vii) $\dfrac{a^2b^2}{p^2}+r^2=a^2+b^2$.

6. Find the radius of curvature at any point on the curves :

(i) $p=a(1+\sin\psi)$. (ii) $p=a\operatorname{cosec}\psi$.

(iii) $p^2+a^2\cos 2\psi=0$.

7. Find the radius of curvature at the origin of the following curves :

(i) $y=x^4-4x^3-18x^2$. (ii) $2x^2-xy+y^2-y=0$.

(iii) $3x^2+4y^2=2x$. (iv) $3x^2+xy+y^2-4x=0$.

(v) $3x^4 - 2y^4 + 5x^2y + 2xy - 2y^2 + 4x = 0$.

(vi) $4x^4 + 3y^3 - 8x^2y + 2x^2 - 3xy - 6y^2 - 8y = 0$.

(vii) $x^3 + y^3 = 3axy$. (viii) $x^2 + 6y^2 + 2x - y = 0$.

(ix) $x^4 + y^2 = 6a(x + y)$. (x) $x^2 + y^2 + 6x + 8y = 0$.

(xi) $y^2 = x^2(a + x)/(a - x)$.

(xii) $ax + by + a'x^2 + 2h'xy + b'y^2 = 0$.

(xiii) $y^2 - 2xy - 3x^2 - 4x^3 - x^2y^2 = 0$.

(xiv) $y^2 - 3xy - 4x^2 + 5x^3 + x^4y - y^5 = 0$.

8. Show that the chord of curvature through the pole for the curve $p = f(r)$ is given by $2f(r)/f'(r)$.

9. Find the chord of curvature through the pole of the curves :

(i) $r = a(1 + \cos\theta)$. (ii) $r^2 = a^2 \cos 2\theta$.

(iii) $r^2 \cos 2\theta = a^2$. (iv) $r = ae^{\theta \cot \alpha}$.

(v) $r^n = a^n \sin n\theta$.

10. Show that the chord of curvature parallel to the axis of y for the curve :

(i) $y = a \log \sec(x/a)$ is constant.

(ii) $y = c \cosh(x/c)$ is double of the ordinate.

11. Show that in a parabola the chord of curvature

(i) through the focus, and (ii) parallel to the axis are each equal to four times the focal distance of the point.

12. Show that for the ellipse $x^2/a^2 + y^2/b^2 = 1$, the radius of curvature at an extremity of the major axis is equal to half the latus rectum.

13. If C be the centre of the ellipse $x^2/a^2 + y^2/b^2 = 1$, show that at any point P,

$$\rho = \frac{CD^3}{ab} = \frac{a^2b^2}{p^3}$$

where CD is the semi-diameter conjugate to CP, and p is the perpendicular from the centre on the tangent P.

14. If ρ_1 and ρ_2 be the radii of curvature at the ends P and D of conjugate diameters of the ellipse $x^2/a^2 + y^2/b^2 = 1$, then

$$\rho_1^{\frac{2}{3}} + \rho_2^{\frac{2}{3}} = (a^2 + b^2)/(ab)^{\frac{2}{3}}.$$

15. Prove that the radius of curvature of the catenary $y = a \cosh (x/a)$ at any point is equal in length to the portion of the normal intercepted between the curve and the axis of x.

16. Show that for the cycloid $x = a(\theta - \sin \theta)$, $y = a(1 - \cos \theta)$, the radius of curvature at any point is twice the portion of the normal intercepted between the curve and the axis of x.

17. Show that in a parabola the radius of curvature is twice the part of the normal intercepted between the curve and the directrix.

18. If ρ_1 and ρ_2 be the radii of curvature at the ends of a focal chord of the parabola $y^2 = 4ax$, then

$$\rho_1^{-\frac{2}{3}} + \rho_2^{-\frac{2}{3}} = (2a)^{-\frac{2}{3}}$$

19. Show that in any curve

(i) $\rho = \left\{\left(\frac{dx}{d\psi}\right)^2 + \left(\frac{dy}{d\psi}\right)^2\right\}^{\frac{1}{2}}.$

(ii) $\dfrac{1}{\rho} = -\dfrac{d^2x}{ds^2}\Big/\dfrac{dy}{ds} = \dfrac{d^2y}{ds^2}\Big/\dfrac{dx}{ds}.$

*(iii) $\dfrac{1}{\rho^3} = \dfrac{d^2x}{ds^2}\cdot\dfrac{d^3y}{ds^3} - \dfrac{d^2y}{ds^2}\cdot\dfrac{d^3x}{ds^3}.$

20. Show that in the curve for which

(i) $y = a \cos{}^m\psi$, ρ is m times the normal.

(ii) $y = ae^{m\psi}$, ρ is m times the tangent.

21. Show that

(i) for the cycloid for which $s^2 = 8ay$,

$$\rho = 4a\sqrt{(1 - y/2a)}.$$

(ii) for the catenary for which $y^2 = c^2 + s^2$, $\rho = y^2/c$.

***22.** Prove that in any curve

(i) $\dfrac{1}{\rho} = \left\{\dfrac{1}{r} - \dfrac{1}{r}\left(\dfrac{dr}{ds}\right)^2 - \dfrac{d^2r}{ds^2}\right\} \div \left\{1 - \left(\dfrac{dr}{ds}\right)^2\right\}^{\frac{1}{2}}.$

(ii) $\rho = r\dfrac{d\theta}{ds} \div \left\{r\left(\dfrac{d\theta}{ds}\right)^2 - \dfrac{d^2r}{ds^2}\right\}.$

23. Show that the radius of curvature at any point of the equiangular spirial subtends a right angle at the pole.

24. Show that at the points in which the curves $r = a\theta$ and $r\theta = a$ intersect, their curvatures are in the ratio 3 : 1.

25. Show that when the angle between the tangent to a curve and the radius vector of the point of contact has a maximum or minimum value, $\rho = r^2/p$.

26. Prove that in any curve

$$\frac{d\rho}{ds} = \frac{3y_1y_2^2 - y_3(1 + y_1^2)}{y_2^2}$$

and show that at every point of a circle

$$3y_1y_2^2 = y_3(1 + y_1^2).$$

ANSWERS

[*In the following examples, generally the magnitudes of the radii of curvature are given.*]

1. (i) a. (ii) $4a \cos \psi$. (iii) $c \sec^2\psi$. (iv) $\frac{4}{3}a \sin \frac{1}{3}\psi$. (v) $am\, e^{m\psi}$.
(vi) $c \tan \psi$. (vii) $3m \sec^3\psi \tan \psi$. (viii) $a \sec \psi$.

2. (i) $2(x+a)^{\frac{3}{2}}/\sqrt{a}$. (ii) $a \sec (x/a)$. (iii) $\operatorname{cosec} x$.
(iv) $(4a+9x)^{\frac{3}{2}}x^{\frac{1}{2}}/6a$. (v) $(x^2+y^2)^{\frac{3}{2}}/2c^2$. (vi) y^2/a.
(vii) $(b^4x^2+a^4y^2)^{\frac{3}{2}}/a^4b^4$. (viii) $3(axy)^{\frac{1}{3}}$. (ix) $\frac{1}{2}(125\sqrt{2})$.
(x) $\frac{4}{4}\sqrt{5}$. (xi) $\frac{4}{3}$. (xii) $\frac{1}{2}$. (xiii) $a/\sqrt{2}$. (xiv) e.

3. (i) a. (ii) $2a(t^2+1)^{\frac{3}{2}}$. (iii) $(a^2 \sin^2\phi+b^2 \cos^2\phi)^{\frac{3}{2}}/ab$.
(iv) $(a^2 \tan^2\phi+b^2 \sec^2\phi)^{\frac{3}{2}}/ab$. (v) at. (vi) $4a \cos 3\theta$.
(vii) $\frac{3}{2}a$. (viii) $4a$.

4. (i) $(r^2+a^2)^{\frac{3}{2}}/(r^2+2a^2)$. (ii) $\frac{1}{2}a$. (iii) $2a \sec^3\frac{1}{2}\theta$.
(iv) $\frac{2}{3}\sqrt{2ar}$. (v) $a^2/3r$. (vi) $r \operatorname{cosec} a$. (vii) $a^3/4r^2$.
(viii) $\dfrac{(a^2+2ab\cos\theta+b^2)^{\frac{3}{2}}}{a^2+3ab\cos\theta+2b^2}$. (ix) $\dfrac{a^m}{m+1}\cdot\dfrac{1}{r^{m-1}}$. (x) r^3/a^2.
(xi) $r(4r^2-3a^2)^{\frac{3}{2}}/3a^3$. (xii) $\frac{1}{3}a$. (xiii) $\frac{1}{2}na$.
(xiv) l. (xv) $\frac{1}{3}a$. (xvi) a.

5. (i) $r \operatorname{cosec} a$. (ii) a. (iii) $2\sqrt{r^3/a}$. (iv) r^3/a^2.
(v) $\frac{2}{3}\sqrt{(2ar)}$. (vi) $a^2/3r$. (vii) a^2b^2/p^3.

6. (i) a. (ii) $2a \operatorname{cosec}^3 \psi$. (iii) a^4/p^3.

7. (i) $\frac{1}{36}$. (ii) $\frac{1}{4}$. (iii) $\frac{1}{4}$. (iv) 2. (v) 1. (vi) 2. (vii) $\frac{3}{2}a$, $\frac{3}{2}a$.
(viii) $\frac{1}{10}\sqrt{5}$. (ix) $6a\sqrt{2}$. (x) 5. (xi) $\pm a\sqrt{2}$.
(xii) $\dfrac{1}{2}\cdot\dfrac{(a^2+b^2)^{\frac{3}{2}}}{a'b^2-2h'ab+b'a^2}$. (xiii) $5\sqrt{10}$; $\sqrt{2}$. (xiv) $\sqrt{2}$, $\frac{17}{2}\sqrt{17}$.

8. (i) $\frac{4}{3}r$. (ii) $\frac{2}{3}r$. (iii) $2r$. (iv) $2r$. (v) $2r/(n+1)$.

11·6. Centre of curvature.

Let $(\bar{x}, \bar{y})$ be the co-ordinates of the centre of curvature C corresponding to any point $P(x, y)$ on the curve.

Since $C(\bar{x}, \bar{y})$ lies on the normal at P *viz.*,

$$(X-x)+(Y-y)y_1=0,$$

$$\therefore\quad (\bar{x}-x)+(\bar{y}-y)y_1=0. \qquad \cdots \qquad \cdots \quad (1)$$

Again, since $PC=\rho$, *i.e.*, $PC^2=\rho^2$,

$$\therefore\quad (\bar{x}-x)^2+(\bar{y}-y)^2=\rho^2=\frac{(1+y_1^{\,2})^3}{y_2^{\,2}}. \qquad \cdots \quad (2)$$

Substituting $-(\bar{y}-y)y_1$ for $(\bar{x}-x)$ from (1) in (2), we get

$$(\bar{y}-y)^2(1+y_1^{\,2})=\rho^2=\frac{(1+y_1^{\,2})^3}{y_2^{\,2}} \qquad \cdots \quad (3)$$

i.e., $$(\bar{y}-y)^2=\frac{(1+y_1^{\,2})^2}{y_2^{\,2}},$$

$$\therefore\quad (\bar{y}-y)=\frac{1+y_1^{\,2}}{y_2}. \qquad \cdots \qquad \cdots \quad (4)$$

Again from (1),

$$\bar{x}-x=-(\bar{y}-y)y_1=-\frac{y_1(1+y_1^{\,2})}{y_2}, \qquad \cdots \quad (5)$$

$\therefore$ from (4) and (5), we get

$$\bar{x}=x-\frac{y_1(1+y_1^{\,2})}{y_2},\quad \bar{y}=y+\frac{1+y_1^{\,2}}{y_2}. \qquad \cdots \quad (6)$$

Cor. Hence the *equation of the circle of curvature* is

$$(x-\bar{x})^2+(y-\bar{y})^2=\rho^2.$$

Note 1. According to our convention, we take the positive sign only in (4); for if y_2 is positive, ρ is positive and hence $\bar{y}-y$ is positive. Similarly if y_2 is negative, ρ is negative and hence $\bar{y}-y$ is negative.

Note 2. Since the normal at P makes an angle $(\frac{1}{2}\pi+\psi)$ with the x-axis, it follows from the definition of the centre of curvature that

$$\frac{\bar{x}-x}{\cos(\frac{1}{2}\pi+\psi)}=\frac{\bar{y}-y}{\sin(\frac{1}{2}\pi+\psi)}=\rho,$$

i.e., $\bar{x}=x-\rho\sin\psi,\quad \bar{y}=y+\rho\cos\psi.$ (7)

Now, since $\tan\psi=y_1$, $\sin\psi=\dfrac{y_1}{\sqrt{1+y_1^2}}$ and $\cos\psi=\dfrac{1}{\sqrt{1+y_1^2}}$.

Thus substituting the values of ρ, $\sin\psi$, $\cos\psi$ in terms of y_1 and y_2 in (7), values (6) of $\bar{x}$, $\bar{y}$ can be obtained.

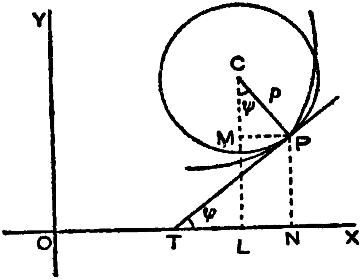
Note 3. By writing the relation (1) as $(\bar{x}-x)x_1+(\bar{y}-y)=0$, and using the values of $\rho^2=(1+x_1^2)^3/x_2^2$ (from *Art. 11·2, Sec, A*) we can similarly obtain

$$\bar{x}=x+\frac{1+x_1^2}{x_2},\quad \bar{y}=y-\frac{x_1(1+x_1^2)}{\;} \qquad (8)$$

where $x_1=\dfrac{dx}{dy}$, $x_2=\dfrac{d^2x}{dy^2}$.

This form is useful when y_1 becomes infinite.

Note 4. The centre of curvature can also be obtained *geometrically* as follows :



Let C $(\bar{x}, \bar{y})$ be the centre of curvature corresponding to the point P (x, y) on the curve.

Then $PC=\rho$.

Let PT be the tangent at P, so that $\angle PTX=\psi$.

Draw PN, CL perpendiculars on OX, PM perpendicular on CL. Then $\angle PCM=\psi$.

$\therefore\quad \bar{x}=OL=ON-MP$
$=x-PC\sin PCM$
$=x-\rho\sin\psi \quad\cdots\quad (1)$

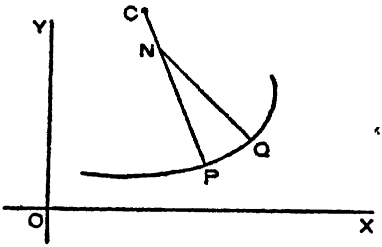
$\bar{y}=LC=LM+MC=PN+MC=y+PC\cos PCM$
$=y+\rho\cos\psi. \quad\cdots\quad (2)$

Since, $\tan\psi==y_1$, $\sin\psi=\dfrac{y_1}{\sqrt{1+y_1^2}}$ and $\cos\psi=\dfrac{1}{\sqrt{1+y_1^2}}$.

Now substituting the values of ρ, $\sin\psi$, $\cos\psi$ in (1) and (2) the required values of $\bar{x}$ and $\bar{y}$ are obtained.

11·7. Property of the Centre of Curvature.

The centre of curvature C for a point P on a curve is the limiting position of the intersection of the normal to the curve at P with a neighbouring normal at Q, as $Q \to P$ along the curve.



Let $P(x, y)$ be the given point and $Q(x+\Delta x, y+\Delta y)$ be a point near P on the curve $y=f(x)$; let us suppose y_1, y_2 exist at P and $y_2 \neq 0$.

The normal at P is

$$(Y-y)\, y_1+(X-x)=0 \qquad (1)$$

or, $$(Y-y)\phi(x)+(X-x)=0. \qquad (2) \text{ putting } y_1=\phi(x).$$

$\therefore$ the normal at Q is

$$(Y-y-\Delta y)\phi(x+\Delta x)+(X-x-\Delta x)=0. \qquad \cdots (3)$$

Suppose the normals at P, Q *i.e.*, (2) and (3) intersect at $N(\xi, \eta)$; and let $(\bar{x}, \bar{y})$ be the point C to which N tends as $Q \to P$.

Subtracting (2) from (3) and putting η for Y, we have

$$(\eta-y)\{\phi(x+\Delta x)-\phi(x)\}-\Delta y\phi(x+\Delta x)-\Delta x=0. \quad \cdots (4)$$

Dividing by Δx, and making $\Delta x \to 0$ and noting that in that case $\eta \to \bar{y}$, we have

$$(\bar{y}-y)\,\phi'(x)-y_1\phi(x)-1=0,$$

i.e., $$(\bar{y}-y)\, y_2-y_1^2-1=0. \qquad \cdots (5)$$

Again, since $(\bar{x}, \bar{y})$ is a point on (1),

$$\therefore \quad (\bar{y}-y)\, y_1+(\bar{x}-x)=0. \qquad \cdots (6)$$

The values of $(\bar{x}, \bar{y})$ obtained from (5) and (6) are identical with those of the co-ordinates of the centre of curvature obtained in Art. 11·6.

Hence, $(\bar{x}, \bar{y})$ *i.e.*, C is the centre of curvature.

11·8. Evolute and Involute.

The locus of the centres of curvature of a given curve is called its *Evolute*.

If the evolute itself be regarded as the original curve, a curve of which it is the evolute is called an *Involute*.

Formulæ (6) and (8) of Art. 11·6, give the co-ordinates of any point $(\bar{x}, \bar{y})$ on the evolute, expressed in terms of the co-ordinates of the corresponding point (x, y) of the given curve ; since y is a function of x, these formulæ give us *the parametric equation of the evolute in terms of the parameter* x.

Ordinary cartesian equation of the evolute is obtained by eliminating x and y between the two expressions for $\bar{x}, \bar{y}$ and the equation of the curve. [See *Art. 11·10, Ex. 2*]

11·9. Properties of the Evolute.

(I) *The normal at any point to the given curve is the tangent of the evolute at the corresponding point of the evolute.*

Let $(\bar{x}, \bar{y})$ be the centre of curvature corresponding to the point (x, y) on the curve. Then from Note 2, Art. 11·6,

$$\bar{x} = x - \rho \sin \psi, \quad \bar{y} = y + \rho \cos \psi.$$

$$\therefore \quad \frac{d\bar{x}}{dx} = 1 - \rho \cos \psi \frac{d\psi}{dx} - \sin \psi \frac{d\rho}{dx}$$

$$= 1 - \frac{ds}{d\psi} \cdot \frac{dx}{ds} \cdot \frac{d\psi}{dx} - \sin \psi \frac{d\rho}{dx} = - \sin \psi \frac{d\rho}{dx}.$$

Thus, $\dfrac{d\bar{x}}{dx} = -\sin \psi \dfrac{d\rho}{dx}$. (1)

Similarly, $\dfrac{d\bar{y}}{dx} = \cos \psi \dfrac{d\rho}{dx}$. (2)

$\therefore$ dividing (2) by (1),

$\dfrac{d\bar{y}}{d\bar{x}} = -\cot \psi = -\dfrac{dx}{dy}$, which is the '$m$' of the normal at (x, y).

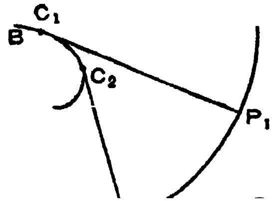
$\therefore$ 'm' of the tangent to the evolute at $(\bar{x}, \bar{y})$ = 'm' of the normal to the given curve at the corresponding point (x, y), and since both the tangent to the evolute and the normal to the curve pass through the same point $(\bar{x}, \bar{y})$, they are identical. Hence the result.

(II) **Length of an arc of the Evolute.**

The length of an arc of the evolute of a curve is the difference between the radii of curvature of the given curve, which are tangents to this arc of the evolute at its extremities.

Let $\bar{s}$ be the length of the arc of the evolute measured from some fixed point on it up to the centre of curvature $(\bar{x}, \bar{y})$. Then from (1) and (2) above, we have

$$\left(\frac{d\bar{x}}{dx}\right)^2 + \left(\frac{d\bar{y}}{dx}\right)^2 = \left(\frac{d\rho}{dx}\right)^2$$



Also we have $\left(\dfrac{d\bar{x}}{dx}\right)^2 + \left(\dfrac{d\bar{y}}{dx}\right)^2 = \left(\dfrac{d\bar{s}}{dx}\right)^2$

$\therefore \dfrac{d\bar{s}}{dx} = \dfrac{d\rho}{dx}$, *i.e.*, $\dfrac{d}{dx}(\bar{s}-\rho)=0$. Hence [*by Art.* 6·6, *Ex.* 1]

$$\bar{s}-\rho = C \text{ (a constant), } i.e., \ \bar{s} = \rho + C.$$

Hence, $\bar{s}_1 - \bar{s}_2 = \rho_1 - \rho_2$, where ρ_1, ρ_2 are the values of ρ at the two points P_1, P_2 on the curve and $\bar{s}_1$, $\bar{s}_2$ are the values of $\bar{s}$ of the corresponding points C_1, C_2 on the evolute.

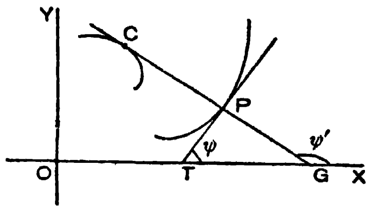
Thus, the arc C_1C_2 of the evolute $= P_1C_1 - P_2C_2$.

Hence, if a string is wrapped round the curve C_1C_2, it is clear that when the string is unwrapped, being kept tight all the time, the point on the thread which was at P_2 will describe the curve P_2P_1.

This is why the curve C_1C_2 is called the evolute of the curve P_1P_2.

(III) **Radius of the Curvature of the Evolute.**

Let ψ' be the angle which the tangent at the point $C(\bar{x}, \bar{y})$ on the evolute [corresponding to the point $P(x, y)$ on the original curve] makes with the x-axis, then ψ' is the angle which the normal at (x, y) on the given curve makes with the x-axis.



$$\psi' = \tfrac{1}{2}\pi + \psi. \quad \therefore \frac{d\psi}{d\psi'} = 1 \text{ ; also from (II) above, } \frac{ds}{d\rho} = 1.$$

Let $\bar{\rho}$ be the radius of curvature of the evolute at $(\bar{x}, \bar{y})$.

$$\therefore \quad \bar{\rho} = \frac{d\bar{s}}{d\psi'} = \frac{d\bar{s}}{d\rho}\cdot\frac{d\rho}{d\psi}\cdot\frac{d\psi}{d\psi'} = \frac{d\rho}{d\psi} = \frac{d}{d\psi}\left(\frac{ds}{d\psi}\right) = \frac{\mathbf{d^2s}}{\mathbf{d\psi^2}}.$$

11·10. Illustrative Examples.

Ex. 1. *Find the centre of curvature at any point* (x, y) *on the parabola* $y^2 = 4ax$.

Here, $\quad y_1 = \sqrt{\dfrac{a}{x}}, \qquad y_2 = -\dfrac{1}{2} \cdot \dfrac{\sqrt{a}}{x^{\frac{3}{2}}}.$

$$\therefore \quad y_1(1+y_1^2) = \sqrt{\frac{a}{x}}\left(1+\frac{a}{x}\right) = \frac{\sqrt{a}\,(x+a)}{x^{\frac{3}{2}}}.$$

If $\bar{x}, \bar{y}$ be the centre of curvature, we have

$$\bar{x} = x - \frac{y_1(1+y_1^2)}{y_2} = x + 2\,(x+a) = 3x + 2a.$$

$$\bar{y} = y + \frac{1+y_1^2}{y_2} = y - \frac{2\sqrt{x}\,(x+a)}{\sqrt{a}}$$

$$= 2\sqrt{ax} - \frac{2\sqrt{x}\,(x+a)}{\sqrt{a}} \quad (\because\ y^2 = 4ax) = -\frac{2}{\sqrt{a}}\,x^{\frac{3}{2}}.$$

Ex. 2. *Find the evolute of the parabola* $y^2 = 4ax$.

As proved above, its centre of curvature $(\bar{x}, \bar{y})$ at any point (x, y) is given by

$$\bar{x} = 3x + 2a, \qquad \cdots \quad (1)$$

$$\bar{y} = -\frac{2}{\sqrt{a}}\,x^{\frac{3}{2}}. \qquad \cdots \quad (2)$$

From (1), $x = \dfrac{\bar{x} - 2a}{3}$.

$\therefore$ from (2), $\bar{y} = -\dfrac{2}{\sqrt{a}}\left(\dfrac{\bar{x}-2a}{3}\right)^{\frac{3}{2}}.$

$\therefore$ squaring and writing x, y for $\bar{x}, \bar{y}$, the required evolute is given by $27ay^2 = 4(x-2a)^3$.

Ex. 3. *Find the equation of the circle of curvature at the point* $(3, 1)$ *on the curve* $y = x^2 - 6x + 10$.

Here, $\quad y_1 = 2x - 6\ ;\ y_2 = 2.$

$\therefore$ at the point $(3, 1)$, $\quad y_1 = 0,\ y_2 = 2.$

If $(\bar{x}, \bar{y})$ be the centre and ρ the radius of curvature at $(3, 1)$,

$$\bar{x} = x - \frac{y_1(1+y_1^2)}{y_2} = 3, \quad \bar{y} = y + \frac{1+y_1^2}{y_2} = 1 + \tfrac{1}{2} = \tfrac{3}{2}.$$

Also, $\rho = \dfrac{(1+y_1{}^2)^{\frac{3}{2}}}{y} = \frac{1}{2}$.

$\therefore$ the equation of the required circle of curvature is

$$(x-3)^2+(y-\tfrac{3}{2})^2=\tfrac{1}{4},$$

or, $$x^2+y^2-6x-3y+11=0.$$

Examples XI(B)

1. Find the centre of curvature of the following curves at the points indicated :

(i) $xy=12$ at $(3, 4)$. [C. P. *1934*]

(ii) $y=x^3+2x^2+x+1$ at $(0, 1)$.

(iii) $xy=x^2+4$ at $(2, 4)$. (iv) $y=\sin^2 x$ at $(0, 0)$.

(v) $x=e^{-2t}\cos 2t,\ y=e^{-2t}\sin 2t$ at $t=0$.

2. Determine the centres of curvature of the following curves at any point (x, y) :

(i) $x^2=4ay$. (ii) $a^2y=x^3$.

(iii) $x^2/a^2+y^2/b^2=1$. (iv) $xy=a^2$.

(v) $x^{\frac{2}{3}}+y^{\frac{2}{3}}=a^{\frac{2}{3}}$. (vi) $y=\frac{1}{2}a\,(e^{x/a}+e^{-x/a})$.

(vii) $x=a\cos\phi,\ y=b\sin\phi$. (viii) $x=at^2,\ y=2at$.

(ix) $x=a\,(\theta-\sin\theta),\ y=a\,(1-\cos\theta)$.

(x) $x=a\,(\cos t+t\sin t),\ y=a\,(\sin t-t\cos t)$.

3. Find the evolutes of the curves (iii), (iv), (v), (ix), (x) of Ex. 2, above.

4. If (α, β) be the co-ordinates of the centre of curvature of the parabola

$$\sqrt{x}+\sqrt{y}=\sqrt{a}, \text{ at } (x, y), \text{ then}$$

$$\alpha+\beta=3(x+y).$$

5. Show that the co-ordinates $(\bar{x}, \bar{y})$ of the centre of curvature at any point (x, y) on a curve are given by

(i) $\bar{x} = x - \dfrac{dy}{d\psi}, \quad \bar{y} = y + \dfrac{dx}{d\psi}$,

(ii) $\bar{x} = x + \rho^2 x'', \ \bar{y} = y + \rho^2 y''$,

where dashes denote differentiation with respect to the arc s.

6. Prove that the distance r_1 between the pole and the centre of curvature corresponding to any point on the curve $r = f(\theta)$ is given by

$$r_1^{\,2} = r^2 + \rho^2 - 2p\rho,$$

where ρ and p have the usual significance.

7. For the equiangular spiral $r = ae^{\theta \cot \alpha}$, prove that the centre of curvature is at the point where the perpendicular to the radius vector through the pole intersects the normal.

8. Find the circle of curvature of the curves :

(i) $y = x + 4/x$ at $(2, 4)$.

(ii) $y = x^3 + 2x^2 + x + 1$ at $(0, 1)$.

(iii) $x = e^y$ at the point where it crosses the x-axis.

(iv) $y^2 = 4x$ at the ends of the latus rectum.

(v) $x + y = ax^2 + by^2 + cx^3$ at the origin.

ANSWERS

1. (i) $(7\frac{1}{6}, 7\frac{1}{8})$. (ii) $(-\frac{1}{2}, \frac{5}{2})$. (iii) $(2, 5)$.

(iv) $(0, \frac{1}{2})$. (v) $(0, -1)$.

2. (i) $\left(-\dfrac{x^3}{4a^2}, \ 2a + \dfrac{3x^2}{4a}\right)$. (ii) $\left\{\dfrac{x}{2}\left(1 - \dfrac{9x^4}{a^4}\right), \left(\dfrac{5}{2}\dfrac{x^3}{a^2} + \dfrac{a^2}{6x}\right)\right\}$.

(iii) $\left(\dfrac{a^2-b^2}{a^4}\,x^3,\ -\dfrac{a^2-b^2}{b^4}\,y^3\right)$.

(iv) $\left(\dfrac{3}{2}\,x+\dfrac{y^3}{2a^2},\ \dfrac{3}{2}\,y+\dfrac{x^3}{2a^2}\right)$.

(v) $(x+3x^{\frac{1}{3}}y^{\frac{2}{3}},\ y+3x^{\frac{2}{3}}y^{\frac{1}{3}})$.

(vi) $\left(x-\dfrac{y\sqrt{y^2-a^4}}{a},\ 2y\right)$.

(vii) $\left(\dfrac{a^2-b^2}{a}\cos^3\phi,\ \dfrac{b^2-a^2}{b}\sin^3\phi\right)$.

(viii) $\{a\,(2+3t^2),\ -2at^3\}$.

(ix) $\{a\,(\theta+\sin\theta), -a\,(1-\cos\theta)\}$.

(x) $(a\cos t,\ a\sin t)$.

3. (i) $(ax)^{\frac{2}{3}}+(by)^{\frac{2}{3}}=(a^2-b^2)^{\frac{2}{3}}$.

(ii) $(x+y)^{\frac{2}{3}}-(x-y)^{\frac{2}{3}}=(4a)^{\frac{2}{3}}$.

(iii) $(x+y)^{\frac{2}{3}}+(x-y)^{\frac{2}{3}}=2a^{\frac{2}{3}}$.

(iv) $x=a\,(\theta+\sin\theta),\ y=-a\,(1-\cos\theta)$.

(v) $x^2+y^2=a^2$.

8. (i) $x^2+y^2-4x-10y+28=0$.

(ii) $x^2+y^2+x-3y+2=0$.

(iii) $x^2+y^2-6x+4y+5=0$.

(iv) $x^2+y^2-10x\pm 4y-3=0$.

(v) $(a+b)(x^2+y^2)=2(x+y)$.

CHAPTER XII

ASYMPTOTES

(*Rectilinear*)

12·1. In some cases a curve may have a branch or branches extending beyond the finite region. In this case if P be a point on such a branch of the curve, having its co-ordinates x, y, and if P moves along the curve so that one at least of x and y tends to $+\infty$ or to $-\infty$, then P is said to tend to infinity, and this we denote by $P \to \infty$.

Def. *If P be a point on a branch of a curve extending beyond the finite region, and a straight line exists at a finite distance from the origin from which the distance of P gradually diminishes and ultimately tends to zero as $P \to \infty$ (moving along the curve), then such a straight line is called an* **asymptote** *of the curve.*

12·2. Asymptotes not parallel to y-axis.

If $y = mx + c$ be an asymptote corresponding to an infinite branch of a curve, where m and c are both finite (including zero), then

$$\mathbf{m} = \underset{\mathbf{x} \to \infty}{\mathbf{Lt}} \frac{\mathbf{y}}{\mathbf{x}} \text{ and } \mathbf{c} = \underset{\mathbf{x} \to \infty}{\mathbf{Lt}} (\mathbf{y} - \mathbf{mx}),$$

where x, y are the co-ordinates of a point P on the branch of the curve.

The distance of the point P from the straight line $y - mx - c = 0$ is given by

$$d = \frac{y - mx - c}{\sqrt{1 + m^2}}, \text{ and if } y = mx + c \text{ be an asymptote,}$$

$d \to 0$ as $x \to \infty$ and since m is finite here,

$$\underset{x\to\infty}{Lt}(y-mx-c)=0, \quad \text{or,} \quad \underset{x\to\infty}{Lt}(y-mx)=c.$$

Again, denoting $y-mx-c$, *i.e.*, $d\sqrt{1+m^2}$ by u,

$$\frac{y}{x}-m=\frac{c+u}{x}.$$

Now making $x \to \infty$, since $u \to 0$ in this case, and c is finite,

$$\underset{x\to\infty}{Lt}\left(\frac{y}{x}-m\right)=0, \quad \text{or,} \quad \underset{x\to\infty}{Lt}\frac{y}{x}=m.$$

Accordingly, to find asymptotes (which are not parallel to the y-axis) of a curve $y=f(x)$ { or $F(x, y)=0$ }, we first of all find out $\underset{x\to\infty}{Lt}\dfrac{y}{x}$ from the equation to the curve, which may have several finite values (inclusive of zero). Corresponding to any such value (m say), we next proceed to find $\underset{x\to\infty}{Lt}(y-mx)$, using the equation to the curve.

If this limit is found to be finite, say c, then $y=mx+c$ is an asymptote. [See *Ex.* 7, § *12·8*]

Note. An alternative definition of a rectilinear asymptote is sometimes given as follows : *If P be a point on a branch of a curve extending to infinity and if a straight line at a finite distance from the origin exists towards which the tangent line to the curve at P approaches as a limit when $P \to \infty$, then the straight line is an asymptote of the curve.*

With this definition also we can prove the results of the above article ; for the equation of the tangent at $P(x, y)$ to the curve is $Y-y=\dfrac{dy}{dx}(X-x)$, or, $Y=\dfrac{dy}{dx}X+\left(y-x\dfrac{dy}{dx}\right)$, and as $x \to \infty$, if this tends to $Y=mX+c$, where m and c are finite, clearly $m=\underset{x\to\infty}{Lt}\dfrac{dy}{dx}$, and $c=\underset{x\to\infty}{Lt}\left(y-x\dfrac{dy}{dx}\right)=\underset{x\to\infty}{Lt}(y-mx)$.*

* A rigorous proof of this last equality requires the use of integration.

It should be noted that when $P \to \infty$, if the tangent line tends to a straight line as its limiting position, that line is an asymptote. The converse however is not true, *i.e.*, even if the tangent line has no definite limiting position when $P \to \infty$, there may be an asymptote.

[See *Ex.* 8, § *12'8*]

12·3. Asymptotes parallel to y-axis.

The necessary and sufficient condition that the straight line $x=a$ *is an asymptote to the curve* $y=f(x)$ *is that* $|f(x)| \to \infty$ *when either* $x \to a+0$ or $x \to a-0$.

For suppose $x \to a-0$. Since $|y| \to \infty$ in this case, P being the point (x, y) on the curve, $P \to \infty$ in this case; [conversely, if $P \to \infty$ in this case, $|y| \to \infty$, and hence the necessity of the condition]. Now the perpendicuar distance of P from the line $x=a$ is $x-a$ (the axes being rectangular), and $|x-a| \to 0$ as $x \to a-0$. Hence, $x=a$ is an asymptote. Similarly, for the case when $x \to a+0$.

Thus to find asymptotes parallel to y-axis, we may write z for $1/y$ in the equation to the curve, and then make $z \to 0$. If then the result leads to finite value or values of x of the type $x=a$, these will give us the corresponding asymptotes parallel to y-axis. [See *Ex.* 7, § *12'8*]

Note. In a similar way we may get *asymptotes parallel to x-axis*; thus if as $y \to b \pm 0$, $|x| \to \infty$, (where x, y is a point on the curve), then $y=b$ is an asymptote.

12·4. Asymptotes of algebraic curves.

The most useful case of determination of asymptotes is for algebraic curves. The general form of the equation of an algebraic curve of the nth degree is, arranging in groups of homogeneous terms,

$$\begin{aligned}(a_0x^n + a_1x^{n-1}y + a_2x^{n-2}y^2 + \cdots\cdots\cdots + a_ny^n)\\ + (b_0x^{n-1} + b_1x^{n-2}y + \cdots\cdots\cdots + b_{n-1}y^{n-1})\\ + (c_0x^{n-2} + c_1x^{n-3}y + \cdots + c_{n-2}y^{n-2})\\ + \cdots\cdots = 0, \qquad \cdots \quad (1)\end{aligned}$$

which can also be written as,

$$x^n\phi_n\left(\frac{y}{x}\right) + x^{n-1}\phi_{n-1}\left(\frac{y}{x}\right) + x^{n-2}\phi_{n-2}\left(\frac{y}{x}\right) + \cdots = 0, \qquad \cdots \quad (2)$$

where ϕ_r is an algebraic polynomial of degree r.

For asymptotes of this curve, we proceed to prove the following rules :

Rule I. **Asymptotes not parallel to y-axis** *will all be given by* $y = mx + c$, *where* m *is any of the real finite roots of* $\Phi_n(m) = 0$ *and for each such value of* m, $c = -\Phi_{n-1}(m)/\Phi'_n(m)$ *provided it gives a definite value of* c.

Proof :

The equation (2) of the curve can be put in the form

$$\phi_n\left(\frac{y}{x}\right) + \frac{1}{x}\phi_{n-1}\left(\frac{y}{x}\right) + \frac{1}{x^2}\phi_{n-2}\left(\frac{y}{x}\right) + \cdots\cdots = 0. \qquad \cdots \quad (3)$$

Now if $y = mx + c$ be an asymptote, where m and c are finite, $\underset{x\to\infty}{Lt}(y/x) = m$ (See § *12·2*). Hence from (3), making $x \to \infty$, since m is finite, and the functions $\phi_n(m)$, $\phi_{n-1}(m)$ etc. which are algebraic polynomials in m are accordingly finite, we get $\phi_n(m) = 0$.

Again, since in this case $\underset{x\to\infty}{Lt}(y - mx) = c$ (See § *12·2*), we can write $y - mx = c + u$, where u is a function of x such that $u \to 0$ when $x \to \infty$. Thus, $\dfrac{y}{x} = m + \dfrac{c+u}{x}$.

From (3) now, we get

$$\phi_n\left(m+\frac{c+u}{x}\right)+\frac{1}{x}\phi_{n-1}\left(m+\frac{c+u}{x}\right)$$
$$+\frac{1}{x^2}\phi_{n-2}\left(m+\frac{c+u}{x}\right)+\cdots=0.$$

Expanding each term by Taylor's theorem, since the function ϕ_r are all algebraic polynomials and will each lead to a finite series, and remembering that $\phi_n(m)=0$, we get,

$$\left\{\frac{c+u}{x}\phi'_n(m)+\frac{(c+u)^2}{x^2 2!}\phi''_n(m)+\frac{(c+u)^3}{x^3 3!}\phi'''_n(m)+\cdots\cdots\right\}$$
$$+\frac{1}{x}\left\{\phi_{n-1}(m)+\frac{c+u}{x}\phi'_{n-1}(m)+\frac{(c+u)^2}{x^2 2!}\phi''_{n-1}(m)+\cdots\right\}$$
$$+\frac{1}{x^2}\left\{\phi_{n-2}(m)+\frac{c+u}{x}\phi'_{n-2}(m)+\cdots\right\}$$
$$+\cdots\cdots=0. \quad \cdots \quad (4)$$

Now multiplying throughout by x and making $x \to \infty$, we get ($\because$ $u \to 0$ now),

$$c\phi'_n(m)+\phi_{n-1}(m)=0, \text{ or, } c=-\phi_{n-1}(m)/\phi'_n(m).$$

Each finite root of $\phi_n(m)=0$ will give one value of c (provided $\phi'_n(m) \neq 0$ for this value), and accordingly we get the corresponding asymptote $y=mx+c$.

Special cases.

If any value of m satisfying $\phi_n(m)=0$ (say $m=m_1$) makes $\phi'_n(m)=0$ also (which requires m_1 to be a multiple root of $\phi_n(m)=0$ as we know from the theory of equations), and if $\phi_{n-1}(m) \neq 0$ for this value, then $c \to \infty$ as $m \to m_1$. Accordingly there is no asymptote corresponding to this value of m.

If for $m=m_1$, we get $\phi_n(m)$, $\phi'_n(m)$, $\phi_{n-1}(m)$, each $=0$, then from (4), multiplying throughout by x^2, and making $x \to \infty$, we derive

$$\tfrac{1}{2}c^2\phi''_n(m)+c\phi'_{n-1}(m)+\phi_{n-2}(m)=0$$

giving two values (say c_1, c_2) of c in general [provided $\phi''_n(m_1) \neq 0$], and thereby giving two parallel asymptotes of the type $y=m_1x+c_1$, $y=m_1x+c_2$.

If $\phi''_n(m_1)$ is also zero {*i.e.*, if m_1 is a triple root of $\phi_n(m)=0$}, and if $\phi'_{n-1}(m_1)$, $\phi_{n-2}(m_1)$ are also identically zero, we shall, proceeding in a similar manner, get three parallel asymptotes in general corresponding to $m=m_1$; and so on.

Rule II. **Asymptotes parallel to y-axis** *exist only when a_n (the coefficient of the highest power of y i.e., of y^n) is zero, and in this case the coefficient of the highest available power of y in the equation (provided it involves x, and is not merely a constant) equated to zero will give us those asymptotes.*

A similar rule will apply to asymptotes parallel to x-axis.

Proof :

After dividing by y^n, and replacing $1/y$ by z, the equation (1) of the curve can be written in ascending powers of z in the form

$$a_n + z(a_{n-1}x + b_{n-1}) + z^2(a_{n-2}x^2 + b_{n-2}x + c_{n-2}) + \cdots = 0. \quad \cdots \quad \cdots \quad (5)$$

This will have an asymptote parallel to y-axis of the type $x=a$ where a is finite, provided $z \to 0$ when $x \to a+0$ or $a-0$ [See § *12·3*].

Hence making $z \to 0$, since x now tends to a finite value, we must have the necessary condition $a_n = 0$.

Assuming this to be satisfied, we get from (5), dividing by z and making $z \to 0$,

$$a_{n-1}x + b_{n-1} = 0 \quad \cdots \quad \cdots \quad (6)$$

giving a finite value of x (provided a_{n-1} is not zero) which makes $|y| \to \infty$ and thus represents an asymptote.

In case a_{n-1} is also zero along with a_n in order that we may have an asymptote parallel to y-axis, since x is to be finite, we must have from (6), $b_{n-1}=0$. Hence, from (5), dividing by z^2 and making $z \to 0$ now we get the asymptotes parallel to y-axis (two in this case) given by

$$a_{n-2}x^2 + b_{n-2}x + c_{n-2} = 0$$

provided this gives finite values of x. In case a_{n-2}, b_{n-2}, c_{n-2} are all identically zero, we proceed in a similar manner with the coefficient of z^3 in (5) *i.e.*, the coefficient of y^{n-3} in the original equation (1), and so on, proving the rule.

Note. By interchanging y and x in arranging the given equation (1), and proceeding in a similar manner, (making $1/x \to 0$) we can prove the corresponding rule for finding the asymptotes parallel to the x-axis.

12·5. Working rule for asymptotes of algebraic curves.

For an algebraic curve of the nth degree with equation given by (1) of the previous article, first of all see if the term involving y^n is absent, in which case, the coefficient of the highest power of y involved in the equation (unless it is merely a constant independent of x) equated to zero will give asymptotes parallel to the y-axis.

Similarly if the term involving x^n is absent, the coefficient of the highest available power of x equated to zero will in general give asymptotes parallel to the x-axis.

Next, replacing x by 1 and y by m in the homogeneous n^{th} degree terms, get $\phi_n(m)$ [as is apparent from the alternative form $x^n\phi^n(y/x)$]. Similarly get $\phi_{n-1}(m)$ from the $n-1^{th}$ degree terms, and if necessary (see later), $\phi_{n-2}(m)$ from the $n-2^{th}$ degree terms, and so on. Now equating $\phi_n(m)$ to zero, obtain the real finite roots m_1, m_2, etc., which will indicate the directions of the corresponding asymptotes (repeated roots giving in general a set of parallel asymptotes).

For each unrepeated root (m_1 say), a definite value c_1 of $c = -\phi_{n-1}(m)/\phi'_n(m)$ is obtained, and the corresponding asymptote $y = m_1 x + c_1$ is determined. For repeated roots, the several values of c may be obtained as explained under the head 'Special cases' on rule I.

12·6. Alternative method of finding asymptotes of algebraic curves.

Let the equation to an algebraic curve be

$$P_n + P_{n-1} + P_{n-2} + \cdots\cdots = 0, \qquad \cdots \quad (1)$$

where $P_n \left[\equiv a_0 x^n + a_1 x^{n-1} y + \cdots\cdots + a_n y^n = x^n \phi_n(y/x) \right]$ consists of homogeneous terms of degree n, P_{n-1} is homogeneous of degree $n-1$, and so on. Now $m_1, m_2, m_3, \ldots\ldots$ being the roots of $\phi_n(m) = 0$, we know from the theory of equations that $m - m_1$, $m - m_2, \ldots\ldots$ are factors of $\phi_n(m)$ and accordingly $P_n \equiv a_n(y - m_1 x)(y - m_2 x)\ldots$ The possible asymptotes are parallel to $y - m_1 x = 0$, $y - m_2 x = 0$, etc., as proved in § 12·4, and their directions are thus all easily found from the factors of P_n.

CASE I. Let $y - m_1 x$ be a non-repeated factor of P_n. Equation (1) can then be written as

$$(y - m_1 x)\, Q_{n-1} + F_{n-1} = 0,$$

$$\text{or,} \quad y - m_1 x + (F_{n-1}/Q_{n-1}) = 0, \qquad \cdots \quad (2)$$

where $Q_{n-1} \left[= (y - m_2 x)(y - m_3 x)\cdots\cdots \right]$ is a homogeneous expression of degree $n-1$ which does not contain $y - m_1 x$ as a factor, and $F_{n-1} \left[\equiv P_{n-1} + P_{n-2} + \cdots\cdots \right]$ consists of $n-1^{th}$ and lower degree terms.

Now the asymptote parallal to $y-m_1x=0$, is $y-m_1x=c_1$ where $c_1=\underset{x\to\infty}{Lt}(y-m_1x)=\underset{x\to\infty}{Lt}(-F_{n-1}/Q_{n-1})$ [*from* (2) *above*], it being remembered that $\underset{x\to\infty}{Lt}(y/x)=m_1$ in this case [See § 12·2]. In other words, the particular asymptote in question is

$$\mathbf{y-m_1x+\underset{x\to\infty}{Lt}(F_{n-1}/Q_{n-1})=0},$$

where in determining the limit involved, we are to put $y=m_1x$ and then make $x\to\infty$.

CASE II. Let P_n have a repeated factor $(y-m_rx)$. The equation (1) can then be written as

$$(y-m_rx)^2\ Q_{n-2}+P_{n-1}+F_{n-2}=0, \qquad \cdots \quad (3)$$

where Q_{n-2} is a homogeneous expression of degree $n-2$, and $F_{n-2}\ \{\equiv P_{n-2}+P_{n-3}+\cdots\cdots\}$ consists of $n-2^{th}$ and lower degree terms.

Now the asymptotes parallel to $y-m_rx=0$ will be $y-m_rx=c_r$, where $c_r=\underset{x\to\infty}{Lt}(y-m_rx)$, and this from (3) is given by

$$c_r^2+\underset{x\to\infty}{Lt}\frac{P_{n-1}+F_{n-2}}{Q_{n-2}}=0,$$

it being remembered that $\underset{x\to\infty}{Lt}(y/x)=m_r$ here.

If P_{n-1} does not contain $y-m_rx$ as a factor then it is easily seen that c_r^2 as given above, does not tend to a finite limit, and accordingly there are no asymptotes parallel to $y=m_rx$.

If on the other hand, P_{n-1} has a factor $y-m_rx$, assuming $P_{n-1}=(y-m_rx)\ R_{n-2}$, we can write (3) in the form

$$(y-m_rx)^2+(y-m_rx)\frac{R_{n-2}}{Q_{n-2}}+\frac{F_{n-2}}{Q_{n-2}}=0,$$

and arguing as before, the required asymptotes will be given by

$$(y - m_r x)^2 + (y - m_r x) \underset{x \to \infty}{Lt} \frac{R_{n-2}}{Q_{n-2}} + \underset{x \to \infty}{Lt} \frac{F_{n-2}}{Q_{n-2}} = 0,$$

it being remembered that in proceeding to determine the limits we are to use $\underset{x \to \infty}{Lt} (y/x) = m_r$ here.

The two parallel asymptotes corresponding to the two repeated factors of P_n are thus obtained.

Similarly we may proceed in cases of factors of P_n repeated more than twice.

Note. If in P_n the term involving y^n be absent, that is $a_n = 0$, clearly P_n will have a factor x, and corresponding to this there will be in general an asymptote parallel to y-axis, *i.e.*, parallel to $x = 0$, $\phi_n(m)$ (which is in general of degree n) will have its degree lower than n in this case. If for instance x^2 be a factor of P_n, $\phi_n(m)$ will be of degree $n-2$, as $x^2 y^{n-2}$ will be the term involving the highest power of y in P_n. In this case, there will be in general two asymptotes parallel to y-axis (*i.e.*, $x = 0$) and $n-2$ asymptotes corresponding to the roots of $\phi_n(m) = 0$, *i.e.*, corresponding to the other factors of P_n.

Thus, *all the possible directions of the asymptotes of the algebraic curve* (including those parallel to y-axis) *will be indicated by the factors of* P_n, and the asymptotes may be very effectively determined by the method of the present article. [For *illustrations*, see *Ex. 1-5*, § *12·8*]

A special case *(Asymptotes by inspection).*

If the equation to an algebraic curve can be put in the form $F_n + F_{n-2} = 0$, *where* F_n *consists of* n^{th} *and lower degree terms which can be expressed as a product of* n *linear factors none of which is repeated, and* F_{n-2} *consists of terms at most of degree* $n-2$, *then all the asymptotes are given by* $F_n = 0$.

For let $F_n \equiv (a_1x + b_1y + c_1)(a_2x + b_2y + c_2)\cdots\cdots$
$$(a_nx + b_ny + c_n)$$
$$= (a_1x + b_1y + c_1)\ Q_{n-1}\ \text{(say)},$$
where Q_{n-1} is of degree $n-1$.

The equation of the curve can then be written as
$$a_1x + b_1y + c_1 + F_{n-2}/Q_{n-1} = 0,$$
and the asymptote parallel to $a_1x + b_1y = 0$ is as shown above,
$$a_1x + b_1y + c_1 + \underset{x\to\infty}{Lt}\ (F_{n-2}/Q_{n-1}) = 0$$
where, in calculating the limit of the last term, we are to put $y = -(a_1/b_1)x$, and then make $x \to \infty$, and this limit is easily seen to be zero, since F_{n-2} is at most of degree $n-2$, and Q_{n-1} is of degree $n-1$.

Thus, $a_1x + b_1y + c_1 = 0$ is an asymptote. Similarly, each of $a_2x + b_2y + c_2 = 0$, etc. will be an asymptote. As there are n asymptotes here, $F_n = 0$ represent all the asymptotes.

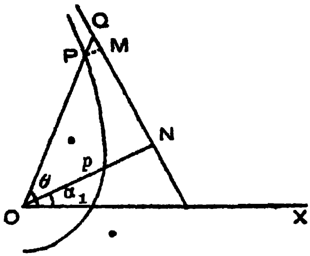
Note. If in the above case, F_n consists of real linear factors, some repeated, and some non-repeated, the non-repeated linear factors equated to zero will be asymptotes to the curve. The asymptotes corresponding to the repeated factors however will have to be obtained as in the general case.

12·7. Asymptote of Polar curves.

Let $r = f(\theta)$ be the polar equation to a curve. This may be written as $u = \dfrac{1}{r} = \dfrac{1}{f(\theta)} = F(\theta)$ (say). $\quad\cdots\quad$ (i)

P being any point (r, θ) on the curve, $P \to \infty$ as $r \to \infty$ which requires $F(\theta) \to 0$. Let the solutions of $F(\theta) = 0$ be $\theta = \alpha, \beta, \gamma, \ldots\ldots$ etc. Then these are the only directions along which the branches of the curve tend to infinity.

Consider the branch corresponding to $\theta=\alpha$. Let the straight line $r\cos(\theta-\alpha_1)=p \cdots$ (ii) be the asymptote to this branch.



Then $p=ON$, the perpendicular from the pole O on the line, and $\angle NOX=\alpha_1$. Let OP produced meet this line at Q. The perpendicular from P on the line is

$$PM=PQ\cos QPM=(OQ-OP)\cos QON$$
$$=\{p\sec(\theta-\alpha_1)-f(\theta)\}\cos(\theta-\alpha_1) \qquad [\text{ From } (i)\ \&\ (ii)\ ]$$
$$=p-f(\theta)\cos(\theta-\alpha_1).$$

Now since (ii) is an asymptote, $PM\to 0$ as $P\to\infty$ *i e.*, as $\theta\to\alpha$ for the branch in question.

$$\therefore \ \underset{\theta\to\alpha}{Lt}\ \{p-f(\theta)\cos(\theta-\alpha_1)\}=0, \text{ or } \underset{\theta\to\alpha}{Lt}\ f(\theta)\cos(\theta-\alpha_1)=p$$

and as p is finite, and $f(\theta)\to\infty$ as $\theta\to\alpha$, $\underset{\theta\to\alpha}{Lt}\cos(\theta-\alpha_1)=0$;

$$\therefore \quad \alpha-\alpha_1=\tfrac{1}{2}\pi, \text{ or } \alpha_1=\alpha-\tfrac{1}{2}\pi.$$

Again, $\quad p=\underset{\theta\to\alpha}{Lt}\ f(\theta)\cos(\theta-\alpha_1)=\underset{\theta\to\alpha}{Lt}\ \dfrac{\cos(\theta-\alpha_1)}{F(\theta)}$

which $\left(\text{being of the form } \dfrac{0}{0}\right)=\underset{\theta\to\alpha}{Lt}\ \dfrac{-\sin(\theta-\alpha_1)}{F'(\theta)}$

$$=-\frac{\sin(\alpha-\alpha_1)}{F'(\alpha)}=-\frac{1}{F'(\alpha)}.$$

Hence, (ii) reduces to $r \cos(\theta - \alpha + \frac{1}{2}\pi) = -1/F'(\alpha)$,

or, $\mathbf{r \sin(\theta - \alpha) = 1/F'(\alpha)}$

which is the required asymptote.

Similarly the other possible asymptotes corresponding to the other branches are $r \sin(\theta - \beta) = 1/F'(\beta)$, $r \sin(\theta - \gamma) = 1/F'(\gamma)$, etc.

12·8. Illustrative Examples.

Ex. 1. *Find the asymptotes of the cubic*

$$x^3 - 2y^3 + xy(2x - y) + y(x - y) + 1 = 0. \qquad \text{[C. P. 1949]}$$

The curve being an algebraic curve of the third degree, since the terms involving x^3 and y^3 are both present, there are no asymptotes parallel to either the x-axis or the y-axis in this case.

To find the asymptotes of the type $y = mx + c$, which are oblique, considering respectively the third and second degree terms (putting 1 for x and m for y), we get here

$$\phi_n(m) = 1 - 2m^3 + m(2 - m) = (1 - m)(1 + m)(1 + 2m),$$

and $\phi_{n-1}(m) = m(1 - m)$.

Now, $\phi_n(m) = 0$ gives $m = 1, -1, -\frac{1}{2}$.

Also $c = -\dfrac{\phi_{n-1}(m)}{\phi_n'(m)} = \dfrac{m(m-1)}{-6m^2 + 2 - 2m}$, and thus for $m = 1$, $c_1 = 0$; for $m = -1$, $c_2 = -1$; and for $m = -\frac{1}{2}$, $c_3 = \frac{1}{2}$.

Hence the required asymptotes are

$$y = x, \; y = -x - 1, \text{ and } y = -\tfrac{1}{2}x + \tfrac{1}{2},$$

$$i.e., \; x - y = 0, \; x + y + 1 = 0 \text{ and } x + 2y = 1.$$

Note. It may be noted that *the equation to determine m and c might be obtained in practice by putting $y = mx + c$ in the given equation, and then equating to zero the coefficients of the two highest powers of x.*

Alternative method :

Writing the highest degree terms in factorised form, the equation can be written as

$$(x - y)(x + y)(x + 2y) + y(x - y) + 1 = 0.$$

Hence the possible asymptotes are parallel to $x-y=0$, $x+y=0$, and $x+2y=0$, and these asymptotes are respectively

$$x-y+\underset{\substack{x\to\infty\\ y=x}}{Lt}\frac{y(x-y)+1}{(x+y)(x+2y)}=0 \quad \cdots \quad \text{(i)}$$

$$x+y+\underset{\substack{x\to\infty\\ y=-x}}{Lt}\frac{y(x-y)+1}{(x-y)(x+2y)}=0 \quad \cdots \quad \text{(ii)}$$

and

$$x+2y+\underset{\substack{x\to\infty\\ y=-\frac{1}{2}x}}{Lt}\frac{y(x-y)+1}{(x-y)(x+y)}=0. \quad \cdots \quad \text{(iii)}$$

The limit involved in (i), $=\underset{x\to\infty}{Lt}\dfrac{x(x-x)+1}{(x+x)(x+2x)}=0$;

that in (ii), $=\underset{x\to\infty}{Lt}\dfrac{-x(x+x)+1}{(x+x)(x-2x)}=\underset{x\to\infty}{Lt}\dfrac{-2x^2+1}{-2x^2}=1$

and that in (iii), $=\underset{x\to\infty}{Lt}\dfrac{-\frac{1}{2}x(x+\frac{1}{2}x)+1}{(x+\frac{1}{2}x)(x-\frac{1}{2}x)}=\underset{x\to\infty}{Lt}\dfrac{-\frac{3}{4}x^2+1}{\frac{3}{4}x^2}=-1.$

Hence the asymptotes are

$$x-y=0,\ x+y+1=0,\ x+2y-1=0.$$

Ex. 2. *Find the asymptotes of* $2x(y-5)^2=3(y-2)(x-1)^2$.

As the curve is algebraic, arranging the terms in descending degrees, the equation can be written as

$$xy(2y-3x)+2x(3x-7y)+38x-3y+6=0. \quad \cdots \quad \text{(i)}$$

The possible asymptotes are parallel to $x=0$, $y=0$ and $2y-3x=0$. The asymptote parallel to $x=0$, *i.e.*, to the y-axis, is (equating to zero the coefficient of y^2, the highest available power of y in (i), since the term involving y^3 is absent here) $2x=0$, *i.e.*, $x=0$, the y-axis itself.

The asymptote parallel to $y=0$, (in a similar manner, since x^3 term is absent here), is $-3y+6=0$, or $y=2$.

The third asymptote is

$$2y-3x+\underset{\substack{x\to\infty\\ y=\frac{3}{2}x}}{Lt}\frac{2x(3x-7y)+38x-3y+6}{xy}=0$$

and since the limit involved $=\underset{x\to\infty}{Lt}\dfrac{2x(3x-\frac{21}{2}x)+(38-\frac{9}{2})x+6}{\frac{3}{2}x^2}=-10$,

the asymptote is $2y-3x-10=0$.

Hence the asymptotes required are $x=0$, $y=2$, $2y=3x+10$.

Ex. 3. *Determine the asymptotes of*

$$x^3+x^2y-xy^2-y^3+2xy+2y^2-3x+y=0.$$

Writing the equation as

$$(x+y)^2\ (x-y)+2y\ (x+y)-3x+y=0, \qquad \cdots \quad \text{(i)}$$

we note that there are presumably two parallel asymptotes parallel to $x+y=0$, and one parallel to $x-y=0$.

The asymptotes parallel to $x+y=0$ are given by

$$(x+y)^2+2\ (x+y)\,.\underset{\substack{x\to\infty \\ y=-x}}{Lt}\ \frac{y}{x-y}-\underset{\substack{x\to\infty \\ y=-x}}{Lt}\ \frac{3x-y}{x-y}=0 \qquad \cdots \quad \text{(ii)}$$

provided the limits involved exist.

Now, $$\underset{\substack{x\to\infty \\ y=-x}}{Lt}\ \frac{y}{x-y}=\underset{x\to\infty}{Lt}\ \frac{-x}{x+x}=-\tfrac{1}{2}$$

and $$\underset{\substack{x\to\infty \\ y=-x}}{Lt}\ \frac{3x-y}{x-y}=\underset{x\to\infty}{Lt}\ \frac{3x+x}{x+x}=2.$$

Hence the asymptotes from (ii) are

$$(x+y)^2-(x+y)-2=0, \text{ or, } (x+y+1)(x+y-2)=0.$$

Again, the asymptote parallel to $x-y=0$ is given from (i), by

$$x-y+\underset{\substack{x\to\infty \\ y=x}}{Lt}\ \frac{2y\ (x+y)-3x+y}{(x+y)^2}=0,$$

i.e., $$x-y+\underset{x\to\infty}{Lt}\ \frac{2x\,.\,(x+x)-3x+x}{(x+x)^2}=0, \text{ i.e., } x-y+1=0.$$

Thus the required asymptotes are

$$x+y+1=0,\ x+y-2=0 \text{ and } x-y+1=0.$$

Ex. 4. *Find the asymptotes of the Folium of De Cartes*

$$x^3+y^3=3axy.$$

The equation can be written as $(x+y)(x^2-xy+y^2)=3axy$, and since the highest degree terms have got only one real linear factor $x+y$, (the linear factors of x^2-xy+y^2 being clearly imaginary), there is only one possible asymptote here, which is parallel to $x+y=0$. The asymptote in question is

$$x+y=\underset{\substack{x\to\infty \\ y=-x}}{Lt}\ \frac{3axy}{x^2-xy+y^2}=\underset{x\to\infty}{Lt}\ \frac{-3ax^2}{x^2+x^2+x^2}=-a,$$

i.e., $x+y+a=0$.

Ex. 5. *Find the asymptotes of* $x(x-y)^2-3(x^2-y^2)+8y=0$.

The possible asymptotes here are, one parallel to $x=0$, and a pair parallel to $x-y=0$.

The first one which is parallel to y-axis, is found by equating the coefficient of y^2 to zero (the term involving y^3 being absent, as it should be under the circumstances), namely, $x+3=0$.

The other two are given by

$$(x-y)^2-3(x-y)\cdot \underset{\substack{x\to\infty \\ y=x}}{Lt} \frac{x+y}{x}+8 \underset{\substack{x\to\infty \\ y=x}}{Lt} \frac{y}{x}=0,$$

i.e., $(x-y)^2-3\,(x-y).2+8=0$, or, $(x-y-4)(x-y-2)=0$.

Thus the required asymptotes are

$$x+3=0,\ x-y=4 \text{ and } x-y=2.$$

Ex. 6. *Prove that the asymptotes of the cubic*

$$(x^2-y^2)y-2ay^2+5x-7=0$$

form a triangle of area a^2.

The equation to the curve may be written as

$$y(x^2-y^2-2ay)+5x-7=0, \text{ or, } y\{x^2-(y+a)^2\}+a^2y+5x-7=0,$$

i.e., $y(x+y+a)(x-y-a)+a^2y+5x-7=0$

which is of the form $F_3+F_1=0$, F_3 having three unrepeated linear factors, and so the required asymptotes are given by equating these factors to zero, namely,

$$y=0,\ x+y+a=0,\ x-y-a=0.$$

By solving in pairs, their points of intersection are easily seen to be

$$(-a,0),\ (a,0) \text{ and } (0,-a).$$

The area of the triangle with these as vertices is

$$\frac{1}{2}\begin{vmatrix} a & 0 & 1 \\ -a & 0 & 1 \\ 0 & -a & 1 \end{vmatrix}=\tfrac{1}{2}\,2a^2=a^2.$$

Ex. 7. *Find the asymptotes, if any, of the curve*

$$y=a \log \sec (x/a).$$

This is not an algebraic curve. To find its asymptotes, if any, which are not parallel to y-axis, we know that $y=mx+c$ will be an asymptote where $m=\underset{x\to\infty}{Lt}\frac{y}{x}$, and $c=\underset{x\to\infty}{Lt}(y-mx)$.

Now in the curve, $\underset{x\to\infty}{Lt}\ \dfrac{y}{x}=\underset{x\to\infty}{Lt}\ \dfrac{a\log\sec(x/a)}{x}$

which limit does not exist.

Hence there is no asymptote non-parallel to y-axis in this case.

To find if there be any asymptote parallel to y-axis, we notice that $y\to\infty$ when $x/a\to 2n\pi\pm\frac{1}{2}\pi$, and accordingly the asymptotes parallel to y-axis are (See § *12·3*)

$$x=(2n\pi\pm\tfrac{1}{2}\pi)a$$

which are the only asymptotes of the given curve.

Ex. 8. *An asymptote is defined in the following two ways :*

(A) *An asymptote is a straight line, the distance of which from a point on a curve diminishes without limit as the point on the curve moves to an infinite distance from the origin.*

(B) *An asymptote to a curve is the limiting position of the tangent when the point of contact moves to an infinite distance from the origin.*

Consider the two curves

$$(i)\ y=ax+b+\frac{c}{x},\quad (ii)\ y=ax+b+\frac{c+\sin x}{x}.$$

Show that for the first curve, an asymptote exists according to both the definitions, but for the second curve an asymptote exists according to the first definition, but not according to the second.

Let us consider the first definition. According to this it has been proved (§ *12·2*) that the straight line $y=mx+c$ will be an asymptote to a curve, where $m=\underset{x\to\infty}{Lt}\ \dfrac{y}{x}$ and $c=\underset{x\to\infty}{Lt}\ (y-mx)$, (x, y) being a point on the curve, provided the limits exist.

Now for curve (i),

$$m=\underset{x\to\infty}{Lt}\ \frac{y}{x}=\underset{x\to\infty}{Lt}\ \left(a+\frac{b}{x}+\frac{c}{x^2}\right)=a,$$

$$\text{and}\ \ c=\underset{x\to\infty}{Lt}\ (y-mx)=\underset{x\to\infty}{Lt}\ (y-ax)=\underset{x\to\infty}{Lt}\ \left(b+\frac{c}{x}\right)=b.$$

Accordingly the asymptote exists, given by $y=ax+b$.

Again, for the curve (ii),

$$m=\underset{x\to\infty}{Lt}\ \frac{y}{x}=\underset{x\to\infty}{Lt}\ \left(a+\frac{b}{x}+\frac{c+\sin x}{x^2}\right)=a\quad [\because |\sin x|\leqslant 1]$$

and $\quad c = \underset{x \to \infty}{Lt} (y - mx) = \underset{x \to \infty}{Lt} (y - ax) = \underset{x \to \infty}{Lt} \left(b + \frac{c + \sin x}{x}\right) = b.$

Thus the asymptote exists here also, given by $y = ax + b$.

Next, consider the second definition.

For curve (i), $\frac{dy}{dx} = a - \frac{c}{x^2}$, and so the equation to the tangent line at (x, y) is $Y - y = \left(a - \frac{c}{x^2}\right)\left(X - x\right)$,

$$\text{or,} \quad Y = \left(a - \frac{c}{x^2}\right) X + y - x\left(a - \frac{c}{x^2}\right) = \left(a - \frac{c}{x^2}\right) X + \left(b + \frac{2c}{x}\right).$$

As $x \to \infty$, the equation becomes $Y = aX + b$, which is then the definite straight line towards which the tangent line approaches, as the point of contact (x, y) moves to an infinite distance. Hence this is the asymptote.

For curve (ii), $\frac{dy}{dx} = a + \frac{x \cos x - (c + \sin x)}{x^2}$

and the equation to the tangent line at (x, y) is

$$Y - y = \left\{a + \frac{x \cos x - (c + \sin x)}{x^2}\right\}\left(X - x\right)$$

or substituting the value of y from (ii),

$$Y = \left\{a + \frac{\cos x}{x} - \frac{c + \sin x}{x^2}\right\} X + \left\{\frac{2(c + \sin x)}{x^2} - \cos x + b\right\}.$$

Now, as $x \to \infty$, $\cos x$ does not tend to any definite limit. Hence the tangent line at (x, y) does not tend to any definite limiting position and so the asymptote does not exist in this case, according to the second definition.

Ex. 9. *Find the asymptote of the curve* $(r - a) \sin \theta = b$.

The equation can be written as $u = \frac{\sin \theta}{b + a \sin \theta} = F(\theta)$ (say).

The directions in which $r \to \infty$ are given by $u = 0$, or $\sin \theta = 0$, giving $\theta = n\pi$.

Now $F'(\theta)$ *i.e.*, $\frac{du}{d\theta} = \frac{\cos \theta \, (b + a \sin \theta) - \sin \theta \, (a \cos \theta)}{(b + a \sin \theta)^2}$

$$= \frac{b \cos \theta}{(b + a \sin \theta)^2}, \text{ and for } \theta = n\pi, \text{ this} = \frac{b \cos n\pi}{b^2} = \frac{\cos n\pi}{b}.$$

Hence, as in § 12·7, the asymptote required is given by

$$r \sin(\theta - n\pi) = 1/F'(n\pi) = b \sec n\pi$$

which, whether n is even or odd, reduces to

$$r \sin \theta = b.$$

Examples XII

Find the asymptotes of the following curves :

1. $y^2 - x^2 - 2x - 2y - 3 = 0.$

2. $y^3 - 6xy^2 + 11x^2y - 6x^3 + y^2 - x^2 + 2x - 3y - 1 = 0.$

3. $x^3 + 3x^2y - xy^2 - 3y^3 + x^2 - 2xy + 3y^2 + 4x + 5 = 0.$

4. $3x^3 + 2x^2y - 7xy^2 + 2y^3 - 14xy + 7y^2 + 4x + 5y = 0.$ [*C. P. 1943*]

5. $x^3 + 2x^2y - xy^2 - 2y^3 + 4y^2 + 2xy - 5y + 6 = 0.$

6. $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1.$ 7. $x^3 - y^3 = 3y(x+y).$

8. $x^4 - y^4 + 3x^2y + 3xy^2 + xy = 0.$

9. $4x^4 - 5x^2y^2 + y^4 + y^3 - 3x^2y + 5x - 8 = 0.$

10. (i) $xy - 2y - 3x = 0.$ (ii) $y^2(x^2 - a^2) = x.$

(iii) $x(x^2 + y^2) + a(x^2 - y^2) = 0.$

11. $x^2y^2 - 4(x-y)^2 + 2y - 3 = 0.$

12. $x^2y^2 - x^2y - xy^2 + x + y + 1 = 0.$ [*C. P. 1937*]

13. $y^2x^2 - 3yx^2 - 5xy^2 + 2x^2 + 6y^2 - x - 3y + 2 = 0.$

14. $x^4 - x^2y^2 + x^2 + y^2 - a^2 = 0.$

15. $y^3 - yx^2 + y^2 + x^2 - 4 = 0.$

16. $x^2(x-y)^2 - a^2(x^2 + y^2) = 0.$ [*C. P. 1945*]

17. $x^3 - 4xy^2 - 3x^2 + 12yx - 12y^2 + 8x + 2y + 4 = 0.$

18. $x^3 + 3x^2y - 4y^3 - x + y + 3 = 0.$

19. $y^3 - xy^2 - x^2y + x^3 + x^2 - y^2 - 1 = 0.$ [*C. P. 1939*]

20. $y^3 + x^2y + 2xy^2 - y + 1 = 0.$ [*C. P. 1941, '44*]

21. $(x^2 - y^2)^2 - 8(x^2 + y^2) + 8x - 16 = 0.$

22. $y(y - x)^2(y - 2x) + 3x^2(y - x) - 2x^2 = 0.$

23. $x^2(x + y)(x - y)^2 + 2x^3(x - y) - 4y^3 = 0.$

24. $(x + y)(x - 2y)(x - y)^2 + 3xy(x - y) + x^2 + y^2 = 0.$

25. $(x + y)^3(x - y)^2 - 2(x + y)^2(x - y)^2 - 2(x^2 + y^2)(x + y)$
$+ 2(x - y)^2 + 4(x - y) = 0.$

26. $y^3 - 5xy^2 + 8x^2y - 4x^3 - 4y^2 + 12xy - 8x^2$
$+ 3y - 3x + 2 = 0.$

27. (i) $xy(x^2 - y^2) = x^2 + y^2.$ (ii) $xy(x^2 + y^2) = x^2 - y^2.$

28. $(x^2 - y^2)(x^2 - 9y^2) + 3xy - 6x - 5y + 2 = 0.$

29. (i) $x^3 - 6x^2y + 11xy^2 - 6y^3 + 2x - y + 1 = 0.$
(ii) $x^4 - 5x^2y^2 + 4y^4 + x^2 - 2y^2 + 2x + y + 7 = 0.$

30. $(x - y + 1)(x + y + 1)(x - 2y + 3) = 2x - 5y + 1.$

*31. (i) $y = \tan x.$ (ii) $y = e^{ax}.$
(iii) $y = e^{-x^2}.$ (iv) $y = \log x.$

32. Show that the asymptotes of the curve
$x^2y^2 = a^2(x^2 + y^2)$ form a square of side $2a$.

33. Show that the asymptotes of the curve
$$x^2y^2 - a^2(x^2 + y^2) - a^3(x + y) + a^4 = 0$$
form a square two of whose angular points lie on the curve.
[*C. P. 1947*]

34. Show that the finite points of intersection of the asymptotes of $xy(x^2-y^2)+a(x^2+y^2)-a^3=0$ with the curve lie on a circle whose centre is at the origin.

35. Find the equation of the cubic which has the same asymptotes as the curve $2x(y-3)^2=3y(x-1)^2$ and which touches the axis of x at the origin and goes through the point (1, 1).

***36.** If any of the asymptotes of the curve

$$ax^2+2hxy+by^2+2gx+2fy+c=0 \ (h^2>ab)$$

pass through the origin, prove that

$$af^2+bg^2=2fgh.$$

***37.** If the equation of a curve can be put in the form

$$y=ax+b+\phi(x), \text{ where } \phi(x)\to 0 \text{ as } x\to\infty,$$

then show that $y=ax+b$ is an asymptote of the curve. Apply this method in determining the asymptotes of the curve $x^2y-x^3-x^2-3x+2=0$.

***38.** 'An asymptote is sometimes defined as a straight line which cuts the curve in two points at infinity without being itself at infinity'. Comment on this definition. Attempt a correct definition and use it to obtain the asymptotes of $(x+y)^2(x+2y+2)=x+9y-2$.

39. Find the asymptotes of :

(i) $r=a(\cos\theta+\sec\theta)$. (ii) $r\cos\theta=2a\sin\theta$.

(iii) $r=a\sec\theta+b\tan\theta$. (iv) $r\cos\theta=a\sin^2\theta$.

(v) $r=a\operatorname{cosec}\theta+b$. (vi) $r\sin n\theta=a$.

(vii) $r\theta=a$. [*C. P. 1937*] (viii) $r^n\sin n\theta=a^n \ (n>1)$.

40. Show that there is an infinite series of parallel asymptotes to the curve

$$r = \frac{a}{\theta \sin \theta} + b.$$

41. Show that all the asymptotes of the curve

$$r \tan n\theta = a$$

touch the circle $r = a/n$.

***42.** Show that the curve

$$r = a \sec n\theta + b \tan n\theta$$

has two sets of asymptotes, members of each set touching a fixed circle.

ANSWERS

1. $y - x = 2$, $y + x = 0$. **2.** $y = x$, $y = 2x + 3$, $y = 3x - 4$.

3. $4x + 12y + 9 = 0$, $2x + 2y - 3 = 0$, $4x - 4y + 1 = 0$.

4. $6y - 6x + 7 = 0$, $2y - 6x + 3 = 0$, $6y + 3x + 5 = 0$.

5. $x + 2y = 0$, $x + y - 1 = 0$, $x - y + 1 = 0$.

6. $y = \pm \dfrac{b}{a} x$. **7.** $x - y = 2$. **8.** $x + y = 0$, $2x - 2y + 3 = 0$.

9. $3x - 3y - 1 = 0$, $3x + 3y + 1 = 0$, $12x - 6y - 1 = 0$, $12x + 6y + 1 = 0$.

10. (i) $x = 2$, $y = 3$. (ii) $y = 0$, $x = \pm a$. (iii) $x = a$.

11. $x = \pm 2$; $y = \pm 2$. **12.** $x = 0$, $x = 1$, $y = 0$, $y = 1$.

13. $x = 2$, $x = 3$, $y = 1$, $y = 2$. **14.** $x = \pm 1$; $y = \pm x$.

15. $y = 1$, $y = x - 1$, $y = -x - 1$. **16.** $x = \pm a$, $x - y = \pm a\sqrt{2}$.

17. $x + 3 = 0$; $x - 2y = 0$; $x + 2y = 6$.

18. $y = x$; $x + 2y - 1 = 0$; $x + 2y + 1 = 0$.

19. $y = \pm x$; $y = x + 1$. **20.** $y = 0$, $x + y = \pm 1$.

21. $y = x \pm 2$; $y = -x \pm 2$.

22. $2y + 3 = 0$; $x - y + 1 = 0$; $x - y + 2 = 0$; $4x - 2y - 3 = 0$.

23. $x=\pm 2$; $x-y+2=0$; $x-y-1=0$; $x+y+1=0$.

24. $x-y-2=0$; $2x-2y+1=0$; $2x+2y-1=0$; $x-2y+2=0$.

25. $x+y-2=0$; $x+y=\pm 1$; $x-y=\pm 1$.

26. $y=x$; $y=2x+1$; $y=2x+3$.

27. (i) $x=0$, $y=0$, $x+y=0$, $x-y=0$. (ii) $x=0$, $y=0$.

28 $x+y=0$, $x-y=0$, $x+3y=0$, $x-3y=0$.

29 (i) $x-y=0$, $x-2y=0$, $x-3y=0$.

(ii) $x+2y=0$, $x-2y=0$, $x+y=0$, $x-y=0$.

30. $x-y+1=0$, $x+y+1=0$, $x-2y+3=0$.

31. (i) $x=(2n+1)\frac{1}{2}\pi$, where n is zero or any integer positive or negative

(ii) $y=0$. (iii) $y=0$. (iv) $x=0$.

35. $2xy^2-3x^2y-6xy+7y=0$. 37. $y=x+1$.

38. $x+2y+2=0$; $x+y=\pm 2\sqrt{2}$.

39. (i) $r\cos\theta=a$. (ii) $r\cos\theta=\pm 2a$. (iii) $r\cos\theta=a\pm b$.

(iv) $r\cos\theta=a$. (v) $r\sin\theta=a$.

(vi) $r\sin\left(\theta-\dfrac{m\pi}{n}\right)=\dfrac{a}{n}\sec m\pi$, where m is an integer.

(vii) $r\sin\theta=a$. (viii) $\theta=\dfrac{m\pi}{n}$, where m is an integer.

CHAPTER XIII

SEC. A. ENVELOPE OF STRAIGHT LINES

13·1. Introduction.

Let us consider the equation

$$x \cos \alpha + y \sin \alpha = a.$$

This represents a straight line ; by giving different values to α, we shall obtain the equations of different straight lines but all these different straight lines have one characteristic feature common to each of them *viz.*, each straight line is at the same distance a from the origin. On account of this property these straight lines are said to constitute *a family*, and α, which is constant for one line but different for different lines, and whose different values give different members of the family, is called the *parameter of the family*. It should be noted that the position of the line varies with α.

As we have a family of straight lines, we have a family of curves. Thus the equation

$$(x-a)^2+y^2=r^2$$

represents a family of circles for different values of a, all the individual members of the family having the common characteristic *viz.*, they are of equal radii 'r' and their centres lie on x-axis. Here a is the parameter of the family.

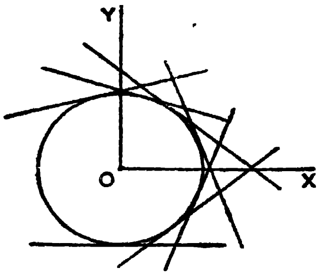
In general, *the equation of a family of curves* is represented by $F(x, y, \alpha)=0$, when α is the parameter.

13·2. Definition of Envelope.

If each of the members of the family of curves $C \equiv F(x, y, \alpha)=0$ touches a fixed curve E, then E is called the *envelope* of the family of curves C. The curve E also, at each point, is touched by some member of the family C.

Illustration :

We know that $x \cos a + y \sin a = a$ touches the circle $x^2 + y^2 = a^2$ at $(a \cos a, a \sin a)$. Thus, each of the members of the family of st. lines $C \equiv x \cos a + y \sin a = a$ (for different values of a) touches the fixed circle $E \equiv x^2 + y^2 = a^2$, and hence *the circle* $x^2 + y^2 = a^2$ *is the envelope of the family of straight lines* $x \cos a + y \sin a = a$; also the circle $x^2 + y^2 = a^2$ at each point $(a \cos a, a \sin a)$ obtained by varying values of a, is touched by some member of the family of straight lines.



In the present section we shall confine ourselves to the determination of the simplest type of envelopes, *i.e.*, the envelopes of straight lines.

13·3. Envelope of straight lines.

The equation of the envelope of the family of straight lines $F(x, y, a) \equiv y - f(a).x - \phi(a) = 0$ *(a being the parameter) is the a-eliminant of* $\mathbf{F = 0}$ *and* $\dfrac{\mathbf{dF}}{\mathbf{d}a} \mathbf{= 0}$.

From $F = 0$, and $\dfrac{dF}{da} = 0$, we have respectively

$$y = f(a)\, x + \phi(a). \qquad \cdots \quad (1)$$

and $$0=f'(a)x+\phi'(a). \qquad \cdots \quad (2)$$

$\therefore$ from (2), $$x=-\frac{\phi'(a)}{f'(a)}=g(a) \text{ say} \qquad \cdots \quad (3)$$

and from (1), $$y=\frac{f'(a)\phi(a)-\phi'(a)f(a)}{f'(a)}=h(a) \text{ say.} \qquad (4)$$

Hence the curve (*i.e.*, the envelope) whose equation is obtained by eliminating a between (1) and (2) is the same as the curve whose equation is given parametrically as

$$\left.\begin{array}{l} x=g(a) \\ y=h(a) \end{array}\right\}. \qquad \cdots \quad (5)$$

Now the equation of the tangent at the point 'a' on the curve (5) is given by

$$y-h(a)=\frac{h'(a)}{g'(a)}\{x-g(a)\}. \qquad \cdots \quad (6)$$

Substituting from (3) and (4) values of $g(a)$, $h(a)$, $g'(a)$, $h'(a)$ in (6) and noting that $h'(a)/g'(a)$ reduces to $f(a)$, and simplifying, we get the equation (6) *i.e.*, the equation of the tangent at 'a' on the curves (5) as

$$y=f(a)x+\phi(a)$$

which is the same as the equation of the given family of straight lines.

Thus every member of the family of straight lines $F(x, y, a)=0$ touches the curve whose equation is given as the a-eliminant of $F=0$ and $\dfrac{dF}{da}=0$, and hence the a-eliminant curve is the envelope of the family of straight lines.

Cor. 1. From the definition, it at once follows that *every curve is the envelope of its tangents.*

Cor. 2. Since we have seen that normals at different points on a curve touch the evolute at the corresponding points, it follows that **the evolute of a curve is the envelope of its normals.**

Thus, if $N(x, y, \alpha)=0$ be the equation of the normal of a curve at a point with parameter α, the evolute is obtained by *eliminating* α between $N(x, y, \alpha)=0$ (i) and $\dfrac{\delta N}{\delta \alpha}=0$ (ii). Since the evolute is the locus of centres of curvature, the co-ordinates of the centre of curvature are obtained in parametric form by *solving* the above two equations for x and y in terms of α.

The above methods of determining the evolute and centre of curvature are much simpler than the methods already given in the chapter XI (curvature). [See *Ex. 4* and *Ex.* 5, *Art. 13·4*]

13·4. Illustrative Examples.

Ex. 1. *Find the envelope of the straight line* $y=mx+\dfrac{a}{m}$, *m being the variable parameter* ($m \neq 0$).

Here, $$mx+\frac{a}{m}-y=0. \qquad \cdots \quad (1)$$

Differentiating with respect to m,

$$x-\frac{a}{m^2}=0, \quad \therefore\ m^2=\frac{a}{x}. \quad \therefore\ m=\pm\sqrt{\frac{a}{x}}.$$

Substituting this value of m in (1),

$$\pm\left(\sqrt{\frac{a}{x}}\cdot x+a\Big/\sqrt{\frac{a}{x}}\right)-y=0,$$

i.e., $\pm 2\sqrt{ax}=y$, or, $y^2=4ax$ (parabola) which is the required envelope.

Ex. 2. *Find the envelope of the family of straight lines*

$$A\alpha^2+B\alpha+C=0,$$

where α is the variable parameter, and A, B, C are linear functions of x, y.

We have $A\alpha^2+B\alpha+C=0$. ... (1)

Differentiating this with respect to α, we have

$$2A\alpha+B=0, \qquad i.e., \quad \alpha=-B/2A.$$

Substituting this value of α in (1), we get

$$A\cdot\frac{B^2}{4A^2}-\frac{B^2}{2A}+C=0, \qquad \text{or,} \qquad B^2=4AC.$$

Thus, the **envelope** of the family of straight lines $\mathbf{A\alpha^2+B\alpha+C=0}$ is the curve $\mathbf{B^2=4AC}$.

Note. When the parameter occurs as a quadratic in any equation, the above result is sometimes used in determining the envelope.

Ex. 3. *Find the envelope of the straight lines*

$$\frac{x}{a}+\frac{y}{b}=1,$$

where a and b are variable parameters, connected by the relation $a+b=c$, c being a non-zero constant.

Since $\quad a+b=c, \quad \therefore \quad b=c-a.$

$\therefore$ the equation of the straight lines becomes

$$\frac{x}{a}+\frac{y}{c-a}=1, \qquad \text{or,} \quad (c-a)x+ay=a(c-a),$$

$$\text{or,} \quad a^2+a(y-x-c)+cx=0.$$

Since it is in the form $Aa^2+Ba+C=0$, its envelope is

$$B^2=4AC, \quad i.e., \quad (y-x-c)^2=4cx,$$

which represents a parabola.

The above equation can be written as

$$x^2+y^2+c^2=2xy+2cx+2cy,$$

or, $\sqrt{x}+\sqrt{y}=\sqrt{c}$, (which represents a parabola).

Otherwise :

The elimination of a and b can also be performed thus :

Differentiating the equation of the line and the given relation with respect to a, we get

$$-\frac{x}{a^2}-\frac{y}{b^2}\frac{db}{da}=0 \text{ and } 1+\frac{db}{da}=0.$$

On equating the values of $\frac{db}{da}$ from these two, we get

$$\frac{a}{\sqrt{x}}=\frac{b}{\sqrt{y}}=\frac{a+b}{\sqrt{x}+\sqrt{y}}=\frac{c}{\sqrt{x}+\sqrt{y}}.$$

$$\therefore \quad a=\frac{c\sqrt{x}}{\sqrt{x}+\sqrt{y}}, \quad b=\frac{c\sqrt{y}}{\sqrt{x}+\sqrt{y}}.$$

Substituting these values of a and b in the equation of the line, we get

$$(\sqrt{x}+\sqrt{y})^2=c, \quad i.e., \quad \sqrt{x}+\sqrt{y}=\sqrt{c}.$$

Ex. 4. *Find the evolute of the parabola $y^2=4ax$ (evolute being regarded as the envelope of its normals).*

The equation of the normal to the parabola at any point 'm' is

$$y=mx-2am-am^3. \quad \cdots \quad (1)$$

Let us find the envelope of this, m being the parameter. Differentiating (1) with respect to m,

$$0=x-2a-3am^2, \text{ or, } m^2=(x-2a)/3a \quad \cdots \quad (2)$$

From (1), $y=m(x-2a-am^2)=m(3am^2-am^2)=2am^3$.

$$\therefore \quad y^2=4a^2m^6=4a^2\cdot\frac{(x-2a)^3}{27a^3} \text{ from (2),}$$

$$i.e., \quad 27ay^2=4(x-2a)^3,$$

which is the envelope of the normals *i.e.*, the required evolute of the parabola.

Ex. 5. *Find the centre of curvature of the ellipse*

The normal at the point 'ϕ' is

$$ax \sec\phi-by \operatorname{cosec}\phi=a^2-b^2. \quad \cdots \quad (1)$$

Differentiating this partially with respect to ϕ,

$$ax \sec\phi \tan\phi+by \operatorname{cosec}\phi \cot\phi=0. \quad \cdots \quad (2)$$

Solving for x and y from (1) and (2), we easily get

$$x=\frac{a^2-b^2}{a}\cos^3\phi, \quad y=-\frac{a^2-b^2}{b}\sin^3\phi.$$

Examples XIII(A)

Find the envelopes of the following families of straight lines :

1. $x \cos \alpha + y \sin \alpha = a$, parameter α.

2. $ax \sec \alpha - by \operatorname{cosec} \alpha = a^2 - b^2$, parameter α.

3. $x \cos 3\theta + y \sin 3\theta = a(\cos 2\theta)^{\frac{3}{2}}$, parameter θ.

4. $x \cos \alpha + y \sin \alpha = a \cos \alpha \sin \alpha$, parameter α.

5. $y = mx + a\sqrt{1 + m^2}$, parameter m.

6. $y = mx + \sqrt{a^2m^2 + b^2}$, parameter m.

7. $x \sec^2\theta + y \operatorname{cosec}^2\theta = a$, parameter θ.

8. $x\sqrt{\cos \theta} + y\sqrt{\sin \theta} = a$, parameter θ.

9. $x \cos^n\theta + y \sin^n\theta = a$, parameter θ.

10. Find the envelopes of the straight line

$$\frac{x}{a} + \frac{y}{b} = 1,$$

where the parameters a and b are connected by the relation

(i) $a^2 + b^2 = c^2$, (ii) $ab = c^2$, c being a constant.

11. Find the envelope of the straight line

$$\frac{x}{l} + \frac{y}{m} = 1,$$

where l and m are parameters connected by the relation

$l/a + m/b = 1$, a and b being constants.

12. Find the envelopes of straight lines at right angles to the radii of the following curves drawn through their extremities.

(i) $r = a(1 + \cos \theta)$. (ii) $r^2 = a^2 \cos 2\theta$. (iii) $r = ae^{m\theta}$.

13. From any point P on a parabola, PM and PN are drawn perpendiculars to the axis and the tangent at the vertex ; show that the envelope of MN is another parabola.

14. Show that the envelope of straight lines which join the extremities of a pair of conjugate diameters of an ellipse is a similar ellipse.

15. If PM, PN be the perpendiculars drawn from any point P on the curve $y = ax^3$ upon the co-ordinate axes, show that the envelope of MN is

$$27y + 4ax^3 = 0.$$

16. From any point P on the ellipse $x^2/a^2 + y^2/b^2 = 1$, perpendiculars PM, PN are drawn upon the co-ordinate axes. Show that MN always touches the curve

$$(x/a)^{\frac{2}{3}} + (y/b)^{\frac{2}{3}} = 1.$$

17. Find the envelope, when t varies, of

$$(a_1t^2 + 2a_2t + a_3)x + (b_1t^2 + 2b_2t + b_3)y + (c_1t^2 + 2c_2t + c_3) = 0.$$

18. Find the evolutes of the following curves (evolute being regarded as envelope of normals).

(i) $x = a \cos \phi$, $y = b \sin \phi$. (ii) $x = at^2$, $y = 2at$.

(iii) $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$. (iv) $x^2/a^2 + y^2/b^2 = 1$.

(v) $x = a(\theta - \sin \theta)$, $y = a(1 - \cos \theta)$.

(vi) $x = a(\cos t + t \sin t)$, $y = a(\sin t - t \cos t)$.

19. Two particles P, Q move along parallel straight lines, one with uniform velocity u and the other with the same initial velocity u but with uniform acceleration f. Show that the line joining them always touches a fixed hyperbola.

20. Show that the radius of curvature of the envelope of the line $x \cos a + y \sin a = f(a)$ is $f(a) + f''(a)$.

ANSWERS

1. $x^2 + y^2 = a^2$. **2.** $(ax)^{\frac{2}{3}} + (by)^{\frac{2}{3}} = (a^2 - b^2)^{\frac{2}{3}}$.

3. $(x^2 + y^2)^2 = a^2(x^2 - y^2)$. **4.** $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$. **5.** $x^2 + y^2 = a^2$.

6. $x^2/a^2 + y^2/b^2 = 1$. **7.** $\sqrt{x} + \sqrt{y} = \sqrt{a}$. **8.** $x^{\frac{4}{5}} + y^{\frac{4}{5}} = a^{\frac{4}{5}}$.

9. $x^{\frac{2}{2-n}} + y^{\frac{2}{2-n}} = a^{\frac{2}{2-n}}$. **10.** (i) $x^{\frac{2}{3}} + y^{\frac{2}{3}} = c^{\frac{2}{3}}$. (ii) $4xy = c^2$.

11. $\sqrt{\frac{x}{a}} + \sqrt{\frac{y}{b}} = 1$.

12. (i) a circle through the pole. (ii) a rectangular hyperbola.
(iii) an equiangular spiral.

17. $(a_1x + b_1y + c_1)(a_3x + b_3y + c_3) = (a_2x + b_2y + c_2)^2$.

18. (i) $(ax)^{\frac{2}{3}} + (by)^{\frac{2}{3}} = (a^2 - b^2)^{\frac{2}{3}}$ (ii) $27ay^2 = 4(x - 2a)^3$.

(iii) $(x + y)^{\frac{2}{3}} + (x - y)^{\frac{2}{3}} = 2a^{\frac{2}{3}}$. (iv) same as (i).

(v) $x = a\,(\theta + \sin\theta),\ y = -a\,(1 - \cos\theta)$ (vi) $x^2 + y^2 = a^2$.

SEC. B. ENVELOPE OF CURVES

13·5. If a curve E exists, which touches each member of a family of curves C [$\equiv f(x, y, a) = 0$], the curve E is called the Envelope of the curves C. Since E is the locus of the points of contact of the family of curves $f(x, y, a) = 0$, the point where the curve $f(x, y, a) = 0$ for a particular value of a touches E, depends upon that value of a. Accordingly the co-ordinates of any point on E are functions of the parameter a [being of the forms $x = \phi(a)$, $y = \psi(a)$] and they satisfy the equation $f(x, y, a) = 0$ of the enveloping curve which touches E at that point.

The equation of the tangent to a C-curve at (x, y) is

$$(X-x)\frac{\partial f}{\partial x}+(Y-y)\frac{\partial f}{\partial y}=0, \quad \cdots \quad (1)$$

and that of the tangent to the E-curve at (x, y) is

$$\frac{X-x}{\dfrac{dx}{da}}=\frac{Y-y}{\dfrac{dy}{da}} \quad \cdots \quad (2)$$ [Since equation of E is of the form $x=\phi(a)$, $y=\psi(a)$]

$$\text{or,} \quad (X-x)\frac{dy}{da}-(Y-y)\frac{dx}{da}=0. \quad \cdots \quad (3)$$

Since the lines (1) and (3) are coincident, coefficients of X and Y in the above two equations are proportional.

$$\therefore \quad \frac{\dfrac{\partial f}{\partial x}}{\dfrac{dy}{da}}=\frac{\dfrac{\partial f}{\partial y}}{-\dfrac{dx}{da}} \; i.e., \; \frac{\partial f}{\partial x}\frac{dx}{da}+\frac{\partial f}{\partial y}\frac{dy}{da}=0. \quad \cdots \quad (4)$$

Now, differentiating $f(x, y, a)=0$ with respect to a, remembering that x and y are now functions of a, we get

$$\frac{\partial f}{\partial a}+\frac{\partial f}{\partial x}\frac{dx}{da}+\frac{\partial f}{\partial y}\frac{dy}{da}=0. \quad \cdots \quad \cdots \quad (5)$$

$$\therefore \quad \text{from (4)}, \frac{\partial f}{\partial a}=0. \quad \cdots \quad \cdots \quad (6)$$

Hence the equation of the envelope, in case an envelope exists, is to be found by eliminating the parameter a between the equations

$$\left.\begin{aligned} f(x, y, a)&=0 \\ \text{and} \quad \frac{\partial f}{\partial a}&=0. \end{aligned}\right\} \quad \cdots \quad \cdots \quad (7)$$

Cor. It is shown in Art. 14·1 (iii) that the circle on the radius vector of a curve as diameter touches the *pedal of the curve*, *so the* *pedal of a curve can be obtained as the envelope of the circles described on the radius vectors of the curve as diameters.* [See *Art. 13·9, Ex. 6.*]

Note. It should be noted that a-eliminant between $f(x, y, a)=0$ and $\frac{\partial f}{\partial a}=0$ may contain other loci, besides the envelope, for instance, nodal locus, cuspidal locus, tac-locus etc. in case, the family of curves C has singular points.

13·6. *The envelope is, in general, the locus of the ultimate points of intersection of neighbouring curves of a family.*

The co-ordinates of the point of intersection P, of two neighbouring curves of the family must satisfy

$$f(x, y, a)=0 \text{ and } f(x, y, a+\Delta a)=0,$$

$$i.e., \; f(x, y, a)=0 \text{ and } \frac{f(x, y, a+\Delta a)-f(x, y, a)}{\Delta a}=0,$$

the second relation by Mean Value Theorem becomes

$$\frac{\partial}{\partial a} f(x, y, a+\theta\Delta a)=0 \text{ where } 0<\theta<1.$$

Now as $\Delta a \to 0$, P satisfies $f=0, \frac{\partial f}{\partial a}=0$.

$\therefore$ the required locus is the a-eliminant of $f=0, \frac{\partial f}{\partial a}=0$.

Note 1. Although the above theorem is generally true, but it is not always true. For example, consider the *family of semi-cubical parabolas* $y=(x-a)^3$. Here for different values of a, we have different semi-cubical parabolas no two of which intersect but every one of which touches the x-axis. So here the x-axis is the envelope, although no two members of the family intersect. From the graphs of the curves, the whole thing becomes at once clear.

Note 2. *Alternative definition of Envelope.*

The points of intersection of the curves $f(x, y, a)=0$ and $\frac{\partial}{\partial a} f(x, y, a)=0$ (a being given) are called *characteristic points* of the family

$f(x, y, a)=0$ (for the given a) if these points exist (*i.e.*, if $f=0$ and $\frac{\partial f}{\partial a}=0$ intersect) and if those points are not singular points of $f(x, y, a)=0$. *The locus of the characteristic points of a family of curves is sometimes called the envelope of the family.*

13·7. Envelope of a special family.

If the curve $f(x, y, a)=0$ be algebraic, the a-eliminant of $f=0, \frac{\partial f}{\partial a}=0$ is the condition that $f=0$ (considered as an equation in a, has equal roots). [*Theorem of Equations*]

Thus, if $f(x, y, a)=A(x, y)\, a^2+B(x, y) a+C=0$, [*i.e.*, if $f(x, y, a)=0$ be a quadratic in a, the parameter] the envelope of the family is given by

$$B^2=4AC.$$

[*For illustration* see *Ex. 2 of Art. 13·9*]

13·8. *Envelope of two parameter family.*

If the equation of a family of curves involves two parameters a and β connected by a given equation, then we can proceed by two methods in finding out the envelope. Suppose the equation of the family is

$$f(x, y, a, \beta)=0, \qquad \cdots \qquad \cdots \quad (1)$$

where a and β are connected by the equation

$$\phi(a, \beta)=0. \qquad \cdots \qquad \cdots \quad (2)$$

First Method: Suppose we can solve $\phi(a, \beta)=0$ for β in terms of a; then we substitute this value of β in (1) and now (1) reduces to one parameter family and we eliminate a between $f=0$ and $\frac{\partial f}{\partial a}=0$.

Second Method :

For a particular point (x, y) on the envelope

$$\frac{\partial f}{\partial \alpha} d\alpha + \frac{\partial f}{\partial \beta} d\beta = 0 \quad \cdots \quad \cdots \quad (3)$$

and from (2), $$\frac{\partial \phi}{\partial \alpha} d\alpha + \frac{\partial \phi}{\partial \beta} d\beta = 0. \quad \cdots \quad \cdots \quad (4)$$

Eliminating $d\alpha$, $d\beta$ between these two equations (we may regard α as the independent variable and β the dependent variable), we get

$$\frac{\dfrac{\partial f}{\partial \alpha}}{\dfrac{\partial \phi}{\partial \alpha}} = \frac{\dfrac{\partial f}{\partial \beta}}{\dfrac{\partial \phi}{\partial \beta}}. \quad \cdots \quad \cdots \quad (5)$$

If we eliminate α and β between (1), (2) and (5), we obtain the equation of the envelope.

Note. This method can be extended to obtain the envelope of a family depending upon n parameters which are connected by $(n-1)$ equations.

13·9. Illustrative Examples.

Ex. 1. *Find the envelope of the family of ellipses*

$$\frac{x^2}{\alpha^2} + \frac{y^2}{(a-\alpha)^2} = 1,$$

α being the parameter.

We have $$x^2 \alpha^{-2} + y^2 (a-\alpha)^{-2} = 1. \quad \cdots \quad \cdots \quad (1)$$

Differentiating with respect to α, we have

$$\frac{x^2}{\alpha^3} = \frac{y^2}{(a-\alpha)^3}. \quad \cdots \quad \cdots \quad (2)$$

$$\therefore \quad \frac{\dfrac{x^2}{\alpha^2}}{\alpha} = \frac{\dfrac{y^2}{(a-\alpha)^2}}{a-\alpha} = \frac{\dfrac{x^2}{\alpha^2} + \dfrac{y^2}{(a-\alpha)^2}}{a} = \frac{1}{a} \quad [\textit{by (1)}]$$

$$i.e., \frac{x^2}{a^2}=\frac{y^2}{(a-a)^2}=\frac{1}{a}. \qquad \therefore \quad \frac{x^{\frac{2}{3}}}{a}=\frac{y^{\frac{2}{3}}}{a-a}=\frac{1}{a^{\frac{1}{3}}}=\frac{x^{\frac{2}{3}}+y^{\frac{2}{3}}}{a}.$$

$\therefore \quad x^{\frac{2}{3}}+y^{\frac{2}{3}}=a^{\frac{2}{3}}$ which is the required a-eliminant between (1) and (2) and hence is the equation of the required envelope.

Ex. 2. *Find the envelope of the system of parabolas* $\lambda x^2+\lambda^2 y=1$, λ *being the parameter.*

Since the equation of the family is

$$\lambda^2 y+\lambda x^2-1=0,$$

and since it is a quadratic in λ, the parameter, by Art. 13·7, its envelope is

$$x^4+4y=0.$$

Ex. 3. *Prove that the envelope of the paths of projectiles in vacuo from the same point with the same velocity in the same vertical plane is a parabola with the point of projection as focus.*

The equation of the path of the projectile with the point of projection O as origin and the horizontal and the vertical lines through O as axes of x and y is

$$y=x\tan a-\tfrac{1}{2}g\frac{x^2}{u^2\cos^2 a}=x\tan a-\tfrac{1}{2}g\frac{x^2}{u^2}(1+\tan^2 a)$$

[See *Authors' Inter. Dynamics* : *Art. 8·5*]

$$=mx-\tfrac{1}{2}g\frac{x^2}{u^2}(1+m^2), \text{ where } \tan a=m,$$

$$i.e., \quad m^2.\tfrac{1}{2}g.\frac{x^2}{u^2}-mx+y+\tfrac{1}{2}g\frac{x^2}{u^2}=0.$$

Here a, and hence $\tan a$ *i.e.*, m being the variable parameter, the equation of the envelope is by Art. 13·7,

$$x^2=4.\frac{1}{2}g\frac{x^2}{u^2}\left(y+\frac{1}{2}g\frac{x^2}{u^2}\right)$$

$$i.e., \frac{u^2}{2g}=y+\frac{1}{2}g\frac{x^2}{u^2}. \qquad \therefore \quad x^2=-\frac{2u^2}{g}\left(y-\frac{u^2}{2g}\right).$$

Transferring the origin to the point $\left(0, \frac{u^2}{2g}\right)$, the equation of the envelope is $\quad x^2 = -\frac{2u^2}{g} y$

which is a parabola with its vertex on the y-axis at the point $\left(0, \frac{u^2}{2g}\right)$ and its concavity turned downwards and latus rectum '$4a$'$=\frac{2u^2}{g}$ and hence 'a' being equal to $\frac{u^2}{2g}$, the focus is at the origin.

Ex. 4. *Find the envelope of circles whose centres lie on the rectangular hyperbola $xy=c^2$ and which pass through its centre.*

Let the equation of a circle having centre at (α, β) and passing through the centre of $xy=c^2$, which is the origin here, be

$$x^2+y^2-2\alpha x-2\beta y=0 \;\ldots\; (1) \quad \text{where } \alpha\beta=c^2. \;\ldots\; (2)$$

This is the case of a two parameter family, where the parameters are connected by a given relation.

Following the first method of Art. 13·8 *i.e.*, eliminating β between (1) and (2), the equation of the circle becomes

$$x^2+y^2-2\alpha x-2\frac{c^2}{\alpha} y=0, \text{ since from (2), } \beta=\frac{c^2}{\alpha},$$

or, $$2\alpha^2 x-\alpha(x^2+y^2)+2c^2 y=0.$$

$\therefore$ by Art. 13·7, the required envelope is

$$(x^2+y^2)^2=4.2x.2c^2y=16c^2xy.$$

Note. By transformation to polars, this equation can be shown to be transformed to $r^2=8c^2 \cos 2\phi$, where $\phi=\frac{1}{4}\pi-\theta$; *i.e.*, the required envelope is a lemniscate.

Ex. 5. *Find the envelope of the parabola*

$$\sqrt{\frac{x}{a}}+\sqrt{\frac{y}{b}}=1,$$

where $ab=k^2$, a and b being variable parameters.

We have $$\sqrt{\frac{x}{a}}+\sqrt{\frac{y}{b}}=1 \quad \ldots \quad \ldots \quad (1)$$

$$ab=k^2. \quad \ldots \quad \ldots \quad (2)$$

We apply here the second method of Art. 13·8. Taking differentials of both (1) and (2) with respect to a and b, we have

$$\frac{\sqrt{x}}{a^{\frac{3}{2}}}\,da+\frac{\sqrt{y}}{b^{\frac{3}{2}}}\,db=0 \qquad \cdots \qquad \cdots \quad (3)$$

$$\frac{da}{a}+\frac{db}{b}=0. \qquad \cdots \qquad \cdots \quad (4)$$

From (3) and (4), $\dfrac{\dfrac{\sqrt{x}}{a^{\frac{3}{2}}}}{\dfrac{1}{a}}=\dfrac{\dfrac{\sqrt{y}}{b^{\frac{3}{2}}}}{\dfrac{1}{b}}$

or, $$\frac{\sqrt{\dfrac{x}{a}}}{1}=\frac{\sqrt{\dfrac{y}{b}}}{1}=\frac{\sqrt{\dfrac{x}{a}}+\sqrt{\dfrac{y}{b}}}{2}=\frac{1}{2} \text{ from (1)}$$

$$\therefore \quad \frac{\sqrt{\dfrac{x}{a}}\cdot\sqrt{\dfrac{y}{b}}}{1.1}=\frac{1}{4} \text{ i.e., } \frac{\sqrt{xy}}{\sqrt{ab}}=\frac{1}{4}.$$

Squaring and using (2), we get (a, b)-eliminant, $16xy=k^2$ and hence this is the required envelope. This obviously represents a hyperbola.

Ex. 6. *Find the pedal of the cardioide $r=a\ (1+\cos\theta)$ with respect to the pole (origin).*

We shall here find out the first positive pedal by considering it as the envelope of the circles described on the radii vectors as diameters.

[See *Cor., Art. 13·5*]

Let ρ, α be the polar co-ordinates of any point on the cardioide. Then $\rho=a\ (1+\cos\alpha)$. $\cdots$ $\cdots$ (1)

Again the equation of the circle on the radius vector ρ as diameter is $r=\rho\cos(\theta-\alpha)$ $\cdots$ $\cdots$ (2)

or, $r=a\ (1+\cos\alpha)\cos(\theta-\alpha)$. $\cdots$ $\cdots$ (3) *from (1)*

Here α is the parameter.

Differentiating (3) with respect to α,

$$0=-\sin\alpha\cos(\theta-\alpha)+(1+\cos\alpha)\sin(\theta-\alpha);$$

$$\therefore \quad \sin\alpha\cos(\theta-\alpha)-\cos\alpha\sin(\theta-\alpha)=\sin(\theta-\alpha);$$

i.e., $\sin(2\alpha-\theta)=\sin(\theta-\alpha)$,

i.e., $2\alpha-\theta=\theta-\alpha$, i.e., $\alpha=\frac{2}{3}\theta$. ... ... (4)

Substituting this value of α in (3), we have the required envelope as $r=a(1+\cos\frac{2}{3}\theta)\cos\frac{1}{3}\theta$

$$=2a\cos^3\tfrac{1}{3}\theta, \quad \text{or,} \quad r^{\frac{1}{3}}=(2a)^{\frac{1}{3}}\cos\tfrac{1}{3}\theta.$$

Ex. 7. *Show that the pedal equation of the envelope of the line*

$$x\cos 2\alpha+y\sin 2\alpha=2a\cos\alpha,$$

where α is the parameter, is $p^2=\frac{4}{3}(r^2-a^2)$.

Let (x, y) be the co-ordinates of any point P on the envelope.

Then (x, y) satisfy the equation, $f(x, y, \alpha)=0$, $\frac{\partial f}{\partial\alpha}=0$,

i.e., $x\cos 2\alpha+y\sin 2\alpha=2a\cos\alpha$ ... ... (1)

$x\sin 2\alpha-y\cos 2\alpha=a\sin\alpha$. ... ... (2)

From the definition of the envelope, it follows that (1) is the tangent to the envelope at $P(x, y)$.

Let p be the length of the perpendicular from the origin O upon the tangent (1) to the envelope at P and r be the distance of P from O.

$\therefore$ $p^2=4a^2\cos^2\alpha$ ... ... (3)

$r^2=x^2+y^2=4a^2\cos^2\alpha+a^2\sin^2\alpha$.

[*squaring and adding (1) and (2)*]

$=3a^2\cos^2\alpha+a^2$. ... ... (4)

Eliminating α between (3) and (4), the required pedal equation of the envelope is obtained.

Examples XIII(B)

1. Find the envelopes of the following curves, α being the parameter

(i) circles $(x-\alpha)^2+y^2-4\alpha=0$,

(ii) parabolas $\alpha y^2=2x+12\alpha^3$,

(iii) ellipses $x^2+\alpha^2y^2=4\alpha$.

2. Find the envelope of the family of curves, θ being the parameter

(i) $x^2 \cos\theta + y^2 \sin\theta = a^2$,

(ii) $P(x, y) \cos\theta + Q(x, y) \sin\theta = R(x, y)$,

(iii) $A(x, y) \cos^m\theta + B(x, y) \sin^m\theta = C(x, y)$.

3. Find the envelope of the family of curves

$$L\lambda^3 + 3M\lambda^2 + 3N\lambda + P = 0,$$

where λ is a parameter and L, M, N, P are functions of x and y.

4. Show that the envelope of the family of ellipses, (a being the parameter)

$$a^2x^2 \sec^4 a + b^2y^2 \operatorname{cosec}^4 a = (a^2 - b^2)^2$$

is the evolute of the ellipse $x^2/a^2 + y^2/b^2 = 1$.

5. Find the envelope of the family of circles which are described on the double ordinates of

(i) the parabola $y^2 = 4ax$ as diameters,

(ii) the ellipse $x^2/a^2 + y^2/b^2 = 1$ as diameters.

6. Find in each case the envelope of circles described upon OP as diameters where O is the origin and P is a point on

(i) the circles $x^2 + y^2 = 2ax$,

(ii) the parabolas $y^2 = 4ax$,

(iii) the ellipses $b^2x^2 + a^2y^2 = a^2b^2$,

(iv) the rectangular hyperbola $xy = c^2$.

7. If the centre of a circle lies upon the parabola $y^2 = 4ax$ and the circle passes through the vertex of the parabola, show that the envelope of the circle is

$$y^2 (2a + x) + x^3 = 0.$$

***8.** Find the envelope of the family of ellipses

$$\frac{x^2}{\alpha^2} + \frac{y^2}{\beta^2} = 1,$$

(i) whose sum of semi-axes is constant $(= c)$,

(ii) whose area is constant $(= \pi c^2)$.

***9.** Show that the envelope of the circles $x^2 + y^2 - 2\alpha x - 2\beta y + \beta^2 = 0$, where α, β are parameters and whose centres lie on the parabola $y^2 = 4ax$ is $x(x^2 + y^2 - 2ax) = 0$.

***10.** Find the envelope of (i) the family of ellipses $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ and (ii) the family of parabolas $\sqrt{\dfrac{x}{a}} + \sqrt{\dfrac{y}{b}} = 1$, where

$$a^n + b^n = c^n. \quad (\ a,\ b \text{ being the parameters })$$

***11.** Show that the envelope of the ellipses

$$\frac{(x - \alpha)^2}{a^2} + \frac{(y - \beta)^2}{b^2} = 1,$$

where the parameters α, β are connected by the relation

$$\frac{\alpha^2}{a^2} + \frac{\beta^2}{b^2} = 1,$$

is the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 4.$

***12.** Find the envelope of the family of curves

$$\frac{x^n}{a^n} + \frac{y^n}{b^n} = 1,$$

where the parameters a and b are connected by the equation

$$a^p + b^p = c^p.$$

***13.** Find the pedal with respect to the pole of the curve $r^2 = a^2 \cos 2\theta$.

***14.** Find the envelope of the circles described on the radii vectors of the curve $r^m = a^m \cos m\theta$ as diameters.

***15.** Show that the pedal equation of the envelope of the line

$x \cos m\alpha + y \sin m\alpha = a \cos n\alpha$, $(m \neq n)$, where α is the parameter,

is $$p^2 = \frac{m^2 r^2 - n^2 a^2}{m^2 - n^2}.$$

16. Given that the astroid $x^{\frac{2}{3}} + y^{\frac{2}{3}} = c^{\frac{2}{3}}$ is the envelope of the family of the ellipses $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$, where a and b are parameters, show that a and b are connected by the relation $a + b = c$.

ANSWERS

1. (i) $y^2 - 4x - 4 = 0$. (ii) $y^3 = \pm 18x$. (iii) $xy = \pm 2$.

2. (i) $x^4 + y^4 = a^4$. (ii) $P^2 + Q^2 = R^2$. (iii) $A^{\frac{2}{2-m}} + B^{\frac{2}{2-m}} = C^{\frac{2}{2-m}}$.

3. $(MN - LP)^2 = 4(MP - N^2)(AN - M^2)$. **5.** (i) $y^2 = 4a(x + a)$.

(ii) $\dfrac{x^2}{a^2 + b^2} + \dfrac{y^2}{b^2} = 1$. **6.** (i) $(x^2 + y^2 - ax)^2 = a^2 (x^2 + y^2)$.

(ii) $x(x^2 + y^2) + ay^2 = 0$.

(iii) $a^2x^2 + b^2y^2 = (x^2 + y^2)^2$. (iv) $(x^2 + y^2)^2 = 4c^2xy$.

8. (i) $x^{\frac{2}{3}} + y^{\frac{2}{3}} = c^{\frac{2}{3}}$. (ii) $2xy = c^2$. **10.** (i) $x^{\frac{2n}{n+2}} + y^{\frac{2n}{n+2}} = c^{\frac{2n}{n+2}}$.

(ii) $x^{\frac{n}{2n+1}} + y^{\frac{n}{2n+1}} = c^{\frac{n}{2n+1}}$. **12.** $x^{\frac{np}{n+p}} + y^{\frac{np}{n+p}} = c^{\frac{np}{n+p}}$.

13. $r^{\frac{2}{3}} = a^{\frac{2}{3}} \cos \frac{2}{3}\theta$. **14.** $r^n = a^n \cos n\theta$, where $n = m/(m+1)$.

CHAPTER XIV

ASSOCIATED LOCI

14·1. Pedal curve.

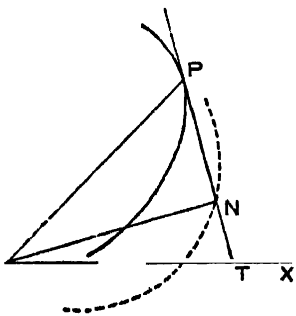
The locus of the foot of the perpendicular drawn from a fixed point on the tangent to a curve, is called *the pedal of the curve* with regard to the fixed point.

(i) *To find the pedal with regard to the origin of any curve whose cartesian equation is given.*

Let the equation of the curve be $f(x, y) = 0$. $\cdots$ (1)

Let $x \cos \alpha + y \sin \alpha = p$ be the equation of the tangent PT to the curve at any point P.

Now, the condition that the line $x \cos \alpha + y \sin \alpha = p$ should touch the curve is of the form $\phi(p, \alpha) = 0$. $\cdots$ (2)



Since (p, α) are the polar co-ordinates of the foot of the perpendicular N on the tangent PT, hence in (2), if r, θ are written for p, a, *the polar equation* of the locus of N *i.e., of the pedal curve* will be obtained as

$$\phi(r, \theta)=0 \qquad \cdots \qquad \cdots \qquad (3)$$

which can now be transformed into cartesian.

Alternative Method :

Let (x, y) be the co-ordinates of P; then the equation of the tangent PT is

$$Y-y=\frac{dy}{dx}(X-x) \qquad \cdots \qquad \cdots \qquad (1)$$

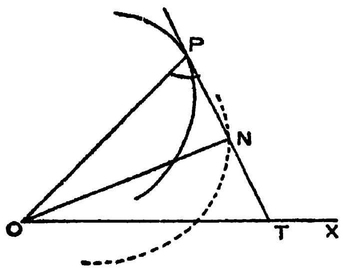
and the equation of ON, which passes through the origin and is perpendicular to PT, is

$$X+\frac{dy}{dx}Y=0, \qquad \cdots \qquad \cdots \qquad (2)$$

Hence the locus of N, (*i.e.*, the pedal) which is the intersection of (1) and (2) is obtained by eliminating x and y from (1) and (2) and from the equation of the curve $f(x, y)=0$.

(*ii*) *To find the pedal with regard to the pole of any curve whose polar equation is given.*

Let the polar equation of the curve be $f(r, \theta)=0$, $\cdots$ (1) and let (r_1, θ_1) be the polar co-ordinates of the foot of the perpendicular N drawn from O on the tangent at P (r, θ).



Now ϕ denoting $\angle OPN$, $\tan\phi=r\dfrac{d\theta}{dr}$. $\qquad \cdots \qquad (2)$

Also $\theta = \angle XOP = \angle XON + \angle NOP = \theta_1 + \frac{1}{2}\pi - \phi$ ⋯ (3)
and since $ON = OP \sin\phi$, $r_1 = r\sin\phi$. ⋯ (4)

If r, θ, ϕ be eliminated between (1), (2), (3) and (4), a relation between r_1 and θ_1 will be obtained, and from this relation by dropping the suffixes, we get the required polar equation of the pedal.

(*iii*) *The circle on radius vector as diameter touches the pedal.*

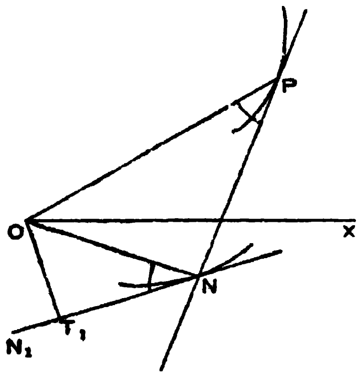
$\angle XON = \angle PTX - \angle ONT.$ [See *Fig. of (ii)*]

$\therefore \quad \theta_1 = \psi - \frac{1}{2}\pi$; also $p = ON = r_1$.

Again, $\dfrac{dp}{d\psi} = r\cos\phi$. [See *Ex.*, *7*, § *10·17*]

If ϕ_1 be the angle between the tangent NT_1 and radius vector ON of the pedal at any point N (*i.e.*, $\angle ONT_1 = \phi_1$), then

$$\tan\phi_1 = r_1\frac{d\theta_1}{dr_1} = r_1\frac{d\theta_1}{d\psi}\cdot\frac{d\psi}{dp}\cdot\frac{dp}{dr_1}.$$



$$= p\frac{d\psi}{dp} = \frac{r\sin\phi}{r\cos\phi} = \tan\phi$$

$\therefore \quad \phi_1 = \phi$, *i.e.*, $\angle ONT_1 = \angle OPN$.

$\therefore \quad T_1N$ touches the circle passing through OPN.

Hence the result.

(*iv*) *If p_1 be the perpendicular from the pole on the tangent to the pedal, then $p_1 r = p^2$.*

Draw OT_1 perpendicular from O on the tangent NT_1 to the pedal.

Since $\angle OPN = \angle ONT_1$, $\therefore$ Δ^s OPN, ONT_1 are similar.

$$\therefore \quad \frac{OP}{ON} = \frac{ON}{OT_1}, \text{ i.e., } \frac{r}{p} = \frac{p}{p_1}, \text{ i.e., } p_1 r = p^2.$$

(*v*) *To find the pedal of a curve when its pedal equation is given.*

Let the pedal equation of the curve be $p = f(r)$ $\quad \cdots \quad (1)$ and let p, r denote the usual entities on the original curve and p_1, r_1 the corresponding things of the pedal curve.

Then $p = r_1$; also from above, $r = \dfrac{p^2}{p_1} = \dfrac{r_1^{\,2}}{p_1}$.

Hence from the equation (1), we get

$$r_1 = f\left(\frac{r_1^{\,2}}{p_1}\right).$$

$\therefore$ the pedal equation of the pedal curve is

$$r = f\left(\frac{r^2}{p}\right).$$

Note. If there be a series of curves designated as

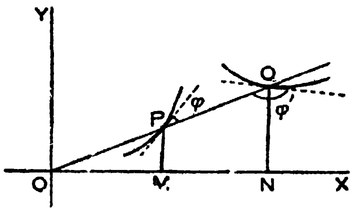
$$\Gamma, \Gamma_1, \Gamma_2 \ldots\ldots\ldots \Gamma_n$$

such that each is the pedal of the one which immediately precedes it, then Γ_1, Γ_2,...Γ_n are called the *first*, the *second*,...... the *nth positive pedal* of Γ. Also regarding any one curve of the series, say Γ_3, as the original curve, the preceding curves Γ_2, Γ_1, Γ are called respectively the *first, second* and *third negative pedal of* Γ_3.

14·2. Inverse curve.

If on the radius vector OP (or OP produced) from the origin O to any point P moving on a curve, a second point Q be taken such that $OP.OQ =$ a constant, say k^2, then the locus of Q is called the *inverse of the curve* along which P moves, with respect to a circle of radius k and centre O, or briefly with respect to O.

(*i*) *To find the inverse of a given curve whose cartesian equation is given.*



Let (x, y) be the co-ordinates of any point P on the curve $f(x, y)=0$, and let $Q(x', y')$ be a point on OP such that $OP.OQ=k^2$.

Draw PM, QN perpendiculars on OX.

Now, $\frac{x}{x'} = \frac{OM}{ON} = \frac{OP}{OQ}$ ($\because$ Δ^s OPM, OQN are similar)

$$= \frac{OP.OQ}{OQ^2} = \frac{k^2}{x'^2 + y'^2}.$$

$$\therefore\ x = \frac{k^2 x'}{x'^2 + y'^2}. \quad \text{Similarly, } y = \frac{k^2 y'}{x'^2 + y'^2}.$$

Since $f(x, y)=0$, $\therefore f\left(\frac{k^2x'}{x'^2+y'^2}, \frac{k^2y'}{x'^2+y'^2}\right) = 0.$

Hence by dropping dashes, the equation of the inverse curve is

$$f\left(\frac{k^2x}{x^2+y^2}, \frac{k^2y}{x^2+y^2}\right)=0,$$

i.e., the equation of the inverse of a curve is obtained by writing $k^2x/(x^2+y^2)$, $k^2y/(x^2+y^2)$ for x, y in the cartesian equation of the curve.

(*ii*) *To find the inverse of a given curve whose polar equation is given.*

Let $f(r, \theta)=0$ be the equation of the given curve and let (r, θ) be the co-ordinates of P and (r', θ) be the co-ordinates of Q.

Since $\quad OP.OQ=k^2, \quad \therefore \quad rr'=k^2. \qquad \therefore \quad r=k^2/r'.$

Again, since $f(r, \theta)=0, \quad \therefore \quad f\left(\frac{k^2}{r'}, \theta\right)=0.$

Hence the polar equation of the inverse curve is

$$f\left(\frac{k^2}{r}, \theta\right)=0.$$

Thus, the equation of the inverse of a curve is obtained by writing k^2/r for r in the polar equation of the curve.

(*iii*) *Tangents to a curve and its inverse are inclined to the radius vector at supplementary angles.*

Let ϕ, ϕ' denote the angles between the tangents and radius vector at the corresponding points of a curve and its inverse.

Then, $\qquad \tan\phi=\frac{rd\theta}{dr}, \quad \tan\phi'=r'\frac{d\theta}{dr'}.$

Now, $\quad (\because\ rr'=k^2),\ \frac{dr'}{d\theta}=\frac{d}{d\theta}\left(\frac{k^2}{r}\right)=-\frac{k^2}{r^2}.\frac{dr}{d\theta}.$

$$\therefore \quad \tan\phi' = r'\frac{d\theta}{dr'} = \frac{k^2}{r}\left(-\frac{r^2}{k^2}\right)\frac{d\theta}{dr} = -r\frac{d\theta}{dr} = -\tan\phi$$

$$= \tan(\pi - \phi).$$

$$\therefore \quad \phi' = \pi - \phi, \text{ i.e., } \phi + \phi' = \pi.$$

(*iv*) *To find the inverse of a curve when its pedal equation is given.*

Let the pedal equation of the given curve be

$$p = f(r). \qquad \cdots \qquad \cdots \quad (1)$$

Let p, r, ϕ denote the usual entities of the original curve, and let p', r', ϕ' denote the corresponding things of the inverse.

Then $\quad rr' = k^2, \quad$ *i.e.*, $\quad r = k^2/r'$.

Also, $\quad \dfrac{p'}{r'} = \sin\phi' = \sin(\pi - \phi) = \sin\phi = \dfrac{p}{r}$.

$$\therefore \qquad p = \frac{r}{r'}\,p' = \frac{rr'}{r'^2}\,p' = \frac{k^2}{r'^2}\,p'.$$

$\therefore$ from equation (1), we get

$$\frac{k^2}{r'^2}\,p' = f\left(\frac{k^2}{r'}\right).$$

Hence the pedal equation of the inverse curve is

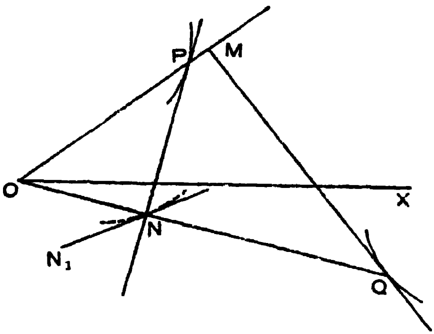
$$p = \frac{r^2}{k^2}\,f\left(\frac{k^2}{r}\right).$$

14·3. Polar reciprocal.

If on the perpendicular ON (or ON produced) from the origin on the tangent at any point P on a curve, a second point Q be taken such that $ON.OQ =$ a constant (say, k^2) then the locus of Q is called *the polar reciprocal of the given curve* with respect to a circle of radius k and centre O.

From the definition, it follows that *the polar-reciprocal of a curve is the inverse of its pedal.* Hence the equation of the polar reciprocal of a curve can be obtained by first finding the pedal of the curve and then its inverse.

Let NN_1 be the tangent to the pedal at N and let QM be the tangent to the polar reciprocal at Q meeting OP produced at M.



Now, $\phi = \angle OPN = \angle ONN_1$ [by § *14·1 (iii)*]. Since QM is the tangent to the inverse of the pedal, hence by § 14·2(iii), $\angle OQM = \angle ONN_1 = \angle OPN$.

Hence the quadrilateral $PMQN$ is cyclic.

$$\therefore \quad OM.OP = OQ.ON = k^2.$$

Also, $\because$ $\angle PNQ = 90°$, $\angle PMQ = 90°$, *i.e.*, OM is perpendicular to QM. Hence the locus of P, *i.e.*, the original curve is the polar reciprocal of the locus of Q, *i.e.*, of the polar reciprocal. Thus, *the polar reciprocal of the polar reciprocal of a curve is the curve itself.*

14·4. Illustrative Examples.

Ex. 1. *Find the pedal of the parabola $y^2 = 4ax$ with respect to the vertex.*

The condition that $X\cos\alpha+Y\sin\alpha=p$ will touch the parabola $y^2=4ax$ is obtained by comparing the equation with the equation of the tangent at (x, y) to the parabola, *i.e.*, with $Yy=2a(X+x)$ or, $-2aX+Yy=2ax$. Hence

$$-\frac{2a}{\cos\alpha}=\frac{y}{\sin\alpha}=\frac{2ax}{p}. \quad \therefore \quad y=-2a\tan\alpha,\ x=-p\sec\alpha.$$

Since $y^2=4ax$, $\therefore$ $4a^2\tan^2\alpha=-4a.\ p\sec\alpha$. Hence the reqd. condition of tangency is $p+a\sin\alpha\tan\alpha=0$.

$\therefore$ the polar equation of the pedal is

$$r+a\sin\theta\tan\theta=0,$$

$$\text{or,}\quad r^2+ar\sin\theta\tan\theta=0.$$

Writing $r^2=x^2+y^2$, $r\sin\theta=y$ and $\tan\theta=y/x$, we get the cartesian equation of the pedal as

$$x(x^2+y^2)+ay^2=0.$$

Alternatively :

$y=mx+a/m$ $\cdots$ (1) is a tangent to the parabola $y^2=4ax$

$y=-(1/m)\,x$ $\cdots$ (2) is the equation of the perpendicular from the origin on the above tangent.

$\therefore$ the locus of the point of intersection of (1) and (2), *i.e.*, the locus of the foot of the perpendicular, *i.e.*, the equation of the pedal is obtained by eliminating m between (1) and (2).

From (2), $m=-\dfrac{x}{y}$; substituting in (1), $y=-\dfrac{x^2}{y}-\dfrac{ay}{x}$,

i.e., $xy^2+x^3+ay^2=0$. Hence the result.

Ex. 2. *Show that the pedal of the circle $r=2a\cos\theta$ with respect to the origin is the cardioide $r=a(1+\cos\theta)$.*

Since the given equation is $r=2a\cos\theta$, $\cdots$ (1)

$$\therefore \quad \tan\phi=r\Big/\frac{dr}{d\theta}=\frac{2a\cos\theta}{-2a\sin\theta}=-\cot\theta=\tan(\tfrac{1}{2}\pi+\theta).$$

$$\therefore \quad \phi=\tfrac{1}{2}\pi+\theta. \qquad \cdots \qquad \cdots \quad (2)$$

Let (r_1, θ_1) be the foot of the perpendicular; then as in (ii) of § 14·1, $\theta=\theta_1+\frac{1}{2}\pi-\phi=\theta_1+\frac{1}{2}\pi-(\frac{1}{2}\pi+\theta)$; $\therefore$ $\theta=\frac{1}{2}\theta_1$ $\cdots$ (3)

Again, $r_1=r\sin\phi=2a\cos\theta.\sin(\frac{1}{2}\pi+\theta)$ [*from (1) and (2)*]

$$=2a\cos^2\theta=2a\cos^2\tfrac{1}{2}\theta_1=a(1+\cos\theta_1).$$

Hence the required locus is $r=a(1+\cos\theta)$.

Ex. 3. *Show that the inverse of the straight line $ax+by+c=0$ is a circle.*

Writing $\dfrac{k^2x}{x^2+y^2}$ and $\dfrac{k^2y}{x^2+y^2}$ for x, y in the given equation of the straight line, the equation of the inverse is

$$a\frac{k^2x}{x^2+y^2}+b\frac{k^2y}{x^2+y^2}+c=0,$$

or, $\quad c(x^2+y^2)+ak^2x+bk^2y=0$

which obviously represents a circle.

Ex. 4. *Find the inverse of the parabola $r=l/(1+\cos\theta)$.*

Writing k^2/r for r in the equation of the parabola, the equation of the inverse is

$$\frac{k^2}{r}=\frac{l}{1+\cos\theta} \text{ or, } r=\frac{k^2}{l}(1+\cos\theta)=a(1+\cos\theta)\left[\text{where } a=\frac{k^2}{l}\right],$$

which represents a cardioide.

Ex. 5. *Find the polar reciprocal of the parabola $y^2=4ax$ with respect to the vertex.*

The pedal of the parabola with respect to the vertex is

$$x(x^2+y^2)+ay^2=0. \qquad [\text{ See } Ex.\ 1\ ]$$

Its inverse is

$$\frac{k^2x}{x^2+y^2}\left\{\frac{k^4x^2}{(x^2+y^2)^2}+\frac{k^4y^2}{(x^2+y^2)^2}\right\}+a\frac{k^4y^2}{(x^2+y^2)^2}=0,$$

or, $\quad k^2x+ay^2=0, \quad$ *i.e.*, $\quad y^2=-(k^2/a)x,$

which represents a parabola.

Ex. 6. *Find the polar reciprocal of the curve $p=f(r)$.*

By Art. 14·1(v), the pedal equation of its pedal is $r=f\left(\dfrac{r^2}{p}\right)$.

Now, to obtain its inverse, writing $\dfrac{k^2}{r'}$ for r and $\dfrac{k^2}{r'^2}p'$ for p [See § *14·2(iv)*], we get

$$\frac{k^2}{r'}=f\left(\frac{k^4}{r'^2}\cdot\frac{r'^2}{k^2p'}\right).$$

Hence on simplifying and dropping the dashes, the pedal equation of the polar reciprocal is

$$\frac{k^2}{r}=f\left(\frac{k^2}{p}\right).$$

Examples XIV

1. Find the pedal of

(i) the ellipse $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$ with respect to the centre, and focus ;

(ii) the parabola $y^2=4ax$ with respect to the focus.

2. Find the equations of the pedals of the following curves with respect to the origin :

(i) $x^n+y^n=a^n$. (ii) $\sqrt{\dfrac{x}{a}}+\sqrt{\dfrac{y}{b}}=1$.

(iii) $\dfrac{x^m}{a^m}+\dfrac{y^m}{b^m}=1$.

3. Show that the first positive pedal of the rectangular hyperbola $x^2-y^2=a^2$ with respect to the centre is the lemniscate $r^2=a^2\cos 2\theta$.

4. Find the pedals with respect to the pole of the curves :

(i) $r^2\cos 2\theta=a^2$. (ii) $r^2=a^2\cos 2\theta$.

(iii) $r=a(1+\cos\theta)$. (iv) $r=ae^{\theta\cot\alpha}$.

(v) $r^m=a^m\cos m\theta$.

5. Show that the pedal of a circle with respect to any point is the curve $r=a+b\cos\theta$, where a is the radius of the circle and b the distance of the centre from the origin.

6. Find the inverses of the following curves with respect to the origin :

(i) $x^2+y^2=a^2$. (ii) $x^2/a^2+y^2/b^2=1$.

(iii) $r=a(1+\cos\theta)$. (iv) $r=ae^{\theta\cot\alpha}$.

23

7. Show that the inverses of the lines $2x+3y=4$ and $3x-2y=6$ are a pair of orthogonal circles.

8. Show that the inverses of the conic $r=\dfrac{l}{1+e\cos\theta}$ with regard to the focus is a curve of the form $r=a+b\cos\theta$.

9. Show that the inverse of a rectangular hyperbola is a lemniscate, and conversely.

10. Find the polar reciprocals with regard to a circle of radius k and centre at the origin, of the curves :

(i) $x^2/a^2+y^2/b^2=1$. (ii) $y^2=4ax$.

(iii) $r=a\cos\theta$. (iv) $r=a(1+\cos\theta)$.

(v) $r^m=a^m\cos m\theta$.

ANSWERS

1. (i) $(x^2+y^2)^2=a^2x^2+b^2y^2$; $x^2+y^2=a^2$. (ii) $x=0$.

2. (i) $a^m(x^m+y^m)=(x^2+y^2)^m$, where $m=n/(n-1)$.

(ii) $(x^2+y^2)(ax+by)=abxy$.

(iii) $(ax)^n+(by)^n=(x^2+y^2)^n$, where $n=m/(m-1)$.

4. (i) $r^2=a^2\cos 2\theta$. (ii) $r^{\frac{2}{3}}=a^{\frac{2}{3}}\cos\frac{2}{3}\theta$. (iii) $r^{\frac{1}{3}}=(2a)^{\frac{1}{3}}\cos\frac{1}{3}\theta$.

(iv) $r=a_1e^{\theta\cot\alpha}$, where $a_1=a\sin\alpha e^{(\frac{1}{2}\pi-\alpha)\cot\alpha}$.

(v) $r^n=a^n\cos n\theta$, where $n=m/(m+1)$.

6. (i) $(x^2+y^2)=k^4/a^2$. (ii) $(x^2+y^2)^2=k^4(x^2/a^2+y^2/b^2)$.

(iii) $r=\dfrac{b}{1+\cos\theta}$, where $b=k^2/a$.

(iv) $r=a_1e^{-\theta\cot\alpha}$, where $a_1=k^2/a$

10. (i) $a^2x^2+b^2y^2=k^4$.

(ii) $ay^2+k^2x=0$. (iii) $r=\dfrac{b}{1+\cos\theta}$, where $b=2k^2/a$.

(iv) $r^{\frac{1}{3}}\cos\frac{1}{3}\theta=(k^2/2a)^{\frac{1}{3}}$. (v) $r^n\cos n\theta=(k^2/a)^n$, where $n=m/(m+1)$.

CHAPTER XV

CONCAVITY AND CONVEXITY
POINT OF INFLEXION

15·1. Concavity and Convexity (with respect to a given point).

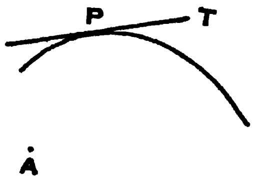


Fig. (i)

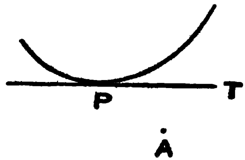


Fig. (ii)

Let PT be the tangent to a curve at P. Then the curve at P is said to be *concave* or *convex* with respect to a point A (not lying on PT), according as a small portion of the curve in the immediate neighbourhood of P (on both sides of it) lies entirely on the same side of PT as A [as in Fig. (i)], or on opposite sides of PT with respect to A [as in Fig. (ii)].

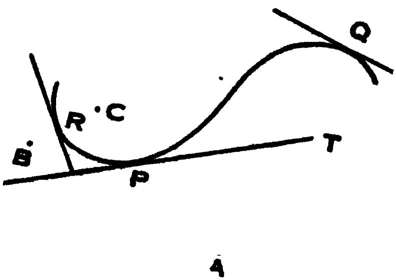


Fig. (iii)

Thus, in Fig. (iii), the curve at P is convex with respect to A, and concave with respect to B or C. The curve at Q is concave with respect to A. Again, the curve at R is convex to B and concave to C.

Note. A curve at a point P on it is **Convex** or **Concave** *with respect to a given line* according as it is convex or concave with respect to the foot of the perpendicular from P on the line.

15·2. Point of Inflexion.

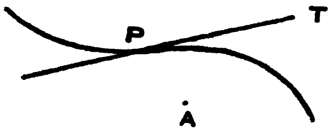


Fig. (iv)

In some curves, at a particular point P on it, the tangent line crosses the curve, as in Fig. (iv). At this point, clearly the curve, on one side of P is convex, and on the other side it is concave to *any* point A (not lying on the tangent line). Such a point on a curve is defined to be a *point of inflexion* (or a point of contrary flexure).

15·3. Analytical Test of Concavity or Convexity (with respect to the x-axis).

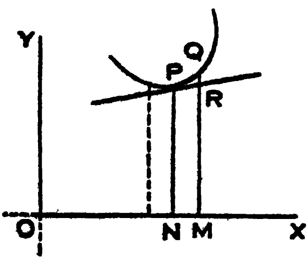


Fig. (i)

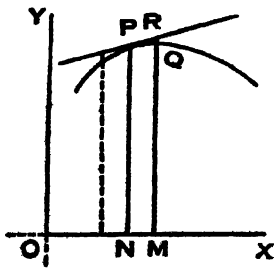


Fig. (ii)

Let P be a point (x, y) on the curve $y=f(x)$, Q a neighbouring point whose abscissa is $x+h$ (h being small, positive or negative). Let PT be the tangent at P, and let the ordinate QM of Q intersect PT at R.

The equation to PT is

$$Y-y=f'(x)(X-x)$$

and abscissa X of R being $x+h$, its ordinate

$$RM=Y=y+hf'(x).$$

Also the ordinate of Q is

$$QM=f(x+h)$$

$$=f(x)+hf'(x)+\frac{h^2}{2\,!}f''(x+\theta h),\ 0<\theta<1.$$

$$\therefore\quad QM-RM=\frac{h^2}{2\,!}f''(x+\theta h). \qquad \cdots \qquad \cdots \quad \text{(i)}$$

Now assuming $f''(x)$ to be continuous at P and $\neq 0$ there, $f''(x+\theta h)$ has the same sign as that of $f''(x)$ when $|h|$ is sufficiently small.

Hence from (i), $QM-RM$ has the same sign as that of $f''(x)$, for positive as well as negative values of h, provided it is sufficiently small in magnitude.

Firstly, let the ordinate PN or y be positive.

Then *if* $f''(x)$ $\left(\text{or } \dfrac{d^2y}{dx^2} \text{ at } P\right)$ *is positive*, from (i) $QM>RM$ for Q on either side of P in its neighbourhood, and so the curve in the neighbourhood of P (on either side of it) is entirely above the tangent, *i.e.*, on the side opposite to the foot N on the x-axis of the ordinate PN, as in Fig. (i). Hence *the curve at P is convex with respect to the x-axis.*

Again, *if $f''(x)$ is negative*, $QM < RM$ on either side of P, and so the curve near P is entirely below the tangent, on the same side of N, as in Fig. (ii). Hence *the curve at P is concave to the x-axis.*

Secondly, let y or PN be negative.

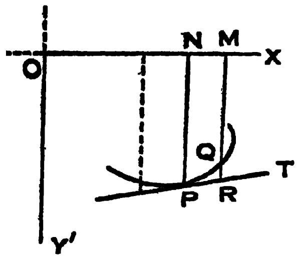


Fig. (iii)

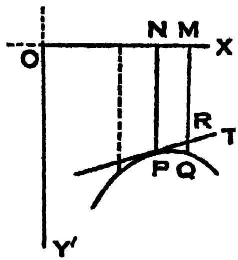


Fig. (iv)

If $f''(x)$ is positive, from (i), as before, $QM > RM$ on either side of P, and as both are negative, QM is numerically less than RM, as in Fig. (iii). The curve therefore, at P lies on the same side as N with respect to the tangent PT. Hence *the curve at P is concave with respect to the x-axis.*

If $f''(x)$ is negative, we similarly get *the curve at P convex with respect to the x-axis*, as in Fig. (iv). Combining the two cases, we get the following criterion for convexity or concavity of a curve at a point with respect to the x-axis :

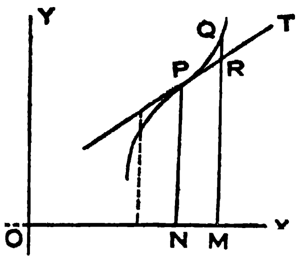
If $y \dfrac{d^2y}{dx^2}$ **is positive** at P, the curve at P is **convex** *to the x-axis.*

If $y \dfrac{d^2y}{dx^2}$ **is negative** at P, the curve at P is **concave** *to the x-axis.*

Note. At a point where the tangent is parallel to the y-axis, $\frac{dy}{dx}$ is infinite. At such a point, instead of considering with respect to the x-axis, we investigate convexity or concavity of the curve with respect to the y-axis. The criterion, as obtained by a method similar to above, is as follows :

The curve at P is *convex or concave with respect to the y-axis* according as $x \frac{d^2x}{dy^2}$ is positive or negative at P.

15·4. Analytical condition for Point of Inflexion.



In the above investigation, let $f''(x)=0$ at P, and $f'''(x) \neq 0$.

Then $$QM=f(x+h)=f(x)+hf'(x)+\frac{h^3}{3!}f'''(x+\theta h).$$

$$\therefore \quad QM-RM=\frac{h^3}{3!}f'''(x+\theta h),$$

and the sign of this for sufficiently small $|h|$ is the same as that of $\frac{h^3}{3!}f'''(x)$, which has got opposite signs for positive and negative values of h, whatever be the sign of $f'''(x)$ at P. Thus near P the curve is above the tangent on one side of P, and below the tangent on the other side, as in the above figure. Hence P is a point of inflexion.

Thus the condition that P is a point of inflexion on the curve $y=f(x)$ is that at P,

$$\frac{d^2y}{dx^2}=0 \text{ and } \frac{d^3y}{dx^3} \neq 0.$$

Note. If $\frac{dy}{dx}$ is infinite at P, the condition that P is a point of inflexion is that, at P,

$$\frac{d^2x}{dy^2}=0 \text{ and } \frac{d^3x}{dy^3} \neq 0.$$

15·5. A more general criterion.

Suppose that at P, $f''(x)=f'''(x)=\cdots\cdots=f^{n-1}(x)=0$, and $f^n(x) \neq 0$.

Then $QM=f(x+h)=f(x)+hf'(x)+\frac{h^n}{n!}f^n(x+\theta h)$.

$\therefore \quad QM-RM=\frac{h^n}{n!}f^n(x+\theta h)$, which for sufficiently small values of $|h|$, has the same sign as that of $\frac{h^n}{n!}f^n(x)$.

If n is even, h^n is positive and the sign is the same as that of $f^n(x)$ or $\frac{d^ny}{dx^n}$ at P for both positive and negative values of h. Considering both the cases when y of P is positive and negative, we find that the curve at P is *convex or concave with respect to the x-axis according as* $y\frac{d^ny}{dx^n}$ is positive or negative.

If n is odd, $\frac{h^n}{n!}f^n(x)$ will have opposite signs for positive and negative values of h, whatever be the sign of $f^n(x)$. Hence Q lies on opposite sides of the tangent for positive and negative values of h. Thus *P is a point of inflexion.*

Note. Since from (i) Art. 15·3, $QM - RM = \frac{h^2}{2!} f''(x+\theta h)$, if $f''(x+\theta h)$ has opposite signs for opposite signs of h when $|h|$ is sufficiently small, $QM > RM$ on one side, and $QM < RM$ on the other side of P on the curve in the immediate neighbourhood, and thus the tangent at P crosses the curve at P, and so P is a point of inflexion. Thus, since θ is positive and numerically less than 1, *an alternative criterion for a point of inflexion is that $f''(x+h)$ should have opposite signs for opposite signs of h when h is numerically sufficiently small*; in other words, $f''(x)$ changes in passing through P from one side to the other.

15·6. Illustrative Examples.

Ex. 1. *Examine the curve $y = \sin x$ regarding its convexity or concavity to the x-axis, and determine its point of inflexion, if any.*

As $y = \sin x$, $\frac{dy}{dx} = \cos x$ and $\frac{d^2y}{dx^2} = -\sin x$. Hence $y\frac{d^2y}{dx^2} = -\sin^2 x$, which is negative for all values of x excepting those which make $\sin x = 0$, *i.e.*, for $x = k\pi$, k being any integer, positive or negative.

Thus, the curve is concave to the x-axis at every point, excepting at points where it crosses the x-axis.

At these points, given by $x = k\pi$, $\frac{d^2y}{dx^2} = 0$, and $\frac{d^3y}{dx^3} = -\cos x \neq 0$. Hence those points where the curve crosses the x-axis are points of inflexion.

Ex. 2. *Show that the curve $y^3 = 8x^2$ is concave to the foot of the ordinate everywhere except at the origin.*

From the given equation, $y = 2x^{\frac{2}{3}}$.

$$\therefore \frac{dy}{dx} = \frac{4}{3}x^{-\frac{1}{3}}, \frac{d^2y}{dx^2} = -\frac{4}{9}x^{-\frac{4}{3}}. \qquad \therefore y\frac{d^2y}{dx^2} = -\frac{8}{9}x^{-\frac{2}{3}} = -\frac{8}{9x^{\frac{2}{3}}}.$$

Thus, excepting at the origin, $x^{\frac{2}{3}}$ being positive for all values of x,

$y \frac{\quad}{dx}$

Hence the curve is concave everywhere to the foot of the ordinate excepting at the origin.

Ex. 3. *Prove that $(a-2, -2/e^2)$ is a point of inflexion of the curve $y=(x-a)e^{x-a}$.*

Here, at points on the curve,

$$\frac{dy}{dx}=e^{x-a}+(x-a)\ e^{x-a}=(1+x-a)\ e^{x-a},$$

$$\frac{d^2y}{dx^2}=e^{x-a}+(1+x-a)\ e^{x-a}=(2+x-a)\ e^{x-a}$$

and similarly $\dfrac{d^3y}{dx^3}=(3+x-a)\ e^{x-a}$.

Hence at $x=a-2$, (where $y=-2e^{-2}$), $\dfrac{d^2y}{dx^2}=0$ and $\dfrac{d^3y}{dx^3}=e^{-2}\neq 0$.

Hence the point $(a-2, -2/e^2)$ is a point of inflexion.

Ex. 4. *Find if there is any point of inflexion on the curve $y-3=6(x-2)^5$.*

Here $\dfrac{dy}{dx}=30\ (x-2)^4$, $\dfrac{d^2y}{dx^2}=120\ (x-2)^3$.

Thus, $\dfrac{d^2y}{dx^2}=0$ when $x=2$ (and so $y=3$).

In the neighbourhood of this point, where $x=2+h$ (h numerically small, positive or negative),

$\dfrac{d^2y}{dx^2}=120h^3$, which has opposite signs for positive and negative values of h. Hence $\dfrac{d^2y}{dx^2}$ changes sign in passing through $x=2$.

Thus, (2, 3) is the only point of inflexion.

Alternatively.

Here $\dfrac{d^3y}{dx^3}=360\ (x-2)^2$, $\dfrac{d^4y}{dx^4}=720\ (x-2)$, $\dfrac{d^5y}{dx^5}=720$. Thus at $x=2$, $\dfrac{d^2y}{dx^2}=\dfrac{d^3y}{dx^3}=\dfrac{d^4y}{dx^4}=0$, and $\dfrac{d^5y}{dx^5}$ (which is of *odd* order) $\neq 0$. Hence $x=2$ gives the point of inflexion.

Examples XV

1. Prove that the curve $y=e^x$ is convex to the x-axis at every point.

2. Prove that the curve $y = \cos^{-1} x$ is everywhere concave to the y-axis excepting where it crosses the y-axis.

3. Prove that the curve $y = \log x$ is convex to the foot of the ordinate in the range $0 < x < 1$, and concave where $x > 1$. Prove also that the curve is convex everywhere to the y-axis.

4. Show that the curve $(y-a)^3 = a^3 - 2a^2x + ax^2$, where $a > 0$, is always concave to the x-axis. How is it situated with respect to the y-axis ?

5. Show that the origin is a point of inflexion on the curves :

(i) $y = x^2 \log(1-x)$. (ii) $y = x \cos 2x$.

6. Find the points of inflexion, if any, on the curves :

(i) $y = \dfrac{x}{(x+1)^2+1}$. (ii) $y^2 = x(x+1)^2$.

(iii) $c^2 y = (x-a)^3$. (iv) $x = ae^{-8x^2}$.

7. Show that the points of inflexion on the curve $y^2 = (x-a)^2(x-b)$ lie on the line $3x + a = 4b$.

8. Show that the curve $y(x^2+a^2) = a^2 x$ has three points of inflexion which lie on a straight line.

ANSWERS

4. Concave where $0 < x < a$, convex everywhere else.

6. (i) $(-2, -1)$, $\left(1+\sqrt{3}, \dfrac{\sqrt{3}-1}{4}\right)$, $\left(1-\sqrt{3}, \dfrac{-1-\sqrt{3}}{4}\right)$.

(ii) $\frac{1}{3}$, $\pm\frac{4}{9}\sqrt{3}$. (iii) a, 0. (iv) $\pm\frac{1}{2}$, ae^{-2}.

Miscellaneous Examples

1. A function $f(x)$ is defined as follows :

$$f(x)=0, \text{ for } x=0\ ;$$
$$f(x)=\tfrac{1}{2}-x, \text{ for } 0<x<\tfrac{1}{2}\ ;$$
$$f(x)=\tfrac{1}{2} \text{ for } x=\tfrac{1}{2}\ ;$$
$$f(x)=\tfrac{3}{2}-x \text{ for } \tfrac{1}{2}<x<1\ ;$$
$$f(x)=1 \text{ for } x=1.$$

Show that $f(x)$ is discontinuous at $x=0$, $\frac{1}{2}$ and 1.

[*C. H. 1965*]

2. A function $f(x)$ is defined as follows :

$$f(x)=2+(x-1)^{2/3} \text{ in the interval } (0, 2).$$

Is Rolle's theorem applicable to $f(x)$?

3. In the Mean Value Theorem

$$f(b)-f(a)=(b-a)f'(\xi),\ a<\xi<b,$$

if $f(x)=x(x-1)(x-2)$ and $a=0$, $b=\frac{1}{2}$, find ξ.

4. If $x=e^u+e^{-v}$, $y=e^{-u}-e^v$, and if z be a function of x, y, show that

$$\frac{\partial z}{\partial u}-\frac{\partial z}{\partial v}=x\frac{\partial z}{\partial x}-y\frac{\partial z}{\partial y}.$$

5. If $V=(x^2+y^2+z^2)^{-\frac{1}{2}}$, show that

$$x\frac{\partial V}{\partial x}+y\frac{\partial V}{\partial y}+z\frac{\partial V}{\partial z}=-V.$$

6. If $V=\log\sqrt{(x^2+y^2+z^2)}$, show that

$$(x^2+y^2+z^2)\left(\frac{\partial^2 V}{\partial x^2}+\frac{\partial^2 V}{\partial y^2}+\frac{\partial^2 V}{\partial z^2}\right)=1.$$

7. If $u = \log r$ where

$$r^2 = (x-a)^2 + (y-b)^2 + (z-c)^2, \text{ show that}$$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = \frac{1}{r^2}.$$

8. If $u = x^2 \tan^{-1} y/x - y^2 \tan^{-1} x/y$, show that

$$\frac{\partial^2 u}{\partial x \partial y} = \frac{x^2 - y^2}{x^2 + y^2}.$$ [*C. H. 1964*]

9. If $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$, prove that

$$\frac{d^2 y}{dx^2} = \frac{\Delta}{(hx + by + f)^3}$$

where $\Delta = abc + 2fgh - af^2 - bg^2 - ch^2$.

10. Show that the points of intersection of the curve

$$xy(x^2 - y^2) + c^2 y^2 + d^2 x^2 - c^2 d^2 = 0$$

with its asymptotes lie on an ellipse.

11. Show that the value of

$$\left(\frac{d}{dx}\right)^n \frac{x^3}{x^2 - 1}$$ for $x = 0$, is 0 if n is even and is $-n\,!$ if n is odd and > 1. [*C. H. 1968*]

12. Establish the formula

$$\frac{d^3 x}{dy^3} = -\left\{\frac{d^3 y}{dx^3}\cdot\frac{dy}{dx} - 3\frac{d^2 y}{dx^2}\right\} \Big/ \left(\frac{dy}{dx}\right)^5.$$ [*C. H. 1968*]

ANSWERS

2. No. 3. $1 - \sqrt{(7/12)}$.

CHAPTER XVI

ON SOME WELL-KNOWN CURVES

16·1. We give below diagrams, equations, and a few characteristics of some well-known curves which have been used in the preceding pages in obtaining their properties. The student is supposed to be familiar with conic sections and graphs of circular functions, so they are not given here.

16·2. Cycloid.

The *cycloid* is the curve traced out by a point on the circumference of a circle which rolls (without sliding) on a straight line.

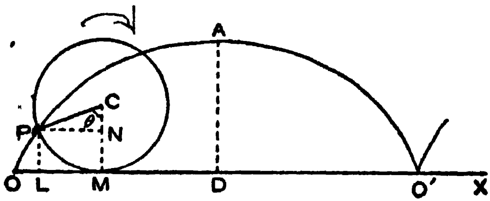


Fig. (i)

$$x = a(\theta - \sin\theta). \qquad y = a(1 - \cos\theta).$$

Let P be the point on the circle MP, called the ***generating circle***, which traces out the cycloid. Let the line OMX on which the circle rolls be taken as x-axis and the point O on OX, with which P was in contact when the circle began rolling, be taken as the origin.

Let a be the radius of the generating circle, and C its centre, P the point (x, y) on it, and let $\angle PCM = \theta$. Then

θ is the angle through which the circle turns as the point P traces out the locus.

$\therefore \quad OM = \text{arc } PM = a\theta.$

Let PL be drawn perpendicular to OX.

$$\therefore \quad x = OL = OM - LM = a\theta - PN = a\theta - a \sin \theta$$
$$= a(\theta - \sin \theta).$$
$$y = PL = NM = CM - CN = a - a \cos \theta$$
$$= a(1 - \cos \theta).$$

Thus, the *parametric equations of the cycloid with the starting point as the origin* and *the line on which the circle rolls*, called the *base, as the x-axis*, are

$$\mathbf{x = a(\theta - \sin \theta), \; y = a(1 - \cos \theta).} \quad \cdots \quad (1)$$

The point A at the greatest distance from the base OX is called the *vertex*. Thus, for the vertex, y i e., $a(1 - \cos \theta)$ is maximum. Hence, $\cos \theta = -1$ *i.e.*, $\theta = \pi$.

$\therefore \quad AD = a(1 - \cos \pi) = 2a. \quad \therefore$ **vertex is $(\mathbf{a\pi, 2a})$.**

For O and O', $y = 0$, $\therefore \cos \theta = 1. \quad \therefore \theta = 0$ and 2π.

As the circle rolls on, arches like OAO' are generated over and over again, and any single arch is called a cycloid.

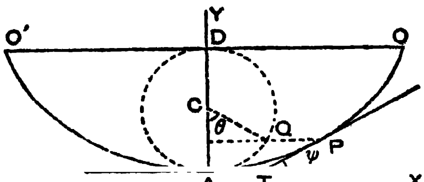


Fig. (ii)

$x = a(\theta + \sin \theta)$. $\qquad y = a(1 - \cos \theta)$

Since the vertex is the point $(a\pi, 2a)$ the equation of the cycloid with the vertex as the origin and the tangent at the vertex as the x-axis can be obtained from the previous equation by transferring the origin to $(a\pi, 2a)$ and turning the axes through π, *i.e.*, by writing

$$a\pi + x' \cos \pi - y' \sin \pi \text{ and } 2a + x' \sin \pi + y' \cos \pi$$

for x and y respectively.

Hence, $a(\theta - \sin \theta) = a\pi - x'$,

or, $x' = a(\pi - \theta) + a \sin \theta = a(\theta' + \sin \theta')$,

where $\theta' = \pi - \theta$

and $a(1 - \cos \theta) = 2a - y'$,

or, $y' = 2a - a + a \cos \theta = a + a \cos \theta$

$= a - a \cos(\pi - \theta) = a(1 - \cos \theta')$.

Hence (replacing θ' by θ) the *equations of the cycloid with the vertex as the origin and the tangent at the vertex as the x-axis* is

$$\mathbf{x = a(\theta + \sin \theta),\ y = a(1 - \cos \theta).} \quad \cdots \quad \text{(ii)}$$

In this equation, $\theta = 0$ *for the vertex*, $\theta = \pi$ *for* O, and $\theta = -\pi$ *for* O'.

The *characteristic properties* are

(i) For the cycloid $x = a(\theta - \sin \theta)$, $y = a(1 - \cos \theta)$, radius of curvature = twice the length of the normal, (the centre of curvature and the x-axis being on the same side of the curvature).

(ii) The evolute of the cycloid is an equal cycloid.

(iii) For the cycloid $x = a(\theta + \sin \theta)$, $y = a(1 - \cos \theta)$, $\psi = \frac{1}{2}\theta$ and $s^2 = 8ay$, s being measured from the vertex.

Note. The above equations (ii) can also be obtained from the Fig. (i) geometrically as follows :

If (x', y') be the co-ordinates of P referred to the vertex as the origin and the tangent at the vertex as the x-axis,

$$x' = LD = OD - OL = a\pi - x = a(\pi - \theta) + a \sin \theta,$$

$$y' = AD - PL = 2a - y = 2a - a(1 - \cos \theta) = a(1 + \cos \theta).$$

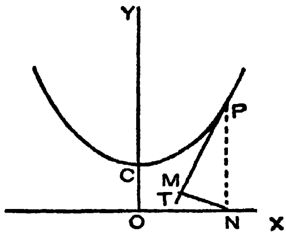
Hence, writing θ' (or θ) for $\pi - \theta$, etc.

16·3. Catenary.

The *catenary* is the curve in which a uniform heavy string will hang under the action of gravity when suspended from two points. It is also called the *chainette*.

Its equation, as shown in books on Statics, is

$$y = c \cosh \frac{x}{c} = \frac{c}{2}\left(e^{\frac{x}{c}} + e^{-\frac{x}{c}}\right)$$



C is called the *vertex*, $OC = c$, OX is called the *directrix*.

The *characteristic properties* are

(i) The perpendicular from the foot of the ordinate upon the tangent at any point is of constant length.

(ii) Radius of curvature at any point = length of the normal at the point (the centre of curvature and the x-axis being on the opposite sides of the curve).

(iii) $y^2 = c^2 + s^2$, s being measured from the vertex C.

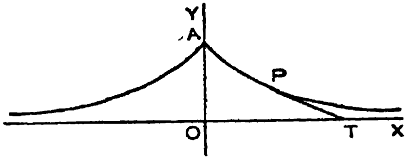
(iv) $s = c \tan \psi$, $y = c \sec \psi$.

(v) $x = c \log (\sec \psi + \tan \psi)$.

16·4. Tractrix.

Its equation is

$$x = \sqrt{a^2 - y^2} + \frac{a}{2} \log \frac{a - \sqrt{a^2 - y^2}}{a + \sqrt{a^2 - y^2}},$$



or, $x = a(\cos t + \log \tan \frac{1}{2}t)$, $y = a \sin t$.

Here $OA = a$.

The *characteristic properties* are

(i) The portion of the tangent intercepted between the curve and the x-axis, is constant.

(ii) The radius of curvature varies inversely as the normal (the centre of curvature and the x-axis being on the opposite sides of the curve).

(iii) The evolute of the tractrix is the catenary

$$y = a \cosh (x/a).$$

16·5. Astroid.

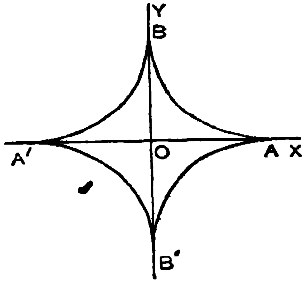
Its equation is $\quad \mathbf{x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}}$,

or, $\quad \mathbf{x = a \cos^3\theta, \; y = a \sin^3\theta}$.

Here $\quad OA = OB = OA' = OB' = a$.

The whole figure lies completely within a circle of radius a and centre O. The points A, A', B, B' are called cusps. It is a special type of a four-cusped hypo-cycloid.

[See § *16·6*]

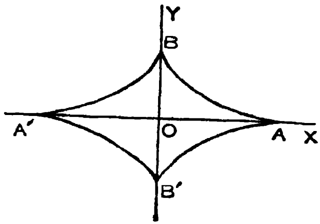


The *characteristic property* of this curve is that the tangent at any point to the curve intercepted between the axes is of constant length.

16·6. Four-cusped Hypo-cycloid.

Its equation is $\left(\frac{x}{a}\right)^{\frac{2}{3}}+\left(\frac{y}{b}\right)^{\frac{2}{3}}=1,$

or, $\quad x=a\cos^3\Phi,\ y=b\sin^3\Phi.$



Here $\quad OA=OA'=a\ ;\ OB=OB'=b.$

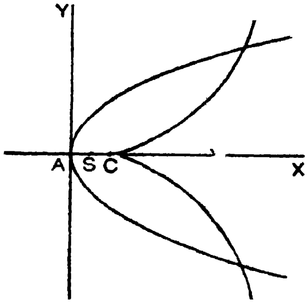
The astroid is a special case of this when $a=b$.

16·7. Evolutes of Parabola and Ellipse.

(i) The evolute of the parabola $y^2=4ax$ is

$$27ay^2=4(x-2a)^3.$$

This curve is called a *semi-cubical parabola.*

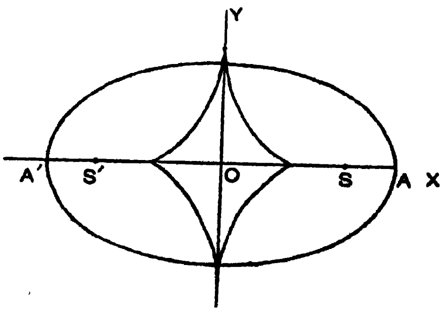


Transferring the origin to $(2a, 0)$, its equation assumes the form $y^2 = kx^3$ where $k = 4/27a$, which is the standard equation of the semi-cubical parabola with its vertex at the origin.

Hence, the vertex C of the evolute is $(2a, 0)$.

(ii) The equation of the evolute of the ellipse
$x^2/a^2 + y^2/b^2 = 1$ is

$$(\mathbf{ax})^{\frac{2}{3}} + (\mathbf{by})^{\frac{2}{3}} = (\mathbf{a}^2 - \mathbf{b}^2)^{\frac{2}{3}}$$



which can be written in the form

$$\left(\frac{x}{\alpha}\right)^{\frac{2}{3}}+\left(\frac{y}{\beta}\right)^{\frac{2}{3}}=1,$$

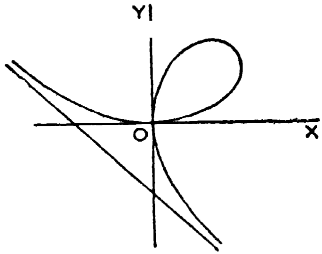
where $\alpha=(a^2-b^2)/a$, $\beta=(a^2-b^2)/b$.

Hence it is a four-cusped hypo-cycloid.

16·8. Folium of Descartes.

Its equation is $\mathbf{x^3+y^3=3axy}$.

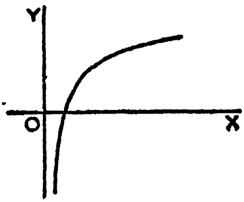
It is symmetrical about the line $y=x$.



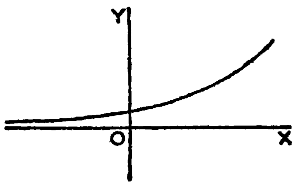
The axes of co-ordinates are tangents at the origin, and there is a loop in the first quadrant.

It has an asymptote $x+y+a=0$ and its radii of curvature at origin are each $=\frac{3}{2}a$.

16·9. Logarithmic and Exponential Curves.



(i) $y=\log x$.

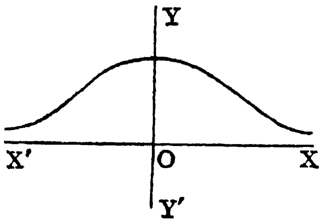


(ii) $y=e^x$.

(i) x is always positive; $y=0$ when $x=1$, and as x becomes smaller and smaller, y, being negative, becomes numerically larger and larger. For $x > 0$, *the curve is continuous.*

(ii) x may be positive or negative but y is always positive, and y becomes smaller and smaller, as x, being negative, becomes numerically larger and larger. *The curve is continuous for all values of x.*

16·10. Probability Curve.



The equation of the probability curve is $y=e^{-x^2}$.

The x-axis is an asymptote.

The area between the curve and the asymptote is

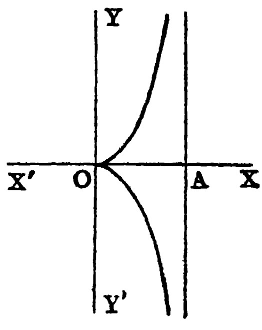
$$=2\int_0^\infty e^{-x^2}\,dx=2.\tfrac{1}{2}\sqrt{\pi}=\sqrt{\pi}.$$

16·11. Cissoid of Diocles.

Its cartesian equation is
$$y^2(2a-x)=x^3.$$

$OA=2a$; $x=2a$ is an asymptote.

Its polar equation is
$$r=\frac{2a\sin^2\theta}{\cos\theta}.$$



16·12. Strophoid.

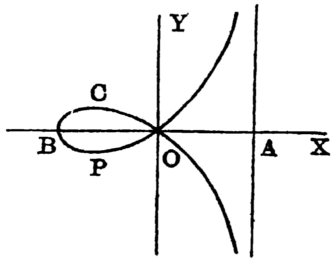
The equation of the curve is

$$y^2 = x^2 . \frac{a+x}{a-x}.$$

$OA = OB = a$.

$OCBP$ is a loop.

$x = a$ is an asymptote.



The curve $y^2 = x^2 \frac{a-x}{a+x}$ is similar, just the reverse of strophoid, the loop being on the right side of the origin and the asymptote on the left side.

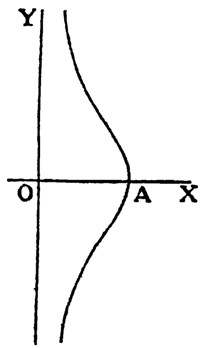
16·13. Witch of Agnesi.

The equation of the curve is

$$xy^2 = 4a^2 (2a - x).$$

Here $OA = 2a$.

This curve was first discussed by the Italian lady mathematician Maria Gactaua Agnesi, Professor of Mathematics at Bologna.



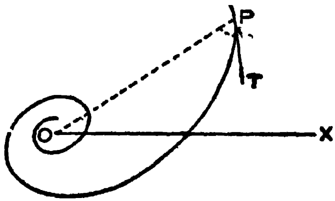
16·14. Logarithmic (or Equi-angular) spiral.

Its equation is $\mathbf{r = ae^{\theta \cot \alpha}}$ (or, $r = ae^{m\theta}$) where $\cot \alpha$ or m is constant.

Characteristic Properties :

(i) The tangent at any point makes a constant angle with the radius vector, ($\phi = a$).

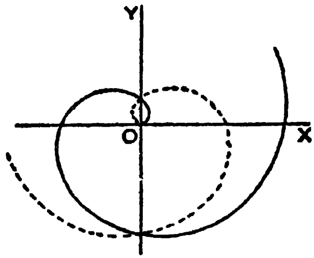
(ii) Its pedal, inverse, polar reciprocal and evolute are all equiangular spirals.



(iii) The radius of curvature subtends a right angle at the pole.

Note. Because of the property (i), the spiral is called *equiangular.*

16·15. Spiral of Archimedes.



Its equation is $\mathbf{r = a\theta}$.

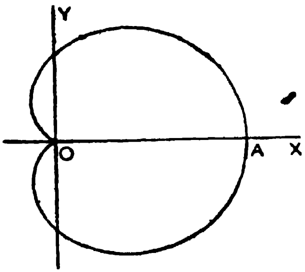
Its *characteristic property* is that its polar subnormal is constant.

16·16. Cardioide.

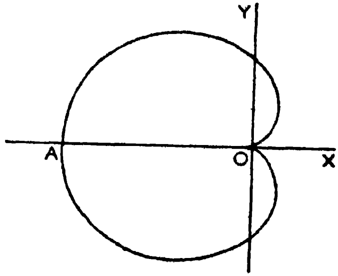
Its equation is (i) $\mathbf{r = a(1 + \cos\theta)}$, or (ii) $\mathbf{r = a(1 - \cos\theta)}$.

In (i), $\theta=0$ for A, and $\theta=\pi$ for O.

In (ii), $\theta=\pi$ for A and $\theta=0$ for O.



(i) $r=a(1+\cos\theta)$.



(ii) $r=a(1-\cos\theta)$.

In both cases, the curve is symmetrical about the initial line, which divides the whole curve into two equal halves and for the upper half, θ varies from 0 to π, and $OA=2a$.

The curve (ii) is really the same as (i) turned through 180°.

The curve passes through the origin, its tangent there being the initial line, and the tangent at A is perpendicular to the initial line.

The evolute of the cardioide is a cardioide.

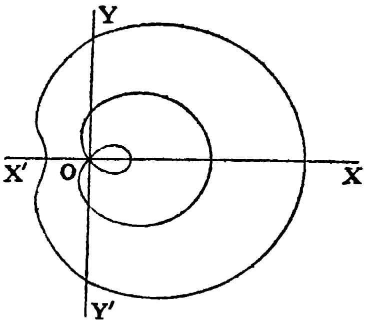
Note. Because of its shape like human heart, it is called a cardioide. The cardioide $r=a(1+\cos\theta)$ is the pedal of the circle $r=2a\cos\theta$ with respect to a point on the circumference of the circle, and inverse of the parabola $r=a/(1+\cos\theta)$.

16·17. Limacon.

The equation of the curve is

$$\mathbf{r=a+b\cos\theta.}$$

When $a > b$, we have the outer curve, and when $a < b$, we have the inner curve with the loop.



When $a = b$, the curve reduces to a cardioide.

[See *fig. in* §*16·16*]

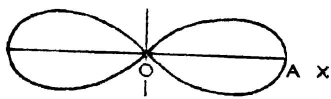
Limacon is the pedal of a circle with respect to a point outside the circumference of the circle.

16·18. Lemniscate.

Its equation is $\mathbf{r^2 = a^2 \cos 2\theta}$,

$$\text{or,} \quad (x^2 + y^2)^2 = a^2(x^2 - y^2).$$

It consists of two equal loops, each symmetrical about the initial line, which divides each loop into two equal halves.



$r^2 = a^2 \cos 2\theta$

Here $OA = OA' = a$.

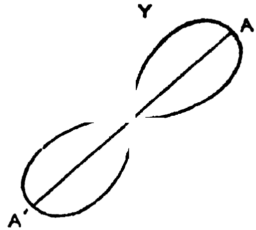
The tangents at the origin are $y = \pm x$.

For the upper half of the right-hand loop, θ varies from 0 to $\frac{1}{4}\pi$.

A characteristic property of it is that the product of the distances of any point on it from $(\pm a/\sqrt{2}, 0)$ is constant.

The lemniscate is the pedal of the rectangular hyperbola $r^2 \cos 2\theta = a^2$. The curve represented by $r^2 = a^2 \sin 2\theta$ is also sometimes called lemniscate or *rose lemniscate*, to distinguish it from the first lemniscate, which is sometimes called *lemniscate of Bernoulli* after the name of the mathematician J. Bernoulli, who first studied its properties.

The curve consists of two equal loops, situated in the first and third quadrants, and symmetrical about the line $y=x$. It is the first curve turned through 45°.

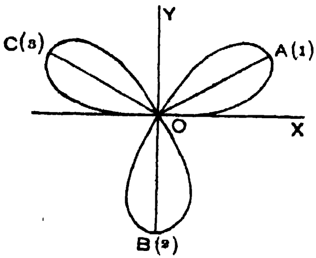


$r^2=a^2 \sin 2\theta$

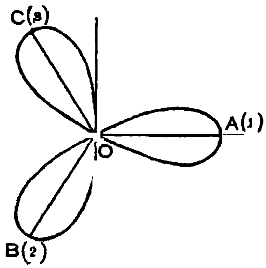
The tangents at the origin are the axes of x and y.

16·19. Rose-Petals ($r=a \sin n\theta$, $r=a \cos n\theta$).

The curve represented by $r=a \sin 3\theta$, or, $r=a \cos 3\theta$ is called a *three-leaved rose*, each consisting of three equal loops. The order in which the loops are described is indicated in the figures by numbers. In each case $OA=OB=OC=a$, and $\angle AOB=\angle BOC=\angle COA=120°$.



$r=a \sin 3\theta$

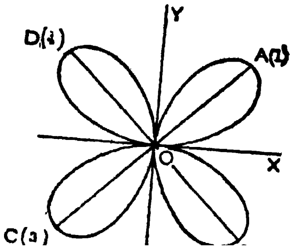


$r=a \cos 3\theta$

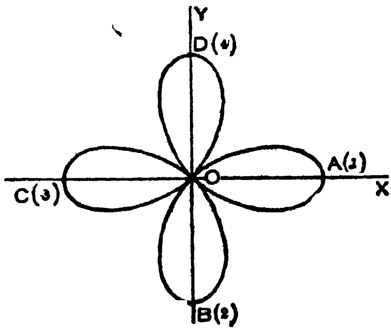
The curve represented by $r=a \sin 2\theta$, or, $r=a \cos 2\theta$ is called a *four-leaved rose*, each consisting of four equal loops.

In each case $OA = OB = OC = OD = a$ and $\angle AOB = \angle BOC = \angle COD = \angle DOA = 90°$.

The class of curves represented by $r = a \sin n\theta$, or, $r = a \cos n\theta$ where n is a positive integer is called *rose-petal*, there being n or $2n$ *equal loops* according as n *is odd* or *even*



$r = a \sin 2\theta$



$r = a \cos 2\theta$

all being arranged symmetrically about the origin and lying entirely within a circle whose centre is the pole and radius a.

16·20. Sine Spiral ($r^n = a^n \sin n\theta$ or $r^n = a^n \cos n\theta$).

The class of curves represented by (i) $r^n = a^n \sin n\theta$, or, (ii) $r^n = a^n \cos n\theta$ is called *sine spiral* and embraces several important and well-known curves as particular cases.

Thus, for the values $n = -1, 1, -2, +2, -\frac{1}{2}$ and $\frac{1}{2}$, the sine spiral is respectively a straight line, a circle, a rectangular hyperbola, a lemniscate, a parabola and a cardioide.

For (i) $\phi = n\theta$; for (ii) $\phi = \frac{1}{2}\pi + n\theta$.

The pedal equation in both the cases is

$$p = r^{n+1}/a^n.$$

APPENDIX

SECTION A

INFINITE SEQUENCE

1. Sequence : An endless succession of numbers arranged in a definite order

$$s_1, s_2, \ldots\ldots s_n \ldots\ldots$$

is called an infinite *sequence*, and is denoted by $\{s_n\}$. By a sequence we shall mean an infinite sequence.

Thus, $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots\ldots$ is an infinite sequence. Again, $2, -1, 4, -3, 6, -5, \ldots$ [$s_n = n+1$ if n is odd, and $-(n-1)$ if n is even] is another infinite sequence.

Monotonic sequence : If in the sequence $\{s_n\}$

(i) $s_{n+1} \geqslant s_n$ for every n,

or else (ii) $s_{n+1} \leqslant s_n$ for every n,

the sequence is said to be *monotonic* or *monotone*.

In case (i), it is called *monotonic increasing* (or strictly, monotonic non-decreasing) ; it would be strictly monotonic increasing if the equality sign be omitted in (i), *i.e.*, if $s_{n+1} > s_n$ for every n.

In case (ii), it is *monotonic decreasing*,

Thus, if $s_n = \dfrac{n}{n+1}$, $\{s_n\}$ is a monotone increasing sequence.

If $s_n = \left(1 + \dfrac{1}{n}\right)^{\frac{1}{2}}$, $\{s_n\}$ is a monotone decreasing sequence.

Bounded sequence : If corresponding to the sequence $\{s_n\}$, we get a finite number K such that $s_n \leqslant K$ for all values of n, the sequence is said to be *bounded above*, or, 'bounded on the right'.

Again, if we get a finite number k such that $s_n \geqslant k$ for all values of n, the sequence $\{s_n\}$ is said to be *bounded below*, or, 'bounded on the left'.

If a sequence is bounded both above and below, it is said to be *bounded*.

Thus, if $s_n = \frac{n}{n+1}$, s_n is bounded both above and below.

Upper bound : If a sequence $\{s_n\}$ is bounded above, then we can prove by Dedekind's Theorem that there exists a number M, such that no member of the sequence exceeds it, but given any pre-assigned positive quantity ε, however small, there is at least one member of the sequence exceeding $M - \varepsilon$.

This number M is called the *upper bound* or 'exact upper bound' of the sequence.

Any number $> M$ is called a rough upper bound.

Lower bound : If a sequence $\{s_n\}$ is bounded below, then there exists a number m such that no member of the sequence $< m$, but given any pre-assigned positive quantity ε, however small, there is at least one member of the sequence $< m + \varepsilon$. This number m is called the *lower bound* of the sequence or 'exact lower bound'.

Convergent sequence : A sequence $\{s_n\}$ is called convergent, and is said to have a limit l, if $\underset{n \to \infty}{Lt}\ s_n = l$.

Analytically, a sequence $\{s_n\}$ is said to be *convergent* and to have a *limit* l, if corresponding to an arbitrary positive number ε, however small, there exists a positive integer N, such that

$$| s_n - l | < \varepsilon \text{ for } n \geqslant N$$

$$i.e., \quad l - \varepsilon < s_n < l + \varepsilon \text{ for } \geqslant N.$$

Thus, if $s_n = \frac{n}{n+1}$, the sequence $\{s_n\}$ is convergent, having the limit 1.

Divergent sequence : The sequence $\{s_n\}$ is called divergent, if $\underset{n\to\infty}{Lt}\ s_n = +\infty$ or $-\infty$.

If $s_n = n^2$, or $s_n = -n$, the sequence $\{s_n\}$ is divergent.

Oscillatory sequence : If a sequence neither converges, nor diverges to $+\infty$ or to $-\infty$, then the sequence is said to be *oscillatory*.

In this case, if it is bounded, it is said to *oscillate finitely*.

If on the other hand it is unbounded (in any direction), it is said to *oscillate infinitely*.

If $s_n = (-1)^n$, the sequence $\{s_n\}$ oscillates finitely.
If $s_n = (-1)^n n^2$, the sequence $\{s_n\}$ oscillates infinitely.

2. Examples :

(i) $2+1,\ 2-\frac{1}{2},\ 2+\frac{1}{3},\ 2-\frac{1}{4},\ 2+\frac{1}{5}, \ldots,\ 2+(-1)^{n-1}.\frac{1}{n}, \ldots$

The sequence is convergent ; limit 2 ; bounded ; upper bound 3, lower bound $\frac{3}{2}$; both the bounds are members of the set here.

(ii) $1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \ldots$

Convergent sequence, with limit 0 ; bounded ; upper bound 1, lower bound 0, which is also equal to the limit ; the lower bound here does not belong to the set.

Note. A convergent sequence with limit 0 is also sometimes called a **null sequence.**

(iii) 1, 2, 3, 4,......

Divergent sequence, $\to +\infty$; bounded below, but not above ; lower bound 1.

(iv) $-5,\ -5^2,\ -5^3, \ldots\ldots$

Divergent, $\to -\infty$; bounded above, upper bound being -5 ; not bounded below.

(v) $2-1,\ -3+\frac{1}{2},\ 2-\frac{1}{3},\ -3+\frac{1}{4},\ 2-\frac{1}{5},\ -3+\frac{1}{6}, \ldots\ldots\ldots,$ s_n being $2-\dfrac{1}{n}$ or $-3+\dfrac{1}{n}$ according as n is odd or even.

Oscillates finitely ; upper bound 2, lower bound -3, both not belonging to the set.

(vi) $1,\ -2,\ 3,\ -4, \ldots, (-1)^{n-1}n, \ldots\ldots$

Oscillates infinitely, between $+\infty$ and $-\infty$.

(vii) $1+1,\ -2,\ 1+\frac{1}{3},\ -4,\ 1+\frac{1}{5},\ -6,\ 1+\frac{1}{7},\ -8, \ldots\ldots\ldots$ s_n being $1+\dfrac{1}{n}$ or $-n$ according as n is odd or even.

Oscillates infinitely, between 1 and $-\infty$ but upper bound is 2, no lower bound existing.

3. Theorems on sequence.

I. *A convergent sequence must be bounded*, and the limit must lie between the upper and the lower bounds (may be equal to one of these, or not).

Let the sequence $\{s_n\}$ be convergent, having limit l. Then by the definition of limit, given any pre-assigned positive quantity ε, however small, we can determine a definite positive integer N, such that

$$l-\varepsilon < s_n < l+\varepsilon \text{ for all } n > N.$$

Now let K be the biggest of the numbers s_1, $s_2, \ldots s_N$ and $l+\varepsilon$. Then evidently $s_n \leqslant K$ for all n. Hence the sequence $\{s_n\}$ is bounded above, having K for a rough upper bound.

Similarly, k, being the least of $s_1, s_2, \ldots s_N$ and $l-\varepsilon$, is a rough lower bound.

Thus the sequence is bounded. Hence it has an exact upper bound M and an exact lower bound m. Hence the theorem. Now l can be easily proved to lie between M and m.

II. *A monotone increasing sequence which is bounded above is convergent.*

Similarly, *a monotone decreasing sequence which is bounded below is convergent.*

The result can be proved with the help of Dedekind's theorem.

Let $\{s_n\}$ be a monotonic increasing sequence, bounded above, so that $s_{n-1} \leqslant s_n \leqslant M$ for all values of n.

Divide the whole set of real numbers into two classes L and R, such that a number belongs to L-class provided it is exceeded by at least one member of the given sequence, and to R-class otherwise. Evidently, members of both classes exist, for any number less than s_1 (say) belongs to the L-class, and the number M belongs to the R-class. Also, as defined, every member of the L-class is clearly less than every member of the R-class. Hence by Dedekind's theorem, there exists a real number l which divides the two classes from one another.

Now, however small a pre-assigned positive quantity ε may be taken, $l-\varepsilon$ being a member of the L-class, there is at least one member of the given sequence, say s_N, such that $l-\varepsilon < s_N$, and as the sequence is monotonic increasing, $l-\varepsilon < s_n$ for every $n \geqslant N$. Again, $l+\varepsilon$ being a member of the R-class, $s_n < l+\varepsilon$ for every n. Hence $l-\varepsilon < s_n < l+\varepsilon$, *i.e.*, $|l-s_n| < \varepsilon$ for every $n \geqslant N$. Hence $\underset{n\to\infty}{Lt}\, s_n = l$, *i.e.*, $\{s_n\}$ is convergent.

Similarly, the other case can be proved.

Note 1. A monotone increasing sequence must either tend to a definite finite limit, or to $+\infty$; it cannot oscillate.

For if it is not bounded above, it $\to +\infty$.

Note 2. A monotone sequence must have all its terms, after at most a finite number of terms, of the same sign.

III. *If* $\{a_n\}$ *and* $\{b_n\}$ *be two convergent sequences having* A *and* B *as their respective limits, then*

(i) *The sequence* $\{a_n \pm b_n\}$ *is also a convergent sequence, having* $A \pm B$ *as the limit.*

(ii) *The sequence* $\{a_n b_n\}$ *is also convergent having* AB *as the limit.*

(iii) *The sequence* $\{a_n/b_n\}$ *is also convergent having* A/B *as the limit, provided* $B \neq 0$.

The proofs of these are similar to the corresponding limit theorem on functions.

[See *Appendix, Section D*, § *1*]

IV. *If* $\{a_n\}$, $\{b_n\}$, $\{c_n\}$ *are three sequences such that*

(i) $a_n < b_n < c_n$ for $n > m_1$ ... (1)

and (ii) $Lt\,\{a_n\} = Lt\,\{c_n\} = l$, as $n \to \infty$

then $Lt\,\{b_n\} = l$, as $n \to \infty$.

Let ε be any positive number.

Since $\{a_n\} \to l$, as $n \to \infty$

$\therefore \quad l - \varepsilon < a_n < l + \varepsilon$, when $n \geqslant m_2$. $\quad \cdots \quad (2)$

Similarly,

$l - \varepsilon < c_n < l + \varepsilon$, when $n \geqslant m_3$. $\quad \cdots \quad (3)$

The inequalities (1), (2) and (3) hold when $n \geqslant m$ where m is the greatest of the numbers m_1, m_2, m_3.

From (1) and (2),

$l - \varepsilon < a_n < b_n$, when $n \geqslant m$.

From (1) and (3),

$b_n < c_n < l + \varepsilon$, when $n \geqslant m$.

$\therefore \quad l - \varepsilon < b_n < l + \varepsilon$, when $n \geqslant m$

$\therefore \quad |b_n - l| < \varepsilon$, when $n \geqslant m$

Hence, $\underset{n \to \infty}{Lt}\ b_n = l$.

[For *illustration* see *Ex. 2, Art. 6, worked out*]

V. *If* $x_n < y_n$ *when* $n \geqslant m_1$,

then $\underset{n \to \infty}{Lt}\ \{x_n\} \leqslant \underset{n \to \infty}{Lt}\ \{y_n\}$, *when they exist.*

Let $\underset{n \to \infty}{Lt}\ \{x_n\} = l_1$, and $\underset{n \to \infty}{Lt}\ \{y_n\} = l_2$.

We are to prove that $l_1 \leqslant l_2$.

If possible suppose $l_1 > l_2$; choose $\varepsilon = \frac{1}{2}(l_1 - l_2)$, a positive number.

Since, $\{x_n\} \to l_1$ as $n \to \infty$

$\therefore \quad l_1 - \varepsilon < x_n < l_1 + \varepsilon$ for $n \geqslant m_2$

i.e., $l_1 - \frac{1}{2}(l_1 - l_2) < x_n$ for $n \geqslant m_2$

i.e., $\frac{1}{2}(l_1 + l_2) < x_n$ for $n \geqslant m_2$. $\quad \cdots \quad (1)$

Since, $\{y_n\} \to l_2$ as $n \to \infty$

$\therefore \quad l_2 - \varepsilon < y_n < l_2 + \varepsilon \quad$ for $n \geqslant m_3$;

$\therefore \quad y_n < l_2 + \frac{1}{2}(l_1 - l_2) \quad$ for $n \geqslant m_3$,

i.e., $y_n < \frac{1}{2}(l_1 + l_2) \quad$ for $n \geqslant m_3$. $\quad \cdots \quad$ (2)

The inequalities (1) and (2) hold where $n \geqslant m$, where m is the greatest of the numbers m_1, m_2, m_3.

$\therefore$ from (1) and (2),

$$x_n > \tfrac{1}{2}(l_1 + l_2) > y_n,$$

i e., $\quad x_n > y_n \quad$ when $n \geqslant m$.

This contradicts our hypothesis that $x_n < y_n$.

Hence our assumption $l_1 > l_2$ is incorrect.

$\therefore \quad l_1 \not> l_2$ *i.e.*, $l_1 \leqslant l_2$.

Illustrations. Consider the sequences $\left\{\frac{1}{n}\right\}, \left\{\frac{1}{n^2}\right\}$.

Here $\frac{1}{n^2} < \frac{1}{n}$ when $n > 1$

i.e., every member of the second sequence except one is less than that of the first but the limits of the two are equal.

Again for the sequences $\left\{\frac{1}{n^2}\right\}, \left\{1 + \frac{1}{n^2}\right\}$

every member of the second sequence is greater than that of the first and the limit of the second is greater than the limit of the first.

VI. Necessary and sufficient conditions that a sequence $\{s_n\}$ may be convergent.

(i) Either, *the sequence is monotonic and bounded* (increasing and bounded above, or decreasing and bounded below),

(ii) or else, *the sequence satisfies* **Cauchy's condition,** namely, "Given any pre-assigned positive quantity ε, however small, we can determine a positive integer N, such that

$$| s_{n+p} - s_n | < \varepsilon, \text{ whenever } n \geqslant N,$$

p being any positive integer".

The proof that Cauchy's condition is necessary is similar to the corresponding case for the existence of the limit of a function. [See *Appendix*, *Section D*, § *3*]

4. An important inequality *(Bernoulli's Inequality)* :

For every positive integer $n \geqslant 2$, *and* $p > -1$ $(p \neq 0)$,

$$(1+p)^n > 1+np.$$

When $n = 2, (1+p)^2 = 1 + 2p + p^2 > 1 + 2p$.

Thus the relation holds for $n = 2$. Suppose it holds for $n = k \geqslant 2$.

$\therefore \quad (1+p)^k > 1 + kp$;

$\therefore \quad (1+p)(1+p)^k > (1+p)(1+kp)$, since $1 + p > 0$,

i.e., $\quad (1+p)^{k+1} > 1 + (k+1)p + kp^2 > 1 + (k+1)p$.

$\therefore$ the relation assumed for $n = k$ is true for $n = k+1$. It holds for $n = 2$; hence it holds generally, for $n > 2$.

Note. The above inequality is true for $n > 1$, even if n be not a positive integer. The proof of this depends upon the well-known inequality in Higher Algebra, namely $(a^n - 1)/n > a - 1$ for $n > 1$, $a > 0$, by putting $a = 1 + p$ here.

5. Some important limits on sequence.

(I) $\underset{n\to\infty}{\mathrm{Lt}} \left(1 + \dfrac{1}{n}\right)^n.$

We shall prove that the above limit exists finitely, and lies between 2 and 3. This limit is generally denoted by e.

By Binomial theorem for a positive integral index,

$$\left(1+\frac{1}{n}\right)^n = 1+n.\frac{1}{n}+\frac{n(n-1)}{2!}\frac{1}{n^2}+\frac{n(n-1)(n-2)}{3!}.\frac{1}{n^3}+\cdots+\frac{1}{n^n}$$

$$= 1+1+\frac{1}{2!}\left(1-\frac{1}{n}\right)+\frac{1}{3!}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)+\cdots$$

$$+\frac{1}{n!}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\cdots\left(1-\frac{n-1}{n}\right). \quad \cdots (1)$$

The number of terms is $n+1$, and each term is clearly positive. If now n be increased (keeping it always a positive integer), since each of $1-\frac{1}{n}$, $1-\frac{2}{n}$, $\cdots$, $1-\frac{n-1}{n}$ is evidently positive, and increases with n, each term of the right-hand series of (1) after the first two increases, and the total number of terms also increasing, $\left(1+\frac{1}{n}\right)^n$ *monotonically increases.*

Again, from (1), ($\because$ each of $1-\frac{1}{n}$, $1-\frac{2}{n}$, $1-\frac{n-1}{n}$ is positive and < 1), it follows that

$$\left(1+\frac{1}{n}\right)^n < 1+1+\frac{1}{2!}+\frac{1}{3!}+\cdots+\frac{1}{n!}$$

$$< 1+1+\frac{1}{2}+\frac{1}{2^2}+\cdots+\frac{1}{2^n}$$

$$[\because r! = 1.2.3\ldots\ldots r > 2^{r-1}]$$

i.e., $< 1+\left(1-\frac{1}{2^n}\right)\Big/\left(1-\frac{1}{2}\right) < 1+1\Big/\left(1-\frac{1}{2}\right)$ *i.e.*, < 3.

Hence the sequence $\left(1+\frac{1}{n}\right)^n$, being monotonic increasing and bounded above, is convergent, *i.e.*, tends to a definite finite limit, which is < 3, and from (1) evidently > 2.

As mentioned before, this limit is denoted by e.

(II) If $s_n = 1 + \frac{1}{1!} + \frac{1}{2!} + \cdots + \frac{1}{n!}$, $\underset{n \to \infty}{Lt}\ s_n = e$.

Evidently here s_n is a monotonic increasing sequence. Also, as shown in I above,

$$s_n < 1 + 1 + \frac{1}{2} + \frac{1}{2^2} + \cdots + \frac{1}{2^{n-1}} < 3.$$

Thus s_n tends to a definite finite limit < 3, and evidently > 2. Call this limit λ.

Let us denote $\left(1 + \frac{1}{n}\right)^n$ by a_n, (n a positive integer).

Then, as we have seen in I above, $a_n < s_n$.

Hence $Lt\ a_n \leqslant Lt\ s_n$; $\therefore\ e \leqslant \lambda$. $\cdots$ (i)

Again, let m be any integer greater than n. Then from (1) of I above, it easily follows that

$$a_m > 1 + \frac{1}{1!} + \frac{1}{2!}\left(1 - \frac{1}{m}\right) + \cdots$$
$$+ \frac{1}{n!}\left(1 - \frac{1}{m}\right)\left(1 - \frac{2}{m}\right)\cdots\left(1 - \frac{n-1}{m}\right).$$

(several terms of right side of a_m which are all positive being left out here)

Now, keeping n fixed, let $m \to \infty$.

When $m \to \infty$, $a_m \to e$, and the right side $\to s_n$.

$\therefore\ e \geqslant s_n$ for all values of n.

Thus $Lt\ s_n$ i.e , $\lambda \leqslant e$. $\cdots$ $\cdots$ (ii)

From (i) and (ii), it is clear that $\lambda = e$,

i.e., $\underset{n \to \infty}{Lt}\ s_n = e$.

(III) **Lt $\sqrt[n]{x}=1$,** *when $n \to \infty$ $(x > 0)$, (n a positive integer).*

For $x=1$, the result is obvious.

Let $x > 1$; then $\sqrt[n]{x} > 1$, so that we may put $\sqrt[n]{x}=1+h_n$ where $h_n > 0$. $\therefore$ $x=(1+h_n)^n > 1+nh_n > nh_n$, $\therefore$ $h_n < x/n$. $\therefore$ $0 < h_n < x/n$. Since when $n \to \infty$, $x/n \to 0$, hence $h_n \to 0$ and therefore $\sqrt[n]{x} \to 1$.

If $x < 1$, we write $x=1/y$, so that $y > 1$. $\therefore$ $\sqrt[n]{x}=1/\sqrt[n]{y}$ which $\to 1$ as $n \to \infty$.

Hence the result.

(IV) **Lt $\sqrt[n]{n}=1$,** *when $n \to \infty$, (n a positive integer).*

For $n > 1$, $\sqrt[n]{n} > 1$. Let $\sqrt[n]{n}=1+h_n$ where $h_n > 0$.

$$\therefore \quad n=(1+h_n)^n=1+\binom{n}{1}h_n+\binom{n}{2}h_n^2+\cdots\cdots\cdots\cdots+h_n^n$$

$> \frac{1}{2}n(n-1)h_n^2$, since all the terms are positive.

$\therefore$ $0 < h_n < \sqrt{2/(n-1)}$. As $n \to \infty$, since right side $\to 0$,

$\therefore$ $h_n \to 0$. $\therefore$ $\sqrt[n]{n} \to 1$.

(V) *As $n \to \infty$,* **Lt $nx^n \to 0$,** $|x| < 1$, *(n a positive integer).*

For $0 < |x| < 1$, $|x|=1/(1+y)$, where y is a finite quantity > 0.

$$\therefore \quad |x^n| = \frac{1}{(1+y)^n}$$

$$= \frac{1}{1+\binom{n}{1}y+\binom{n}{2}y^2+\cdots+\binom{n}{n}y^n} < \frac{1}{\binom{n}{2}y^2} \text{ for } n > 1$$

since in the denominator each term is positive.

$$\therefore \quad |nx^n| < \frac{2}{(n-1)y^2}, \text{ which } \to 0 \text{ as } n \to \infty.$$

$$\therefore \quad nx^n \to 0 \text{ as } n \to \infty.$$

(VI) *As* $n \to \infty$, $\dfrac{\log n}{n^p} \to 0$, $(p > 0)$, (*n a positive integer*).

Since $e > 1$ and $p > 0$, $\therefore\ e^p > 1$, hence $1/e^p < 1$.

If g denotes the characteristic of $\log n$,

$$g \leqslant \log n < g+1. \quad \therefore\ e^g \leqslant n.$$

$$\therefore\ \text{for } n > 1,\ 0 < \frac{\log n}{n^p} < \frac{g+1}{(e^g)^p} < e^p.\frac{g+1}{(e^p)^{g+1}}.$$

When $n \to \infty$, $(g+1) \to \infty$, and hence right extreme $\to 0$ [*by V, writing x for* $1/e^p$].

$\therefore\ (\log n)/n^p \to 0$ as $n \to \infty$.

(VII) *If* $\{s_n\}$ *is a sequence such that* $\underset{n\to\infty}{\text{Lt}} \left|\dfrac{s_{n+1}}{s_n}\right| = l$, *where* $0 \leqslant l < 1$, *then* Lt $s_n = 0$ *as* $n \to \infty$.

From the given condition it follows that we can determine a positive integer n_0 such that when $n \geqslant n_0$, $|s_{n+1}/s_n| < l+\varepsilon$, *i.e.*, $< k$, where $0 < k < 1$ [choosing $\varepsilon < 1-l$].

$$\therefore\quad |s_n| = \left|\frac{s_n}{s_{n-1}}.\frac{s_{n-1}}{s_{n-2}} \cdots \frac{s_{n_0+1}}{s_{n_0}}\right|.|s_{n_0}| < k^{n-n_0}|s_{n_0}|$$

i.e., $< k^n.\{|s_{n_0}|/k^{n_0}\} \to 0$

since $k^n \to 0$ when $n \to \infty$, and the other factor is finite.

$\therefore\quad s_n \to 0$ as $n \to \infty$.

Note. If $\{s_n\}$ be a sequence such that $s_n > 0$, and $Lt\ (s_{n+1}/s_n) = l$, where $l > 1$, then $Lt\ s_n = \infty$ as $n \to \infty$.

Similarly if $\{s_n\}$ be a sequence such that $s_n < 0$ and $Lt\ (s_{n+1}/s_n) = l$, where $l > 1$, then $Lt\ s_n = -\infty$, as $n \to \infty$.

Illus. (i) $\underset{n\to\infty}{\mathrm{Lt}} \dfrac{x^n}{n!}=0$ for all values of x.

Denoting the given expression by s_n, we have here

$$Lt \left| \frac{s_{n+1}}{s_n} \right| = Lt \left| \frac{1}{n+1} \cdot x \right| = 0 \text{ for all values of } x.$$

Hence $Lt\ s_n = 0$ for all values of x.

(ii) $\underset{n\to\infty}{\mathrm{Lt}} \dfrac{x^n}{n!}=0$ if $|x| \leqslant 1$.

Here $Lt \left| \dfrac{s_{n+1}}{s_n} \right| = Lt \left| \dfrac{n}{n+1} \cdot x \right| = Lt \left| \dfrac{1}{1+1/n} \cdot x \right| = |x|.$

Thus if $|x| < 1$, $Lt\ s_n = 0$; when $|x| = 1$, since $Lt\ 1/n = 0$, we get the reqd. result.

(iii) $\underset{n\to\infty}{\mathrm{Lt}} \dfrac{m(m-1)\ldots(m-n+1)}{n!} x^n = 0$ if $|x| < 1$.

Denoting the given expression by s_n, we have here

$$\underset{n\to\infty}{Lt} \left| \frac{s_{n+1}}{s_n} \right| = \underset{n\to\infty}{Lt} \left| \frac{m-n}{n+1} x \right| = \underset{n\to\infty}{Lt} \left| \frac{m/n-1}{1+1/n} \cdot x \right| = |x|.$$

Hence if $|x| < 1$, $Lt\ s_n = 0$.

(iv) By the above theorem we can also easily show that

$\mathrm{Lt}\ x^n = 0$ and $\mathrm{Lt}\ nx^n = 0$ when $|x| < 1$.

(VIII) *If $Lt\ s_n = 0$ as $n \to \infty$, then*

$$\underset{n\to\infty}{\mathrm{Lt}} \frac{s_1+s_2+\cdots+s_n}{n} = 0.$$

Since $Lt\ s_n = 0$, for a given $\varepsilon > 0$, m can be so chosen that for $n > m$, we have $|s_n| < \frac{1}{2}\varepsilon$. For these n's we have

$$\left| \frac{\Sigma s_n}{n} \right| \leqslant \frac{|s_1+s_2+\cdots+s_m|}{n} + \frac{n-m}{n} \cdot \tfrac{1}{2}\varepsilon$$

$$< \frac{|s_1+s_2+\cdots+s_m|}{n} + \tfrac{1}{2}\varepsilon. \quad [\because n-m < n]$$

Since the numerator of the first fraction on the right side now contains a fixed number of terms, we can determine a number μ, such that for $n > \mu$, the fraction remains $< \frac{1}{2}\varepsilon$. Let N be the number greater than both m and μ; then for $n > N$, $|\Sigma s_n/n| < \frac{1}{2}\varepsilon + \frac{1}{2}\varepsilon$, *i.e.*, $< \varepsilon$.

Hence the result follows.

(IX) *If* $Lt\ s_n = l$ *as* $n \to \infty$, *then*

$$\underset{n\to\infty}{\mathrm{Lt}} \frac{s_1 + s_2 + \cdots + s_n}{n} = l.$$

Let $s_n = l + t_n$, then $Lt\ (s_n - l) = Lt\ t_n = 0$

$$\therefore \underset{n\to\infty}{Lt} \frac{\Sigma t_n}{n} = 0, \text{ i.e., } \underset{n\to\infty}{Lt} \frac{\Sigma(s_n - l)}{n}, \text{ i.e., } \underset{n\to\infty}{Lt} \left(\frac{\Sigma s_n}{n} - l\right) = 0.$$

Hence the result.

This is known as *Cauchy's first theorem on limits.*

Illus: $\underset{n\to\infty}{Lt} \dfrac{1 + 1/2 + \cdots + 1/n}{n} = 0$, since lim. $1/n = 0$.

$\underset{n\to\infty}{Lt} \dfrac{1 + \sqrt{2} + \sqrt[3]{3} + \cdots + \sqrt[n]{n}}{n} = 1$, since lim. $\sqrt[n]{n} = 1$.

(X) The following important theorems regarding limits of sequences are added without proof :

(i) *If* $\{b_n\}$ *is a monotonically increasing sequence such that* $b_{n+1} > b_n$ *and if* $b_n \to \infty$ *and* $\{a_n\}$ *be any sequence, then,*

$$\underset{n\to\infty}{\mathrm{Lt}} \frac{a_n}{b_n} = \underset{n\to\infty}{\mathrm{Lt}} \frac{a_n - a_{n-1}}{b_n - b_{n-1}},$$

provided the limit on the right exists, whether finite or infinite.

(ii) *If the sequences* $\{a_n\}$, $\{b_n\}$ *tend to zero and if* $\{b_n\}$ *is a monotonically decreasing sequence such that* $b_n > b_{n+1} > 0$, *then*

$$\underset{n\to\infty}{\mathrm{Lt}} \frac{a_n}{b_n} = \underset{n\to\infty}{\mathrm{Lt}} \frac{a_n - a_{n-1}}{b_n - b_{n-1}},$$

provided the limit on the right exists whether finite or infinite.

(iii) *If* $\{a_n\}$ *be a sequence of positive terms, then*

$$\underset{n\to\infty}{\mathrm{Lt}} \left[\sqrt[n]{a_n}\right] = \underset{n\to\infty}{\mathrm{Lt}} \left[\frac{a_{n+1}}{a_n}\right],$$

provided the limit on the right exists whether finite or infinite.

This is known as *Cauchy's second theorem on limits.*

Illus. : (i) $\underset{n\to\infty}{Lt} \dfrac{1^4+2^4+\cdots+n^4}{n^5} = \underset{n\to\infty}{Lt} \dfrac{n^4}{n^5-(n-1)^5} = \dfrac{1}{5}.$

(ii) $\underset{n\to\infty}{Lt} (n^{\frac{1}{n}}) = \underset{n\to\infty}{Lt} \dfrac{n+1}{n} = 1.$

(iii) $$\underset{n\to\infty}{Lt} \frac{(n\,!)^{\frac{1}{n}}}{n} = \underset{n\to\infty}{Lt} \left(\frac{n\,!}{n^n}\right)^{\frac{1}{n}} = \underset{n\to\infty}{Lt} \frac{(n+1)\,!\,n^n}{(n+1)^{n+1}n\,!}$$

$$= \underset{n\to\infty}{Lt} \left(\frac{n}{n+1}\right)^n = \underset{n\to\infty}{Lt} \left(1+\frac{1}{n}\right)^{-n} = \frac{1}{e}.$$

6. Illustrative Examples.

Ex. 1. *Show that the sequence* $\{x_n\}$ *is monotone ascending*

where $x_n = \left(1+\dfrac{1}{n}\right)^n.$

$$x_n > x_{n-1} \text{ if } \left(1+\frac{1}{n}\right)^n > \left(1+\frac{1}{n-1}\right)^{n-1},$$

i.e., if $\left(\dfrac{1+\dfrac{1}{n}}{1+\dfrac{1}{n-1}}\right)^n > \left(1+\dfrac{1}{n-1}\right)^{-1}$ *i.e.*, $> \dfrac{n-1}{n}$ *i.e.*, $> 1-\dfrac{1}{n}$,

i.e., if $\left(\dfrac{n^2-1}{n^2}\right)^n > 1-\dfrac{1}{n}$ *i.e.*, if $\left(1-\dfrac{1}{n^2}\right)^n > 1-\dfrac{1}{n}$

which is true by Bernoulli's inequality.

Note. That the above sequence is monotone ascending has been proved by a different method in Art. 5 (1).

Ex. 2. *Evaluate* $\underset{n\to\infty}{Lt}\left[\dfrac{1}{\sqrt{n^2+1}}+\dfrac{1}{\sqrt{n^2+2}}+\cdots+\dfrac{1}{\sqrt{n^2+n}}\right]$.

Let $\{b_n\}$ be the sequence where

$$b_n=\frac{1}{\sqrt{n^2+1}}+\frac{1}{\sqrt{n^2+2}}+\cdots+\frac{1}{\sqrt{n^2+n}}$$

and $\{a_n\}$ be the sequence where $a_n=\dfrac{n}{\sqrt{n^2+n}}$,

and $\{c_n\}$ be the sequence where $c_n=\dfrac{n}{\sqrt{n^2+1}}$.

$\therefore\ a_n < b_n < c_n$.

Also $Lt\ \{a_n\}=Lt\ \dfrac{1}{\sqrt{1+\dfrac{1}{n}}}=1$

and $Lt\ \{c_n\}=Lt\ \dfrac{1}{\sqrt{1+\dfrac{1}{n^2}}}=1$.

$\therefore$ by theorem (IV) of Art. 3, $Lt\ \{b_n\}=1$.

Ex. 3. *Assuming that the following sequence of positive terms possesses finite limit, find it.*

$$\sqrt{2},\ \sqrt{2+\sqrt{2}},\ \sqrt{2+\sqrt{2+\sqrt{2}}},\ldots$$

Here $a_n=\sqrt{2+a_{n-1}}$. ... (1)

Let l be the limit of the above sequence.

$$\therefore \quad \underset{n\to\infty}{Lt}\ a_n = \underset{n\to\infty}{Lt}\ a_{n-1} = l. \qquad \cdots \quad (2)$$

Squaring (1) and taking limit, we have

$$l^2 - l - 2 = 0. \quad \therefore \quad (l-2)(l+1) = 0.$$

$\therefore$ $l = 2$, since l cannot be -1, every term of the sequence being positive.

$\therefore$ required limit of the sequence $= 2$.

Section A—Examples

1. Discuss the behaviour of the following sequences $\{x_n\}$ as $n \to \infty$, where

(i) $x_n = \dfrac{\sin n}{\sqrt{n}}$. (ii) $x_n = \cos \frac{1}{2}n\pi + \dfrac{1}{n}$.

(iii) $x_n = (\sqrt{n}) \cos n\pi$.

2. Show that the sequence $\{a_n\}$ is monotone ascending where

$$a_n = \frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{2n},$$

and show also that it is bounded.

3. Show that the sequence $\{a_n\}$ is monotone decreasing where

$$a_n = \left(1 + \frac{1}{n}\right)^{n+1}. \qquad [\ C.\ H.\ 1955\ ]$$

4. Show that the sequence $\{b_n\}$ where

$$b_n = \frac{a_1 + a_2 + \cdots + a_n}{n}, \quad a_n > 0$$

is monotone ascending or descending according as the sequence $\{a_n\}$ is so.

5. If $\{x_n\}$, $\{y_n\}$ denote two positive sequences such that

$$x_{n+1} = \tfrac{1}{2}(x_n + y_n) \text{ and } \frac{2}{y_{n+1}} = \frac{1}{x_n} + \frac{1}{y_n},$$

show that the sequences are monotones, the former descending and the latter ascending.

6. If $x_n = (-1)^n$ and $y_n = \dfrac{x_1 + x_2 + \cdots + x_n}{n}$, prove that the sequence $\{y_n\}$ converges, although $\{x_n\}$ does not do so.

7. If $a_n > 0$ and if $a_n \to a\ (>0)$, then show that the sequence $\{b_n\} \to a$ where

$$b_n = \quad \frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n}$$

8. If $a_n > 0$ and if $a_n \to a\ (>0)$, then show that the sequence $\{b_n\} \to a$ where

$$b_n = \sqrt[n]{a_1 a_2 \cdots a_n}.$$

[*Use A.M. > G.M. > H.M. and apply (V) of Art. 3*]

ANSWERS

1. (i) Convergent : limit is 0. (ii) Oscillates finitely.
   (iii) Oscillates infinitely.

SECTION B

INFINITE SERIES

1. Definitions.

An endless expression of the form

$$u_1 + u_2 + u_3 + \cdots + u_n + \cdots$$

where u_1, u_2,...... have definite values (positive or negative, constant or variable) is called an *infinite series*, and is denoted by Σu_n.

For such a series, let $s_n \equiv u_1 + u_2 + \cdots + u_n$ be the sum of first n terms of the series. Then considering the sequence $\{s_n\}$,

(i) if $s_n \to S$ (a definite finite limit) as $n \to \infty$, the series Σu_n is said to be *convergent*, and S is called its sum,

(ii) if $s_n \to +\infty$, or to $-\infty$, as $n \to \infty$,
the series Σu_n is said to be *divergent*,

(iii) if s_n oscillates (finitely or infinitely) as $n \to \infty$, the series Σu_n is said to be the *oscillatory*.

If a series Σu_n consisting of positive as well as negative terms, be convergent, but the same series, when all the terms are taken with positive sign, *i e.*, the series $\Sigma \mid u_n \mid$ be divergent, then the original series Σu_n is said to be *semi-convergent.*

If on the other hand, $\Sigma \mid u_n \mid$ is convergent, it can be shown by Cauchy's condition for the convergence of a sequence, that Σu_n, (no matter what the signs of u_1, u_2 etc. may be), is also convergent. In such a case Σu_n is said to be *absolutely convergent.*

Divergent and oscillatory series together will sometimes be referred to as 'non-convergent'.

Examples :

(i) $1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\cdots$ is convergent, sum 2.

For here, $s_n=\dfrac{1-\dfrac{1}{2^n}}{1-\dfrac{1}{2}}=2-\dfrac{2}{2^{n-1}}\to 2$ as $n\to\infty$.

$$\frac{1}{1.2}+\frac{1}{2.3}+\frac{1}{3.4}+\cdots \text{ is convergent, sum } 1.$$

For here, $s_n=\left(1-\frac{1}{2}\right)+\left(\frac{1}{2}-\frac{1}{3}\right)+\left(\frac{1}{3}-\frac{1}{4}\right)+\cdots+\left(\frac{1}{n}-\frac{1}{n+1}\right)$

$$=1-\frac{1}{n+1}\to 1 \text{ as } n\to\infty.$$

Similarly, $1-\dfrac{1}{3}+\dfrac{1}{3^2}-\dfrac{1}{3^3}+\cdots$ is convergent, sum $\frac{3}{4}$.

(ii) $1+2+3+4+\cdots$ is divergent, $\to+\infty$.

(iii) $-5-5^2-5^3-\cdots$ is divergent, $\to-\infty$.

(iv) $1+2-3+1+2-3+1+2-3+\cdots\cdots$ is oscillatory, oscillating finitely between 3 and 0.

(v) $1-2+3-4+5-6+\cdots$ is oscillatory, oscillating infinitely between $-\infty$ and $+\infty$.

Here, $s_n=-\frac{1}{2}n$ or $\frac{1}{2}(n+1)$ according as n is even or odd.

(vi) $1-1+2-2+3-3+\cdots$ is oscillatory, oscillating infinitely between 0 and ∞.

(vii) $1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\frac{1}{6}+\cdots$ is semi-convergent.

[See § *2*, (*iii*) *below*.]

(viii) The geometric series

$$1+x+x^2+x^3+\cdots$$

is convergent if $|x| < 1$, divergent if $x \geq 1$, oscillatory if $x = -1$ (oscillating finitely), and oscillatory if $x < -1$ (oscillating infinitely).

Note. The word sum in connection with an infinite series is not a sum in the ordinary sense of the term. It does never represent the actual sum of any number of terms of the series, but only represents a limiting value which is never actually reached. *Sum of an infinite series is a matter of definition or convention*, and has no other existence in any previous notion.

2. Tests of convergence : In order to ascertain the convergence of infinite series, it is not always very convenient to find the limiting value of s_n as $n \to \infty$. So various methods and rules have been devised for testing the convergence. We give below (without proof) a few important of these tests. For proofs, see any book on Higher Algebra (*e.g.*, Barnard & Child).

(i) A *necessary* condition for convergence of Σu_n is that $Lt\ u_n = 0$.

(ii) If a series Σu_n *of positive and decreasing* terms be convergent, then $Lt\ nu_n = 0$. [*Pringsheim's theorem*]

For the harmonic series $\Sigma(1/n)$, since $nu_n = 1$, so $Lt\ nu_n \neq 0$, and hence the series is not convergent. Thus (s_n being monotone increasing here), the series is divergent. Similarly the series $\Sigma 1/(2n-1)$ is divergent.

(iii) In the *alternating series* $u_1 - u_2 + u_3 - \cdots$, in which the terms are alternately positive and negative, if $u_n > u_{n+1}$ for all values of n, and if $u_n \to 0$ as $n \to \infty$, the series is convergent.

Consider the series $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$

Here, $u_n = 1/n$; $u_{n+1} = 1/(n+1)$. $\therefore$ $u_n > u_{n+1}$.

Also $u_n \to 0$ as $n \to \infty$. Hence the series is convergent.

From (ii), it follows that

$$1-\tfrac{1}{2}+\tfrac{1}{3}-\tfrac{1}{4}+\cdots \text{ is semi-convergent.}$$

Similarly, the series $1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\cdots$ is semi-convergent.

(iv) **Comparison Test :** If Σu_n and Σv_n be two infinite series of positive terms, and if $\underset{n\to\infty}{Lt}(u_n/v_n)=\rho$, ρ being a finite quantity, then the series are either both convergent or both divergent.

Note An important series used very often for comparison is

$$\frac{1}{1^p}+\frac{1}{2^p}+\frac{1}{3^p}+\cdots+\frac{1}{n^p}+\cdots\cdots$$

which is convergent if $p>1$, and divergent if $p\leq 1$.

For proof, see any book on Higher Algebra.

(v) **D' Alembert's Ratio Test :** If Σu_n be a series of positive terms, and if

$$\underset{n\to\infty}{Lt}\frac{u_{n+1}}{u_n}<1,\ \Sigma u_n \text{ is convergent,}$$

$$>1,\ \Sigma u_n \text{ is divergent,}$$

$$=1, \text{ the test fails.}$$

In the last case some other method must be devised to test the convergence of the series. In this case (vi) may be tried.

(vi) **Raabe's Test :** The series Σu_n of positive terms is convergent or divergent according as

$$\underset{n\to\infty}{Lt}\left\{n\left(\frac{u_n}{u_{n+1}}-1\right)\right\}>1 \text{ or } <1.$$

The test fails if the limit $=1$.

(vii) **Gauss's Test:** If for a series Σu_n of positive terms, u_n/u_{n+1} be expressed in powers of $1/n$, so that

$$\frac{u_n}{u_{n+1}} = 1 + \frac{\mu}{n} + O\left(\frac{1}{n^2}\right)$$

then Σu_n is convergent if $\mu > 1$, and divergent if $\mu \leqslant 1$.

Note. The notation $O(1/n^2)$ denotes such a function $f(n)$ that for every $n \geqslant n_0$ (a definite positive integer), $| f(n) | < k \frac{1}{n^2}$, where k is a finite quantity independent of n.

(viii) **Cauchy's Root Test:** If Σu_n be a series of positive terms, and if

$$\underset{n\to\infty}{Lt} (u_n)^{\frac{1}{n}} < 1, \ \Sigma u_n \text{ is convergent,}$$

$$> 1, \ \Sigma u_n \text{ is divergent,}$$

$$= 1, \text{ the test fails.}$$

(ix) **Tests for absolute convergence:** If for a series Σu_n,

$$\underset{n\to\infty}{Lt} \left| \frac{u_{n+1}}{u_n} \right| < 1, \ \Sigma u_n \text{ is absolutely convergent,}$$

$$> 1, \ \Sigma u_n \text{ is non-convergent,}$$

$$= 1, \text{ the test fails.}$$

For other properties of infinite series, any book on Higher Algebra may be consulted.

3. Illustrative Examples.

Ex. 1. *Prove that the series*

$$\sqrt{\tfrac{1}{2}} + \sqrt{\tfrac{2}{3}} + \sqrt{\tfrac{3}{4}} + \sqrt{\tfrac{4}{5}} + \cdots$$

is divergent.

Here, $u_n = \sqrt{\frac{n}{n+1}} = 1 \Big/ \sqrt{1+\frac{1}{n}} \to 1$ as $n \to \infty$.

As $Lt\, u_n \neq 0$, the necessary condition for the convergence of the series is not satisfied.

Again, the terms here being all positive, the sum s_n of the first n terms evidently monotonically increases with n.

Hence the series, being non-convergent here must be divergent.

Ex. 2. *Test the convergence or divergence of the series whose n^{th} term is $\sqrt{n^2+1} - n$.*

Here, $u_n = \sqrt{n^2+1} - n$. Assume $v_n = \frac{1}{n}$.

Then $$\frac{u_n}{v_n} = n(\sqrt{n^2+1} - n) = \frac{n.\{(n^2+1) - n^2\}}{\sqrt{n^2+1}+n}$$

$$= \frac{1}{\sqrt{1+\frac{1}{n^2}}+1} \to \frac{1}{2} \text{ (a finite quantity) as } n \to \infty.$$

Thus, Σu_n and Σv_n are either both convergent, or both divergent. But $\Sigma v_n = \Sigma \frac{1}{n}$ is divergent. Hence Σu_n is divergent.

Ex. 3. *Find whether the series*

$$2x + \frac{3x^2}{8} + \frac{4x^3}{27} + \cdots + \frac{n+1}{n^3}x^n + \cdots$$

is convergent or divergent, $(x > 0)$.

Here, $$\frac{u_{n+1}}{u_n} = \frac{n+2}{(n+1)^3}x^{n+1} \times \frac{n^3}{n+1}\cdot\frac{1}{x^n}$$

$$= \frac{n^3(n+2)}{(n+1)^4}\cdot x = \frac{1+\frac{2}{n}}{\left(1+\frac{1}{n}\right)^4}\cdot x \to x \text{ as } n \to \infty.$$

$\therefore$ by D' Alembert's test, the series is convergent if $x < 1$, and divergent if $x > 1$. For $x = 1$, the test fails.

If $x = 1$, for the series $u_n = \frac{n+1}{n^3}$.

Take $v_n = \frac{1}{n^2}$; then Σv_n is known to be convergent.

Now $\dfrac{u_n}{v_n}=\dfrac{n+1}{n}=1+\dfrac{1}{n}\to 1$ (a finite quantity) as $n\to\infty$.

$\therefore$ Σu_n is also convergent.

Thus the given series is convergent if $x\leqslant 1$, and divergent if $x>1$.

Ex. 4. *Test the convergence or otherwise of the series*

$$\frac{x}{1}+\frac{1}{2}\cdot\frac{x^3}{3}+\frac{1.3}{2.4}\cdot\frac{x^5}{5}+\frac{1.3.5}{2.4.6}\cdot\frac{x^7}{7}+\ldots\ldots$$

Here, $$\frac{u_{n+1}}{u_n}=\frac{1.3.5\ldots(2n-1)}{2.4.6\cdots 2n}\cdot\frac{x^{2n+1}}{2n+1}\times\frac{2.4.6\cdots(2n-2)}{1.3.5\cdots(2n-3)}\cdot\frac{2n-1}{x^{2n-1}}$$

$$=\frac{(2n-1)^2}{2n(2n+1)}\cdot x^2\to x^2 \text{ as } n\to\infty.$$

$\therefore$ by D' Alembert's test, the series is convergent if $x^2<1$ *i.e.*, $|x|<1$, and divergent, if $x^2>1$, *i.e.*, $|x|>1$.

If $x=1$, the test fails. Let us apply Raabe's test here.

Here, $$n\left\{\frac{u_n}{u_{n+1}}-1\right\}=n\left\{\frac{2n(2n+1)}{(2n-1)^2}-1\right\}$$

$$=\frac{n(6n-1)}{(2n-1)^2}=\frac{6-\dfrac{1}{n}}{\left(2-\dfrac{1}{n}\right)^2}\to\frac{6}{4} \text{ i.e., } \frac{3}{2} \text{ which is } >1.$$

Hence, the series is convergent in this case.

If $x=-1$, all the terms being reversed in sign as compared to the case when $x=1$, the series remains convergent.

Thus the given series is convergent when $|x|\leqslant 1$, and divergent when $|x|>1$.

Ex. 5. *Prove that the series*

$$\frac{1+2}{2.1}+\left(\frac{2+2}{2.2}\right)^2+\left(\frac{3+2}{2.3}\right)^3+\cdots+\left(\frac{n+2}{2.n}\right)^n+\cdots$$

is convergent.

Here, $u_n=\left(\dfrac{1}{2}+\dfrac{1}{n}\right)^n$

$\therefore$ $\displaystyle\mathop{Lt}_{n\to\infty} u_n^{\frac{1}{n}}=\mathop{Lt}_{n\to\infty}\left(\frac{1}{2}+\frac{1}{n}\right)=\tfrac{1}{2}<1.$

$\therefore$ by Cauchy's root test, the series is convergent.

SECTION C

POWER SERIES

1. Definition.

A series in which the successive terms are in ascending positive integral powers of a variable, say x, of the form

$$a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n + \cdots$$

or shortly, $\Sigma a_n x^n$

where the coefficients $a_0, a_1, a_2, \ldots$ are independent of x, is called a *power series in x*, and the numbers a_n are called the *coefficients* of the different powers of x.

By a power series we shall mean an infinite power series and the study of such series is of great importance in Calculus. It is clear that in the discussion of infinite power series, the mere statement 'convergent' (as in the case of constant-term power series, *e.g.*, $\Sigma 1/n^n$) is of little significance unless it is stated *for what values of x*, the series is convergent.

Let us consider the most important and simplest power series (*viz.*, the *geometric series*)

$$1 + x + x^2 + x^3 + \cdots\cdots \text{ to } \infty.$$

Here, $s_n = 1 + x + x^2 + \cdots + x^{n-1}$

$$= \frac{1-x^n}{1-x} \ (\text{for } x \neq 1) = \frac{1}{1-x} - \frac{x^n}{1-x}.$$

Now, $\underset{n\to\infty}{Lt}\, s_n = \underset{n\to\infty}{Lt}\left[\frac{1}{1-x} - \frac{x^n}{1-x}\right] = \frac{1}{1-x} - \frac{1}{1-x} \cdot \underset{n\to\infty}{Lt}\, x^n.$

Since for $|x| < 1$, $Lt\ x^n = 0$, $\therefore\ \underset{n\to\infty}{Lt}\, s_n = \frac{1}{1-x}$ for $|x| < 1$.

Thus the geometric series is convergent (also absolutely convergent) for $|x| < 1$ *i.e.*, $-1 < x < 1$.

For $x \geqslant 1$, $s_n \geqslant n$. $\therefore$ $s_n \to \infty$ as $n \to \infty$; *i.e.*, the series diverges to $+\infty$.

For $x = -1$, $s_n = 1$ or 0 according as n is odd or even *i.e.*, s_n does not tend to a definite limit.

For $x < -1$, $Lt\ s_n \to +\infty$ or $-\infty$ according as n is odd or even.

Every power series is convergent for $x=0$. A power series $\Sigma a_n x^n$ is convergent either for all values of x, or for a certain range of values of x, or for no value of x except 0.

2. Interval of convergence.

If a power series $\Sigma a_n x^n$ converges for every $|x| < r$ (indeed absolutely) but diverges for $|x| > r$, then $(-r, r)$ is called the *interval of convergence* of the given power series.

Note 1. r is very often called the *radius of convergence*.

Note 2. If a power series is convergent for all values of x, its interval of convergence is sometimes written as $(-\infty, +\infty)$ and in such cases the series is said to be *everywhere convergent*.

Note 3. Every power series is absolutely convergent in the open interval $(-r, r)$, the interval of convergence.

Note 4. The behaviour of the series *at the end-points of the interval of convergence* (*i.e.*, for $x=r$ and $-r$) has to be studied separately in every individual case, for there may be diverse possibilities, the series may be convergent at one end-point, and divergent (or oscillatory) at the other, or convergent at the both end-points, or divergent at both end-points, and so on.

Note 5. Although by actual division, we get

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots\cdots \text{ to } \infty$$

it must not be supposed that the series represents the function from which it is derived for *all values* of x.

For putting $x=2$, we get

$$-1=1+2+2^2+2^3+\cdots\cdots$$

which is evidently absurd.

The series represents the function only for those values of x for which the series is convergent.

We know

$$\frac{1}{\sqrt{(1+x)}}=1-\frac{1}{2}x+\frac{1.3}{2.4}x^2-\frac{1.3.5}{2.4.6}x^3+\cdots\cdots \text{ is convergent for } x=1.$$

If we multiply the series by itself and rearrange it in powers of x, we get

$$\frac{1}{1+x}=1-x+x^2-x^3+\cdots\cdots$$

which is obviously not true for $x=1$.

This happens because the original series is not absolutely convergent for $x=1$.

Again, if we differentiate term by term

$$\log(1+x)=x-\tfrac{1}{2}x^2+\tfrac{1}{3}x^3-\tfrac{1}{4}x^4+\cdots\cdots$$

for $x=1$, we get

$$\tfrac{1}{2}=1-1+1-1+1-1+\cdots\cdots$$

which is obviously absurd.

Herein lies the supreme importance of knowing beforehand the conditions under which a power series, whenever it is being used in any investigation, is convergent, and also under what conditions, the mathematical operations, performed on the series, are valid ; otherwise we would be led to absurdities as above.

3. Determination of interval of convergence.

For the series $\Sigma a_n x^n$,

(i) if $\underset{n\to\infty}{Lt}\left|\dfrac{a_{n+1}}{a_n}\right| \to$ a finite quantity λ ($\lambda \neq 0$),

(ii) if $\underset{n\to\infty}{Lt}\left|a_n\right|^{\frac{1}{n}}=\lambda$ ($\lambda \neq 0$),

then the interval of convergence of the series is $(-r, r)$ where $r=1/\lambda$.

For the first case, using D' Alembert's ratio test to the series (omitting the first term), we have for any fixed value of x,

$$\underset{n\to\infty}{Lt}\left|\frac{u_{n+1}}{u_n}\right| = \underset{n\to\infty}{Lt}\left|\frac{a_{n+1}x^{n+1}}{a_n x^n}\right| = \underset{n\to\infty}{Lt}\left|\frac{a_{n+1}}{a_n}\right| \cdot |x|$$
$$= \lambda\,|x|$$

which is less or greater than 1, according as $|x| <$ or $> 1/\lambda$. Hence the result.

If $\lambda = 0$, the test is satisfied for all values of x. Hence in this case, the series is convergent for all values of x, and the interval of convergence is $(-\infty, +\infty)$.

When however $|x| = r$ *i.e.*, $1/\lambda$ (*i.e.*, for the end-points of the interval of convergence), the limiting value of the ratio being 1, the test fails. In such cases we must use some other constant term series test to determine whether the series is convergent or not for these values of x.

For the second case we can similarly use Cauchy's test.

Illustrative Examples.

Find the interval of convergence of the following series :

Ex. 1. $1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots\cdots + \frac{x^n}{n!} + \cdots\cdots$ (*i.e.*, $\mathbf{e^x}$).

Here, $\underset{n\to\infty}{Lt}\left|\frac{u_{n+1}}{u_n}\right| = Lt\left|\frac{x^n}{n!} \div \frac{x^{n-1}}{(n-1)!}\right| = Lt\,\frac{1}{n}\,|x|$

$= 0$, for all values of x.

$\therefore$ the series is absolutely convergent for all values of x.

Here the interval of convergence is $(-\infty, +\infty)$.

Ex. 2. $x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots\cdots$ (*i.e.*, **sin x**).

$1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots\cdots$ (*i.e.*, **cos x**).

For the 1st series

$$\underset{n\to\infty}{Lt}\left|\frac{u_{n+1}}{u_n}\right| = Lt \left|\frac{(-1)^n x^{2n+1}}{(2n+1)\,!} \div \frac{(-1)^{n-1} x^{2n-1}}{(2n-1)\,!}\right|$$

$$= Lt \frac{x^2}{2n\,(2n+1)} = 0, \text{ for all values of } x.$$

$\therefore$ the series is absolutely convergent for all values of x.

Similarly, the cosine series is absolutely convergent for all values of x.

The interval of convergence in each case is $(-\infty, \infty)$.

Ex. 3. $x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots\cdots$, *i.e.*, **log (1+x).**

Here, $\underset{n\to\infty}{Lt}\left|\frac{u_{n+1}}{u_n}\right| = Lt \left|\frac{(-1)^n x^{n+1}}{n+1} \div \frac{(-1)^{n-1} x^n}{n}\right| = Lt \frac{n}{n+1} \cdot |x|$

$$= |x|\, Lt \frac{1}{1+1/n} = |x|.$$

Hence the series is absolutely convergent for $|x| < 1$, *i.e.*, $-1 < x < 1$.

The series is divergent for $|x| > 1$, *i.e.*, for $x > 1$ and $x < -1$.

For $|x| = 1$, the test fails.

Putting $x = -1$, in the same series, we have

$$-(1+\tfrac{1}{2}+\tfrac{1}{3}+\tfrac{1}{4}+\cdots) \text{ which tends to } -\infty.$$

Putting $x = 1$, we have

$$1-\tfrac{1}{2}+\tfrac{1}{3}-\tfrac{1}{4}+\cdots$$

which is convergent. [See *Sec. B*, § *2*, *Appendix.*]

Thus, the interval of convergence is $(-1, 1)$; the series is also convergent for $x = 1$.

Ex. 4. $x - \frac{x^3}{3} + \frac{x^5}{5} - \cdots\cdots$ *i.e.*, **(tan⁻¹ x).**

$$\underset{n\to\infty}{Lt}\left|\frac{u_{n+1}}{u_n}\right| = Lt \left|(-1)^n \frac{x^{2n+1}}{2n+1} \Big/ (-1)^{n-1} \frac{x^{2n-1}}{2n-1}\right| = Lt \frac{2n-1}{2n+1} x^2$$

$$= Lt \frac{2-1/n}{2+1/n} x^2 = x^2.$$

Hence the series is absolutely convergent for $x^2 < 1$, *i.e.*, for $-1 < x < 1$ and divergent for $x^2 > 1$, *i.e.*, $x > 1$ and $x < -1$.

When $x=1$, and -1 the series becomes

$$(1-\tfrac{1}{3}+\tfrac{1}{5}-\cdots) \text{ and } -(1-\tfrac{1}{3}+\tfrac{1}{5}-\cdots) \text{ respectively,}$$

both of which are convergent. [See *Sec. B*, § *2*, *Appendix*]

Hence for this series, the interval of convergence is $(-1, 1)$ the series being also convergent at both end points $1, -1$.

Ex. 5. $x+\dfrac{1}{2}\cdot\dfrac{x^3}{3}+\dfrac{1.3}{2.4}\cdot\dfrac{x^5}{5}+\dfrac{1.3.5}{2.4.6}\cdot\dfrac{x^7}{7}+\cdots\cdots$ *i.e.*, ($\sin^{-1}x$).

Here, $u_{n+1}=\dfrac{1.3.5\ .\ (2n-1)}{2.4.6\ldots 2n}\ \dfrac{x^{2n+1}}{2n+1}$,

$$\therefore \quad \underset{n\to\infty}{Lt}\left|\frac{u_{n+1}}{u_n}\right|=Lt\left|\frac{(2n-1)^2}{2n\ (2n+1)}\ x^2\right|=x^2.$$

$\therefore$ the series is absolutely convergent for $x^2 < 1$, *i.e.*, $-1 < x < 1$ and divergent for $x^2 > 1$, *i.e*, $x > 1$ and $x < -1$.

When $x=1$, the series becomes

$$S\equiv 1+\frac{1}{2}\cdot\frac{1}{3}+\frac{1.3}{2.4}\cdot\frac{1}{5}+\frac{1.3.5}{2.4.6}\cdot\frac{1}{7}+\cdots\cdots$$

Now, $$\frac{u_n}{u_{n+1}}=\frac{2n\ (2n+1)}{(2n-1)^2}=\left(\frac{2n+1}{2n}\right)\left(\frac{2n}{2n-1}\right)^2=\left(1+\frac{1}{2n}\right)\left(1-\frac{1}{2n}\right)^{-2}$$

$$=\left(1+\frac{1}{2n}\right)\left\{1+2\cdot\frac{1}{2n}+3\cdot\left(\frac{1}{2n}\right)^2+\cdots\cdots\right\}$$

$$=1+\frac{3}{2n}+O\left(\frac{1}{n^2}\right).$$

Hence by Gauss's test, this series is convergent since here $\mu=\frac{3}{2} > 1$.

When $x=-1$, the series becomes $-S$ and hence convergent.

Thus, the series is also convergent for $x=\pm 1$.

Ex. 6. (i) $1-x+x^2-x^3+\cdots\cdots$ $\quad\left(i.e., \dfrac{1}{1+x}\right)$.

(ii) $1-x^2+x^4-x^6+\cdots\cdots$ $\quad\left(i.e., \dfrac{1}{1+x^2}\right)$.

(iii) $1+\dfrac{1}{2}x^2+\dfrac{1.3}{2.4}x^4+\dfrac{1.3.5}{2.4.6}x^6+\cdots\cdots$ $\quad\left(i.e., \dfrac{1}{\sqrt{1-x^2}}\right)$.

Interval of convergence (i) $-1 < x < 1$. (ii) $-1 < x < 1$. (iii) $-1 < x < 1$.

Ex. 7. (i) $x + \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots\cdots$ (*i.e.*, **sinh x**).

(ii) $1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots\cdots$ (*i.e.*, **cosh x**).

As in the case of sine and cosine series the above two series are absolutely convergent for all values of x.

Ex. 8. $\frac{x-3}{3} + \frac{1}{2}\frac{(x-3)^2}{3^2} + \frac{1}{3}\frac{(x-3)^3}{3^3} + \cdots\cdots$

$$\underset{n\to\infty}{Lt}\left|\frac{u_{n+1}}{u_n}\right| = Lt\left|\frac{(x-3)^{n+1}}{(n+1)\,3^{n+1}} \div \frac{(x-3)^n}{n3^n}\right| = \left|\frac{x-3}{3}\right| Lt\,\frac{n}{n+1}$$

$$= \left|\frac{x-3}{3}\right|.$$

Thus, the series is convergent if $\left|\frac{x-3}{3}\right| < 1$,

i.e., for $-1 < \frac{x-3}{3} < 1$,

i.e., $-3 < x-3 < 3$ or $0 < x < 6$.

The series is also convergent for $x-3=-3$, *i.e.*, $x=0$ but not for $x-3=3$, *i.e.*, for $x=6$.

Ex. 9. $1!\,x + 2!\,x^2 + 3!\,x^3 + \cdots + n!\,x^n + \cdots$

Here $\underset{n\to\infty}{Lt}\left|\frac{u_{n+1}}{u_n}\right| = Lt\left|\frac{(n+1)!\,x^{n+1}}{n!\,x^n}\right| = Lt\,(n+1).|x|$

$$= 0, \text{ if } x=0, \text{ and } \to \infty \text{ if } x \neq 0.$$

$\therefore$ the series is convergent only for $x=0$.

4. Properties of power series.

We give below certain statements (without proof) regarding the properties of infinite power series, which are often used in obtaining many new series.

(i) *Two power series, as long as they converge, may be added and subtracted term by term.*

Thus, if $f(x)=\Sigma a_n x^n$, $\phi(x)=\Sigma b_n x^n$, $f(x)\pm\phi(x)=\Sigma a_n x^n \pm \Sigma b_n x^n$
$$=\Sigma(a_n \pm b_n)\, x^n.$$

(ii) *Two power series in x can be multiplied out term by term for values of x in the interior of the intervals of convergence of both the series.*

Thus, if $f(x)=\Sigma a_n x^n$, $\phi(x)=\Sigma b_n x^n$, then $f(x)\,.\,\phi(x)=\Sigma a_n x^n\,.\,\Sigma b_n x^n$ $=\Sigma(a_0 b_n + a_1 b_{n-1} + \cdots\cdots + a_n b_0)\, x^n$, provided *$x$ is interior to the intervals of convergence of both the series.* The product series is absolutely convergent in the same range (*i.e.*, in the common part of the intervals of convergence).

Cor. Every power series may be multiplied (any number of times) by itself, within its interval of convergence. Thus, if $f(x)=\Sigma a_n x^n$, $\{f(x)\}^2=(\Sigma a_n x^n)^2=(\Sigma a_n x^n).(\Sigma a_n x^n)=\Sigma(a_0 a_n + a_1 a_{n-1} + \cdots + a_n a_0)x^n$; similarly for $(\Sigma a_n x^n)^3$, etc.

(iii) *The quotient of two power series $\Sigma a_n x^n$, $\Sigma b_n x^n$ ($b_0 \neq 0$) is another power series $\Sigma c_n x^n$, provided x remains within a sufficiently small interval in which the denominator does not vanish and both numerator and denominator are convergent series.*

Thus, $$\frac{a_0 + a_1 x + a_2 x^2 + \cdots}{b_0 + b_1 x + b_2 x^2 + \cdots} = c_0 + c_1 x + c_2 x^2 + \cdots$$

Since $\Sigma a_n x^n = \Sigma b_n x^n . \Sigma c_n x^n$,

$\therefore$ $a_0 = b_0 c_0$, $a_1 = c_0 b_1 + c_1 b_0$, etc.

from which $c_0, c_1, c_2, \ldots\ldots$ can be easily calculated.

As a particular case, we have under the same conditions

$$\frac{1}{b_0 + b_1 x + b_2 x^2 + \cdots} = c_0 + c_1 x + c_2 x^2 + \cdots \quad (b_0 \neq 0)$$

the coefficients $c_0, c_1, \ldots\ldots$, being calculated as before.

Note 1. For the expansion to be valid it is necessary that

$$\lambda = \left|\frac{b_1}{b_0} x\right| + \left|\frac{b_2}{b_0} x^2\right| + \cdots\cdots < 1,$$

whence the required condition follows.

Note 2. The coefficients $c_0, c_1, c_2, \ldots\ldots$ are the same as those found by long division.

Note 3. If some of the initial terms in the denominator happen to be zero, the quotient may still be found as a power series *together with a rational fraction.*

Thus, suppose $b_0=0$, $b_1=0$ but $b_2 \neq 0$; then we write

$$\frac{\Sigma a_n x^n}{\Sigma b_n x^n}=\frac{\Sigma a_n x^n}{b_2 x^2 (1+B_1 x+B_2 x^2+\cdots)}$$

where $B_1=b_3/b_2$, $B_2=b_4/b_2$, etc.

(iv) *Limits, term by term, are permissible in case of power series within its interval of convergence.*

Thus, if $f(x)=\Sigma a_n x^n$, $\underset{x\to a}{Lt} f(x)=\underset{x\to a}{Lt} \Sigma a_n x^n$.

Note. Strictly speaking, it involves an inversion of limiting process. The above relation really means

$$\underset{x\to a}{Lt}\left[\underset{n\to\infty}{Lt} S_n\right]=\underset{n\to\infty}{Lt}\ \underset{x\to a}{Lt} S_n.$$

Although the limit of the sum of *a finite* number of terms is *always* equal to the sum of the limit of each term, in case the number of terms is *infinite*, the equality holds *only under certain condition.*

(v) *A function represented by a power series viz., $f(x)=\Sigma a_n x^n$ is continuous and differentiable (any number of times) at every point x_1 interior to the interval of convergence, and its differential coefficient may be obtained by term by term differentiation.*

Thus, we have, $f'(x_1)=\Sigma n a_n x_1^{n-1}$; $f''(x_1)=\Sigma n(n-1)\ a_n x_1^{n-2}$; etc.

Note. By taking the nth derivative of $f(x)=\Sigma a_n x^n$ and putting $x=0$, we obtain $a_n=\frac{1}{n!}f^n(0)$. Hence generally a power series in its interval of convergence is the Maclaurin's series of the function which it represents.

(vi) *The integral of a function $f(x)$ represented by the power series $\Sigma a_n x^n$, over every interval inside the interval of convergence, can be obtained by term by term integration.*

Thus, $\int_{x_1}^{x_2} f(x)\ dx = \int_{x_1}^{x_2} \Sigma a_n x^n\ dx = \Sigma \frac{a_n}{n+1}(x_2^{n+1} - x_1^{n+1})$
provided x_1 and x_2 are both interior to the interval of convergence.

Note 1. The point x_2 may be taken at the boundary of the interval of convergence, provided that the integrated series converges there, whether the original series does so or not.

Note 2. The series obtained by the derivation or integration of a power series term by term, has the same radius of convergence as of the original series.

(vii) *If two power series $\Sigma a_n x^n$ and $\Sigma b_n x^n$ both converge in the same interval $(-r, r)$ and both represent the same function $f(x)$, then they are identical, i.e., $a_n = b_n$ is true for all values of n.*

Note. A function $f(x)$ can be represented by a power series in only one way, if at all.

(viii) If $y = f(x) = \Sigma a_n x^n$, and $F(y) = \Sigma b_n y^n$, then

$$\begin{aligned} F\{f(x)\} &= b_0 + b_1 (\Sigma a_n x^n) + b_2 (\Sigma a_n x^n)^2 + \cdots\cdots \\ &= b_0 + b_1 (a_0 + a_1 x + a_2 x^2 + \cdots) \\ &\qquad + b_2 (a_0 + a_1 x + \cdots)^2 + \cdots \\ &= c_0 + c_1 x + c_2 x^2 + \cdots\cdots \end{aligned}$$

for every value of x for which $\Sigma |a_n x^n|$ converges and has a sum less than the radius of convergence of $\Sigma b_n y^n$.

Note. As a particular case, if $\Sigma b_n y^n$ is convergent for all values of y, then the theorem holds for every x for which $\Sigma a_n x^n$ converges absolutely.

Thus, in the series $\Sigma \frac{y^n}{n!}$, we may substitute $y = \Sigma x^n$ for $|x| < 1$ or $y = \Sigma \frac{x^n}{n!}$ for every x, and then re-arrange in powers of x.

This is the case of the substitution of a power series for another power series.

SECTION D

PROOFS OF SOME IMPORTANT THEOREMS AND RESULTS ON LIMITS, CONTINUITY, ETC.

1. Proof of Fundamental Theorems on Limits *(Art. 2·8).*

We are given $\underset{x \to a}{Lt} f(x) = l$, and $\underset{x \to a}{Lt} \phi(x) = l'$, where l and l' are finite quantities.

(i) Since $\underset{x \to a}{Lt} f(x) = l$, $\underset{x \to a}{Lt} \phi(x) = l'$, we can, when *any* positive number ε is given, choose positive numbers δ_1, δ_2, such that

$$|f(x) - l| < \tfrac{1}{2}\varepsilon \text{ when } 0 < |x - a| \leqslant \delta_1 \quad \cdots \quad (1)$$

$$|\phi(x) - l'| < \tfrac{1}{2}\varepsilon \text{ when } 0 < |x - a| \leqslant \delta_2. \quad \cdots \quad (2)$$

Let δ be any positive number which is smaller than both δ_1 and δ_2; then the inequalities (1) and (2) both hold good when $0 < |x - a| \leqslant \delta$.

Now $|\{f(x) - l\} + \{\phi(x) - l'\}| \leqslant |f(x) - l| + |\phi(x) - l'|$

$\therefore \quad |\{f(x) + \phi(x)\} - (l + l')| < \tfrac{1}{2}\varepsilon + \tfrac{1}{2}\varepsilon$ *i.e.*, $< \varepsilon$,

when $0 < |x - a| \leqslant \delta$.

$\therefore$ by definition, $l + l'$ is the limit of $\{f(x) + \phi(x)\}$ as $x \to a$.

Similarly, it can be shown that $l - l'$ is the limit of $\{f(x) - \phi(x)\}$ as $x \to a$.

Hence $\underset{x \to a}{\mathbf{Lt}} \{\mathbf{f(x)} \pm \boldsymbol{\phi}\mathbf{(x)}\} = \boldsymbol{l \pm l'}$.

(ii) We have, $f(x)\,\phi(x) - ll' = \{f(x) - l\}\{\phi(x) - l'\}$

$$+\, l'\{f(x) - l\} + l\{\phi(x) - l'\}.$$

$$\therefore \ |f(x)\phi(x)-ll'| \leqslant |f(x)-l|\,|\phi(x)-l'| + |l'|\,|f(x)-l| + |l|\,|\phi(x)-l'|$$

Now ε being any pre-assigned positive quantity, and choosing any other positive quantity k,

$$k, \frac{\varepsilon}{3|l|}, \frac{\varepsilon}{3|l'|}, \frac{\varepsilon}{3k}$$

are known positive quantities, and since $\underset{x\to a}{Lt}\ f(x)=l$, $\underset{x\to a}{Lt}\ \phi(x)=l'$ we can choose positive numbers $\delta_1, \delta_2, \delta_3, \delta_4$, such that

$$|f(x)-l| < k \text{ when } 0<|x-a|\leqslant \delta_1,$$

$$|\phi(x)-l'| < \frac{\varepsilon}{3k} \text{ when } 0<|x-a|\leqslant \delta_2,$$

$$|f(x)-l| < \frac{\varepsilon}{3|l'|} \text{ when } 0<|x-a|\leqslant \delta_3,$$

$$\text{and } |\phi(x)-l'| < \frac{\varepsilon}{3|l|} \text{ when } 0<|x-a|\leqslant \delta_4.$$

Hence, if δ be the least of the positive numbers $\delta_1, \delta_2, \delta_3, \delta_4$, all the above four inequalities hold when

$$0<|x-a|\leqslant \delta,$$

and so from (1),

$$|f(x)\phi(x)-ll'| < k\cdot\frac{\varepsilon}{3k} + |l'|\cdot\frac{\varepsilon}{3|l'|} + |l|\cdot\frac{\varepsilon}{3|l|}$$

i.e., $< \frac{\varepsilon}{3}+\frac{\varepsilon}{3}+\frac{\varepsilon}{3}$ *i.e.*, $<\varepsilon$ when $0<|x-a|\leqslant\delta$.

$\therefore$ by definition, $\underset{x\to a}{Lt}\ f(x)\,\phi(x) = ll'$.

(iii) We have $\left|\dfrac{f(x)}{\phi(x)}-\dfrac{l}{l'}\right| = \left|\dfrac{l'\{f(x)-l\}-l\{\phi(x)-l'\}}{l'\,\phi(x)}\right|$

$$\leqslant \frac{|l'|\,|f(x)-l| + |l|\,|\phi(x)-l'|}{|l'|\,|\phi(x)|} \qquad \dots \quad (1)$$

Now since $\underset{x \to a}{Lt}\ \phi(x) = l'$, there exists a positive number δ_1 such that $| \phi(x) - l' | < \frac{1}{2} | l' |$ when $0 < | x - a | \leqslant \delta_1$, for $l' \neq 0$.

$$\therefore \quad | l' | - | \phi(x) | \leqslant | \phi(x) - l' | < \tfrac{1}{2} | l' |$$

or, $| \phi(x) | > \frac{1}{2} | l' |$ when $0 < | x - a | \leqslant \delta_1$. ⋯ (2)

Also, there exist positive numbers δ_2, δ_3, such that

$$| f(x) - l | < \varepsilon' \text{ for } 0 < | x - a | \leqslant \delta_2,$$

$$| \phi(x) - l' | < \varepsilon' \text{ for } 0 < | x - a | \leqslant \delta_3 \qquad \cdots \quad (3)$$

where ε' is any chosen positive number.

If δ be the smallest of δ_1, δ_2, δ_3, then it follows from (1), (2), (3) that when $0 < | x - a | \leqslant \delta$,

$$\left| \frac{f(x)}{\phi(x)} - \frac{l}{l'} \right| < \frac{\{ | l' | + | l | \} \varepsilon'}{\frac{1}{2}\{ | l' | \}^2}.$$

Now ε being any pre-assigned positive number, if we choose $\varepsilon' = \frac{1}{2}\varepsilon \{ | l' | \}^2 / \{ | l | + | l' | \}$, we get

$$\left| \frac{f(x)}{\phi(x)} - \frac{l}{l'} \right| < \varepsilon \text{ when } 0 < | x - a | \leqslant \delta.$$

Hence $\quad \underset{x \to a}{\mathbf{Lt}}\ \dfrac{\mathbf{f(x)}}{\boldsymbol{\phi}\mathbf{(x)}} = \dfrac{\boldsymbol{l}}{\boldsymbol{l'}}$.

(iv) Let $u = f(x)$; since $F(u)$ is continuous for $u = l$,

$$| F(u) - F(l) | < \varepsilon \text{ when } | u - l | \leqslant \delta_1$$

i.e., when $| f(x) - l | \leqslant \delta_1$. ⋯ (1)

Again, since $f(x) \to l$ as $x \to a$,

$$| f(x) - l | < \delta_1 \text{ when } 0 < | x - a | \leqslant \delta. \qquad \cdots \quad (2)$$

Combining (1) and (2),

$$| F\{f(x)\} - F(l) | < \varepsilon \text{ when } 0 < | x - a | \leqslant \delta$$

i.e., $\quad \underset{x \to a}{\mathbf{Lt}}\ \mathbf{F\{f(x)\} = F(}\boldsymbol{l}\mathbf{)}$.

(v) Assume that the inequalities $\phi(x) < f(x) < \psi(x)$ are satisfied when $0 < |x-a| < \delta_1$.

Since $\underset{x \to a}{Lt}\, \phi(x) = l$, $|\phi(x) - l| < \varepsilon$ when $0 < |x-a| \leqslant \delta_2$

i.e., $l - \varepsilon < \phi(x) < l + \varepsilon$ when $0 < |x-a| \leqslant \delta_2$.

Similarly, $l - \varepsilon < \psi(x) < l + \varepsilon$ when $0 < |x-a| \leqslant \delta_3$.

If δ be the smallest of the numbers $\delta_1, \delta_2, \delta_3$, then all the above inequalities are satisfied when $0 < |x-a| \leqslant \delta$. Under these conditions $l - \varepsilon < \phi(x) < f(x)$.

Also $\quad l + \varepsilon > \psi(x) > f(x), \quad \therefore \quad l - \varepsilon < f(x) < l + \varepsilon$

i.e., $|f(x) - l| < \varepsilon$ when $0 < |x-a| \leqslant \delta$.

$\therefore \; f(x) \to l$ as $x \to a$.

***2.** $\underset{x \to \pm\infty}{\mathbf{Lt}} \left(1 + \dfrac{1}{x}\right)^x = e.$ [*Art.* 2·9(*ii*), (*b*)]

We have already seen [See *Appendix, Sec. A*, § *5(1)*] that $\underset{n \to \infty}{Lt} \left(1 + \dfrac{1}{n}\right)^n = e$ (n a positive integer).

Now, let x be any large positive number. Then we can get two consecutive positive integers n, $n+1$, such that

$$n \leqslant x < n + 1. \quad \therefore \quad 1 + \frac{1}{n} \geqslant 1 + \frac{1}{x} > 1 + \frac{1}{n+1},$$

and each being > 1, and as $n + 1 > x \geqslant n$,

$$\left(1 + \frac{1}{n}\right)^{n+1} > \left(1 + \frac{1}{x}\right)^x > \left(1 + \frac{1}{n+1}\right)^n,$$

or, $\left(1 + \dfrac{1}{n}\right)\left(1 + \dfrac{1}{n}\right)^n > \left(1 + \dfrac{1}{x}\right)^x > \left(1 + \dfrac{1}{n+1}\right)^{n+1} \Big/ \left(1 + \dfrac{1}{n+1}\right).$

Now when $x \to \infty$, $n \to \infty$ also, and n being positive integer, both $\left(1 + \dfrac{1}{n}\right)^n$ and $\left(1 + \dfrac{1}{n+1}\right)^{n+1} \to e$ as proved before.

Also $1+\frac{1}{n} \to 1$ and $1+\frac{1}{n+1} \to 1$. Hence the two extremes in the above inequality tend to a common limit e, and $\therefore \left(1+\frac{1}{x}\right)^x \to e$.

Lastly, suppose $x=-p$, where p is a large positive number; then as $p \to \infty$, $x \to -\infty$.

Then, $\left(1+\frac{1}{x}\right)^x = \left(1-\frac{1}{p}\right)^{-p} = \left(\frac{p}{p-1}\right)^p = \left(1+\frac{1}{p-1}\right)^p$

$$= \left(1+\frac{1}{q}\right)^{q+1} \text{ (where } q=p-1 \text{)}$$

Now, if $p \to \infty$, $q \to \infty$ and hence

$$\left(1+\frac{1}{q}\right)^{q+1} = \left(1+\frac{1}{q}\right)^q \cdot \left(1+\frac{1}{q}\right) \to e.$$

Thus, $\left(1+\frac{1}{x}\right)^x \to e$ as $x \to -\infty$.

Hence, we see that $\underset{x\to\pm\infty}{Lt} \left(1+\frac{1}{x}\right)^x = e$,

x being not confined to be integral here.

Cor. Replacing y by $\frac{1}{x}$, as $x \to \pm\infty$, $y \to 0$, and we get

$$\underset{y\to 0}{Lt} (1+y)^{\frac{1}{y}} = e, \quad \text{or} \quad \underset{x\to 0}{\mathbf{Lt}} (1+x)^{\frac{1}{x}} = e.$$

3. Cauchy's necessary and sufficient condition for the existence of a limit. [*Art.* 2·10]

To prove that the condition is necessary. Let $\underset{x\to a}{Lt} f(x)$ exist, and be finite, and $= l$ (say).

Then given any pre-assigned positive number ε, we can find a positive number δ, such that

$$|f(x)-l|<\tfrac{1}{2}\varepsilon$$

when $0<|x-a|\leqslant\delta$. If now x_1 and x_2 be any two quantities satisfying $0<|x-a|\leqslant\delta$, then

$$\begin{aligned}|f(x_1)-f(x_2)| &= |\{f(x_1)-l\}-\{f(x_2)-l\}| \\ &\leqslant |f(x_1)-l|+|f(x_2)-l| \\ &< \tfrac{1}{2}\varepsilon+\tfrac{1}{2}\varepsilon,\ i.e., <\varepsilon.\end{aligned}$$

Hence the condition is necessary.

To prove that the condition is sufficient is beyond the scope of the present treatise.

4. An important theorem on limit.

Theorem : *If* $\underset{x\to a}{Lt}\ \phi(x)=l_1$ *and* $\underset{x\to a}{Lt}\ \psi(x)=l_2$ *and if* $\phi(x)<\psi(x)$ *in a certain neighbourhood of a except a, then* $l_1\leqslant l_2$.

Let us suppose that the inequality $\phi(x)<\psi(x)$ holds good when $0<|x-a|<\delta_1$, *i.e.*, in the neighbourhood

$$a-\delta_1<x<a+\delta_1,\ x\neq a. \qquad \cdots \quad (1)$$

If possible, suppose $l_1>l_2$.

Let us choose $\varepsilon=\tfrac{1}{2}(l_1-l_2)$, a positive number.

Since $\underset{x\to a}{Lt}\ \phi(x)=l_1$,

$\therefore \quad |\phi(x)-l_1|<\varepsilon$, when $0<|x-a|\leqslant\delta_2$

i.e., $\quad l_1-\varepsilon<\phi(x)<l_1+\varepsilon$, when $a-\delta_2\leqslant x\leqslant a+\delta_2$.

$\therefore \quad l_1-\tfrac{1}{2}(l_1-l_2)<\phi(x)$, when $a-\delta_2\leqslant x\leqslant a+\delta_2$

i.e., $\quad \tfrac{1}{2}(l_1+l_2)<\phi(x). \qquad \cdots \qquad \cdots \quad (2)$

Again, since $\underset{x\to a}{Lt}\ \psi(x)=l_2$,

$\therefore \quad l_2 - \varepsilon < \psi(x) < l_2 + \varepsilon$, when $a - \delta_3 \leqslant x \leqslant a + \delta_3$.

$\therefore \quad \psi(x) < l_2 + \frac{1}{2}(l_1 - l_2)$

i.e., $\psi(x) < \frac{1}{2}(l_1 + l_2)$, when $a - \delta_3 \leqslant x \leqslant a + \delta_3$. $\cdots$ (3)

Let δ be the smallest of $\delta_1, \delta_2, \delta_3$; then the inequalities (1), (2), (3) hold good in the interval $a - \delta \leqslant x \leqslant a + \delta$.

$\therefore$ from (2) and (3), $\phi(x) > \frac{1}{2}(l_1 + l_2) > \psi(x)$,

i.e., $\phi(x) > \psi(x)$ in $a - \delta \leqslant x \leqslant a + \delta$

which contradicts our hypothesis that $\phi(x) < \psi(x)$ in that interval. Hence our assumption $l_1 > l_2$ is incorrect.

$\therefore \quad l_1 \not> l_2$ *i.e.*, $l_1 \leqslant l_2$.

***5. Proofs of some properties of continuous functions.** [*Art. 3·4*]

(iv) Since $f(x)$ is continuous at $x = a$, from definition, if ε be any chosen positive number, we can get a positive quantity δ, such that

$$|f(x) - f(a)| < \varepsilon \text{ i.e., } f(a) - \varepsilon < f(x) < f(a) + \varepsilon \quad \cdots \quad (1)$$

for all values of x satisfying $a - \delta < x < a + \delta$.

As $f(a) \neq 0$ here, if $f(a)$ be positive, choose $\varepsilon = \frac{1}{2} f(a)$; then from (1), $f(x) > f(a) - \varepsilon$ *i e.*, $> \frac{1}{2} f(a)$, and is accordingly positive when $a - \delta < x < a + \delta$.

If $f(a)$ be negative, choose $\varepsilon = -\frac{1}{2} f(a)$, and then we have from (1), $f(x) < f(a) + \varepsilon$ *i.e.*, $< f(a) - \frac{1}{2} f(a)$, *i.e.*, $< \frac{1}{2} f(a)$ and is accordingly negative when $a - \delta < x < a + \delta$.

Thus whatever be the sign of $f(a)$, we can find δ such that $f(x)$ has the same sign as that of $f(a)$ in the range $a - \delta < x < a + \delta$.

(v) Let $OA = a$, $OB = b$. Bisect the interval AB at C_1. If $f(x)$ be not zero at C_1, it must be opposite in sign to one of $f(a)$ and $f(b)$ which are given to be of opposite signs. Suppose $f(x)$ has opposite signs at C_1 and B. Bisect C_1B at C_2. If $f(x)$ be not zero at C_2, it must have opposite signs at the extremities of one of the intervals C_1C_2 or C_2B. Bisect that particular interval at C_3. Proceeding in this manner n times, unless $f(x)$ is zero at one of these points of bisection, we can get an interval C_{n-1}, C_n (say) within AB, at the opposite extremities of which $f(x)$ will have opposite signs. This interval C_{n-1}, C_n is clearly $1/2^n$ of the interval AB, *i.e.*, $=(b-a)/2^n$, and taking n large enough, can be made as small as we like.

But $f(x)$ being a continuous function for every value of x within the interval AB, corresponding to any point C_n in it, given by $x=c$ say, if $f(c)$ be not zero, it must be possible [*by (iv) above*] to get a positive quantity δ such that $f(x)$ will retain the same sign, namely that of $f(c)$, in the interval $(c-\delta,\ c+\delta)$. Now whatever δ we may choose, $(b-a)/2^n$ can be made less than δ by taking n large enough, and it has been shown that at the extremities of the interval C_{n-1}, C_n, which falls within $(c-\delta,\ c+\delta)$, $f(x)$ has got opposite signs. We are thus led to a contradiction if $f(x)$ is not zero anywhere within the interval AB. Hence there must be some point in the interval, given by $x=\xi$ (say) where $f(\xi)=0$ under the circumstances.

(vi) Let k *be any* quantity intermediate between $f(a)$ and $f(b)$ which are given to be unequal. Let $\phi(x)=f(x)-k$. Then since $f(x)$ is continuous in the interval (a, b), $\phi(x)$ is also continuous. Also $\phi(a)=f(a)-k$ and $\phi(b)=f(b)-k$ are of opposite signs, since k lies between $f(a)$ and $f(b)$. Hence by (v) above, there is a value $x=\xi$ in the interval, for which $\phi(\xi)=0$, *i.e.*, $f(\xi)=k$. In other words, $f(x)$ assumes the value k at some point in the interval.

(vii) Let the function $f(x)$ be continuous throughout the closed interval (a, b). Let us divide all the real numbers in the interval into two classes L, R, putting a number ξ in L if $f(x)$ is bounded in (a, ξ), and in R otherwise. Members of L-class exist in this case, since $f(x)$ being continuous at a (to the right), corresponding to any pre-assigned positive number ε, we can get a positive number δ such that $f(a)-\varepsilon < f(x) < f(a)+\varepsilon$ (and accordingly $f(x)$ is bounded) in the interval $(a, a+\delta)$, so that $a+\delta$ belongs to L-class. If now numbers of R-class also exist in the interval, then by Dedekind's theorem, there exists a definite number c (say) in the interval, which represents the section. [To include all real numbers in the classification, we put all numbers less than a in L, and all numbers greater than b in R here.]

Now since $f(x)$ is continuous at c, for any given positive quantity ε, we can determine a positive number δ such that $f(c)-\varepsilon < f(x) < f(c)+\varepsilon$ within the interval $(c-\delta, c+\delta)$, *i.e.*, $f(x)$ is bounded therein. Also $c-\delta$ belonging to L-class, $f(x)$ is bounded in $(a, c-\delta)$. Hence $f(x)$ is bounded throughout the interval $(a, c+\delta)$. But $c+\delta$ belonging to R-class, $f(x)$ is not bounded in $(a, c+\delta)$. This contradiction shows that no number of the R-class can exist in the interval (a, b); in other words, $f(x)$ is bounded throughout the interval (a, b).

(viii) and (ix) Let $f(x)$ be continuous in the closed interval (a, b), and let M and m be its upper and lower bounds in the interval. If possible, let there be no point in the interval where $f(x)=M$. Then $M-f(x)>0$ for all points in the interval. Now, since $f(x)$ is continuous, $M-f(x)$ is also continuous, and so $1/\{M-f(x)\}$ is also continuous in the interval. Thus $1/\{M-f(x)\}$ is bounded in the interval, *i.e.*, $1/\{M-f(x)\} \leqslant k$, where k is a fixed positive number.

$$\therefore \quad M-f(x) \geqslant \frac{1}{k}, \text{ or, } f(x) \leqslant M-\frac{1}{k}.$$

This contradicts the assumption that M is the upper bound of $f(x)$ in the interval.

Hence $f(x)$ must assume the value M at some point in the interval.

Similarly it may be proved that $f(x)$ assumes the value m also in the interval.

It now follows from (vi) that $f(x)$ assumes every intermediate value between M and m.

*6. Proof of the equality $\dfrac{\partial^2 f}{\partial x\, \partial y}=\dfrac{\partial^2 f}{\partial y\, \partial x}$ [*Art. 9·3*]

If f_x, f_y, f_{xy}, f_{yx} all exist, and f_{yx} (or f_{xy}) *is continuous, then $f_{xy}=f_{yx}$.*

Proof : Let $\phi(x)=f(x, y+k)-f(x, y)$. $\quad\cdots\quad$ (1)

Now applying Mean Value Theorem to $\phi(x)$, we get

$$\begin{aligned}\phi(x+h)-\phi(x) &= h\phi_x(x+\theta h), \qquad 0<\theta<1,\\ &= h[f_x(x+\theta h, y+k)-f_x(x+\theta h, y)]\\ &= h[F(y+k)-F(y)] \text{ say}\\ &\qquad [\text{ where } F(y)=f_x(x+\theta h, y)\ ]\\ &= h[kF_y(y+\theta' k)], \qquad 0<\theta'<1,\\ &\qquad \text{by Mean Value Theorem}\\ &= hk[f_{yx}(x+\theta h, y+\theta' k)]. \qquad\cdots\quad (2)\end{aligned}$$

Again from (1), $\phi(x+h)=f(x+h, y+k)-f(x+h, y)$.

$$\therefore \quad \phi(x+h)-\phi(x)=f(x+h, y+k)-f(x+h, y) -f(x, y+k)+f(x, y) \qquad \cdots \quad (3)$$

Now, $$f_y(x, y)=\underset{k\to 0}{Lt} \frac{f(x, y+k)-f(x, y)}{k}$$

and $$f_{xy}(x, y)=\underset{h\to 0}{Lt} \frac{f_y(x+h, y)-f_y(x, y)}{h}$$

$$=\underset{h\to 0}{Lt}\ \underset{k\to 0}{Lt}\left[\frac{f(x+h, y+k)-f(x+h, y)-f(x, y+k)+f(x, y)}{hk}\right]$$

$$=\underset{h\to 0}{Lt}\ \underset{k\to 0}{Lt} \frac{\phi(x+h)-\phi(x)}{hk} \qquad [\textit{from } (3)]$$

$$=\underset{h\to 0}{Lt}\ \underset{k\to 0}{Lt} f_{yx}(x+\theta h, y+\theta' k) \qquad [\textit{from } (2)]$$

$=f_{yx}(x, y)$, since f_{yx} is continuous

Illus.: *If* $f(x, y)=xy\dfrac{x^2-y^2}{x^2+y^2}$, *when* $x\neq 0$, *or* $y \neq 0$,

$=0$, *when* $x=0$, $y=0$,

show that at $(0, 0)$, $\dfrac{\delta^2 f}{\delta x\, \delta y} \neq \dfrac{\delta^2 f}{\delta y\, \delta x}$, *i.e.*, $f_{xy}(0, 0) \neq f_{yx}(0, 0)$.

When $x \neq 0$, $y \neq 0$

$$f_x(x, y)=y\left\{\frac{x^2-y^2}{x^2+y^2}+x\frac{(x^2+y^2)\,2x-(x^2-y^2)\,2x}{(x^2+y^2)^2}\right\}$$

$$=y\left\{\frac{x^2-y^2}{x^2+y^2}+\frac{4x^2y^2}{(x^2+y^2)^2}\right\}. \qquad \cdots \quad (1)$$

Similarly, $$f_y(x, y)=x\left\{\frac{x^2-y^2}{x^2+y^2}-\frac{4x^2y^2}{(x^2+y^2)^2}\right\} \qquad \cdots \quad (2)$$

$$f_x(0, 0)=\underset{h\to 0}{Lt}\frac{f(h, 0)-f(0, 0)}{h}=0. \text{ Similarly, } f_y(0, 0)=0.$$

From (1) and (2), we see that

$$f_x(0, y)=-y\ (y\neq 0),\ f_y(x, 0)=x\ (x\neq 0).$$

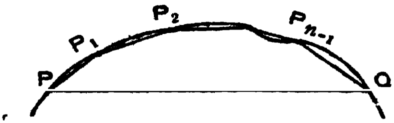
Again, $f_{xy}(0,0)=\underset{h\to 0}{Lt}\dfrac{f_y(h,0)-f_y(0,0)}{h}=\underset{h\to 0}{Lt}\dfrac{h}{h}=1$

$f_{yx}(0,0)=\underset{k\to 0}{Lt}\dfrac{f_x(0,k)-f_x(0,0)}{k}=\underset{k\to 0}{Lt}\dfrac{-k}{k}=-1.$

$\therefore\ f_{xy}(0,0)\neq f_{yx}(0,0).$

***7. On the proof of $\underset{Q\to P}{\text{Lt}}\dfrac{\text{chord } PQ}{\text{arc } PQ}=1.$** [*Art. 10·8*]

Let $PP_1,\ P_1P_2,\ \dots,\ P_{n-1}Q$ be the sides of an open polygon inscribed in arc PQ of the curve $y=f(x)$. If the sum of the n sides ΣPP_1 tends to a definite limit when $n\to\infty$ and the length of each side tends to zero, *that limit is defined as the length of the arc PQ.*



Let $\theta_1, \theta_2, \dots \theta_n$ be the angles which the sides make with the chord PQ, and let $f'(x)$ *be continuous* throughout PQ.

Projecting the sides on PQ, we have

$$PQ=\text{proj. } PP_1+\text{proj } P_1P_2+\cdots+\text{proj. } P_{n-1}Q$$
$$=PP_1\cos\theta_1+P_1P_2\cos\theta_2+\cdots+P_{n-1}Q\cos\theta_n.$$

$\therefore$ it follows that $PQ < PP_1+P_1P_2+\cdots+P_{n-1}Q$

and $>(PP_1+P_1P_2+\cdots+P_{n-1}Q)\cos\theta$

where θ is numerically the greatest of the angles $\theta_1,\ \theta_2,\dots\theta_n$.

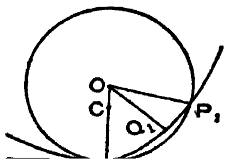
Hence $\cos\theta<\dfrac{PQ}{PP_1+P_1P_2+\cdots P_{n-1}Q}<1.$

Since the chords $PP_1,\ P_1P_2,\ \dots\ P_{n-1}Q$ as well as PQ are parallel to tangent to the arcs at points between their respective extremities (by the Mean Value Theorem), it follows from the continuity of $f'(x)$, that the numerical value of θ can be made as small as we please by taking Q sufficiently near to P, and in the limiting position, $\cos\theta \to 1$ and $\Sigma PP_1 \to$ arc PQ,

$$\therefore \quad \underset{Q\to P}{Lt} \frac{\text{chord } PQ}{\text{arc } PQ} = 1.$$

8. A Theorem on curvature.

If a circle be drawn touching a curve at P and cutting



it at another point P_1, then as $P_1 \to P$, the circle tends to the circle of curvature. (*Art. 11·3*)

Let C be the centre of curvature at P, and let $y = f(x)$ be the equation of the curve, where $f'(x)$ and $f''(x)$ exist.

Let $O(\xi, \eta)$ be the centre of a circle touching the curve at P and cutting it again at P_1, and let r be its radius; also let $Q(x, y)$ be any point on the arc PP_1 of the curve.

$$\therefore \qquad OQ^2 = (\xi - x)^2 + (\eta - y)^2 \equiv F(x).$$

Since $OP^2 = OP_1{}^2 = r^2$, it follows that $F(x)$ has the same value both at P and P_1. Hence by Rolle's Theorem, there exists a point $Q_1(x_1, y_1)$ between P and P_1, such that $F'(x_1) = 0$, *i.e.*, $(\xi - x_1) + (\eta - y_1)\left(\dfrac{dy}{dx}\right)_{x_1} = 0$ which is evidently the condition that OQ_1 is normal to the curve at Q_1. Now let $P_1 \to P$, then Q_1 also $\to P$ and hence by Art. 11·7, O, the point of intersection of the normals at Q_1 and P, tends to C, the centre of curvature and thus r also tends to CP *i.e.*, ρ.

Thus, the circle tends to the circle of curvature.

9. Contact.

1. DEF. Two curves $y=\phi(x)$, $y=\psi(x)$ are said to have *a contact of nth order at* $x=a$, if

$$\phi(a)=\psi(a),\ \phi'(a)=\psi'(a),\ \phi''(a),=\psi''(a),\ \cdots,\ \phi^n(a)=\psi^n(a)$$

and $\phi^{n+1}(a) \neq \psi^{n+1}(a)$.

Illus :

Find the parabola $y=a+bx+cx^2$ *so that it shall have a contact of the second order with the curve* $y=x^4$ *at the point (1, 1).*

For the curve $y=x^4$, values of y, y_1, y_2 at (1, 1) are 1, 4, 12.

For the parabola, $y=a+bx+cx^2$, values of y, y_1, y_2 at (1, 1) are $a+b+c$, $b+2c$ and $2c$. Hence for a second order contact $a+b+c=1$, $b+2c=4$, $2c=12$. $\therefore$ $a=3$, $b=-8$, $c=6$. $\therefore$ the reqd. parabola is $y=3-8x+6x^2$.

Obviously values of y_3 for the two curves at (1, 1) are not the same.

2. The circle having a second order contact with a given curve at a given point is called the **osculating circle.**

To find the equation of the osculating circle of the curve $y=f(x)$ *at* (x, y). ... ... (1)

Let the equation of the osculating circle be

$$(X-\alpha)^2+(Y-\beta)^2=r^2. \qquad \cdots \quad (2)$$

Since (2) has a second order contact with (1) at (x, y), the values of y, y_1, y_2 must be the same for the two curves when $X=x$.

Differentiating (2) twice and putting $X=x$, $Y=y$, we obtain $(x-\alpha)^2+(y-\beta)^2=r^2$, $(x-\alpha)+(y-\beta)y_1=0$, $1+y_1^2+(y-\beta)y_2=0$, where y, y_1, y_2 are equal to $f(x), f'(x), f''(x)$. These equations give

$$\alpha=x-\frac{y_1(1+y_1^2)}{y^2},\ \beta=y+\frac{1+y_1^2}{y_2},\ r=\frac{(1+y_1^2)^{\frac{3}{2}}}{y_2}.$$

Thus the centre and radius of the osculating circle of a curve at a point being the same as those of the circle of curvature at the point, the two circles are the same.

SECTION E

PARTIAL FRACTIONS

We shall not here enter into a detailed discussion of the theory of partial fractions for which the student is referred to treaties on Higher Algebra, but we shall briefly indicate the different methods adopted in breaking up an algebraical fraction into partial fractions according to the nature of the factors of the denominator of the fraction.

Let $\dfrac{f(x)}{\phi(x)}$ be a rational algebraic fraction [*i.e.*, $f(x)$ and $\phi(x)$ are polynomials] in which the *numerator is of lower degree than the denominator*. We know from the Theory of Equations that $\phi(x)$ can always be broken up into real factors which may be linear or quadratic and some of which may be repeated.

CASE I. *When the denominator contains factors, real, linear, but none repeated.*

To each non-repeated linear factor of the denominator such as $x-a$, there corresponds a partial fraction of the form $\dfrac{A}{x-a}$, where A is constant. The given fraction can be expressed as a sum of fractions of this type and the unknown constants A's can be determined easily as shown by the following example.

Ex. 1. *Resolve into partial fractions*

$$\frac{x}{(x-1)(x-2)(x-3)}.$$

Let $$\frac{x}{(x-1)(x-2)(x-3)}=\frac{A}{x-1}+\frac{B}{x-2}+\frac{C}{x-3}.$$

Multiplying both sides by $(x-1)(x-2)(x-3)$, we get

$$x=A\,(x-2)(x-3)+B(x-1)(x-3)+C(x-1)(x-2). \quad \cdots \quad (1)$$

First Method :

Since (1) is an identity, the coefficients of the different powers of x on both sides must be equal.

Equating coefficients of x^2, x and constant terms on both sides, we get

$$A+B+C=0\ ;\ -(5A+4B+3C)=1\ ;\ 6A+3B+2C=0$$

whence we can determine the values of A, B, C.

Otherwise :

Since (1) is an identity, it must be true for all values of x. Hence putting $x=1, 2, 3$ respectively on both sides, we get

$$1=2A,\ 2=-B,\ 3=2C,$$

$\therefore\ A=\frac{1}{2},\ B=-2,\ C=\frac{3}{2}.$

$\therefore$ the given expression

$$=\frac{1}{2}\cdot\frac{1}{x-1}-2\cdot\frac{1}{x-2}+\frac{3}{2}\cdot\frac{1}{x-3}.$$

CASE II. *When the denominator contains factors, real, linear, but some repeated.*

To each p-fold linear factor such as $(x-a)^p$ there will correspond the sum of p partial fractions of the form

$$\frac{A_p}{(x-a)^p}+\frac{A_{p-1}}{(x-a)^{p-1}}+\cdots\cdots+\frac{A_1}{(x-a)}$$

where the constants $A_p,\ A_{p-1},\ldots\ldots A_1$ can be evaluated as follows.

Ex. 2. *Resolve into partial fractions*

$$\frac{x^2}{(x+1)^2(x+2)}.$$

Let $\dfrac{x^2}{(x+1)^2(x+2)}=\dfrac{A}{(x+1)^2}+\dfrac{B}{(x+1)}+\dfrac{C}{(x+2)}$.

Multiplying both sides by $(x+1)^2(x+2)$, we get

$$x^2=A\ (x+2)+B\ (x+1)(x+2)+C\ (x+1)^2.$$

Putting $x=-1,-2$, successively we get $A=1$, $C=4$, and equating coefficients of x^2 on both sides, we get $B+C=1$, $\therefore\ B=-3$.

$\therefore$ the given expression $=\dfrac{1}{(x+1)^2}-\dfrac{3}{x+1}+\dfrac{4}{x+2}$.

Note. The partial fraction in the above case can also be obtained in the following way.

Denote the first power of the repeated factor *i.e.*, $x+1$ by z; then the fraction $=\dfrac{1}{z^2}\cdot\dfrac{(z-1)^2}{z+1}$. Now divide the numerator by the denominator of the 2nd fraction after writing them in ascending powers of z, till the highest power of the repeated factor *viz.*, z^2 appears in the remainder. Thus the fraction

$$=\frac{1}{z^2}\left(1-3z+\frac{4z^2}{1+z}\right)=\frac{1}{z^2}-\frac{3}{z}+\frac{4}{1+z}.$$

Now replace z by $x+1$ and the required partial fractions are obtained.

CASE III. *When the denominator contains factors, real, quadratic, but none repeated.*

To each non-repeated quadratic factor such as x^2+px+q (or x^2+q), $q\neq 0$, there corresponds a partial fraction of the form $\dfrac{Ax+B}{x^2+px+q}$, where the unknown constants A and B can be determined as follows.

Ex. 3. *Resolve into partial fractions* $\dfrac{x}{(x-1)(x^2+4)}$.

Let $$\frac{x}{(x-1)(x^2+4)}=\frac{A}{x-1}+\frac{Bx+C}{x^2+4}.$$

$\therefore$ $$x=A(x^2+4)+(Bx+C)(x-1).$$

Putting $x=1$, on both sides, we get $A=\frac{1}{5}$.

Equating coefficients of x^2 and x on both sides, we get
$$A+B=0 \text{ and } C-B=1\text{ ; hence } B=-\tfrac{1}{5},\ C=\tfrac{4}{5}.$$

$\therefore$ the given fraction $=\dfrac{1}{5}\cdot\dfrac{1}{x-1}-\dfrac{1}{5}\dfrac{x}{x^2+4}+\dfrac{4}{5}\cdot\dfrac{1}{x^2+4}$.

Ex. 4. *Resolve into partial fractions* $\dfrac{1}{x^3+1}$.

Let $$\frac{1}{x^3+1}=\frac{1}{(x+1)(x^2-x+1)}=\frac{A}{x+1}+\frac{Bx+C}{x^2-x+1}.$$

$\therefore$ $$1=A(x^2-x+1)+(Bx+C)(x+1).$$

Putting $x=-1$, we get $A=\frac{1}{3}$.

Equating coefficients of x^2 and the constant terms, we have
$$A+B=0 \text{ and } A+C=1. \quad \therefore\ B=-\tfrac{1}{3},\ C=\tfrac{2}{3}.$$

$\therefore$ $$\frac{1}{x^3+1}=\frac{1}{3}\cdot\frac{1}{x+1}-\frac{1}{3}\cdot\frac{x-2}{x^2-x+1}.$$

CASE IV. *When the denominator contains factors, real, quadratic, but some repeated.*

To each quadratic factor $(x^2+px+q)^r$, repeated r-times, there will correspond the sum of r partial fractions of the form

$$\frac{L_r x+M_r}{(x^2+px+q)^r}+\frac{L_{r-1}x+M_{r-1}}{(x^2+px+q)^{r-1}}+\cdots\cdots+\frac{L_1x+M_1}{x^2+px+q},$$

where we may proceed to obtain the coefficients L_r, M L_{r-1}, etc. as in the previous examples.

INDEX

UNIVERSITY QUESTIONS

1967 (Pass—C. U.)

1. (*a*) Explain what is meant by the *linear continuum.*

(*b*) Define the *differential coefficient* of a function $f(x)$ at $x=a$.

Prove that if $f(x)$ has a finite and definite differential coefficient at $x=a$, $f(x)$ must be continuous at $x=a$.

(*c*) Explain what is meant by the continuity of $f(x, y)$ at a point (a, b).

2. (*a*) Find from first principles the differential coefficient of $\sin x$.

(*b*) Prove *any two* of the following :

(*i*) $\dfrac{dy}{dx}=\dfrac{\sin^2(a+y)}{\sin a}$, if $\sin y=x\sin(a+y)$.

(*ii*) $\dfrac{dy}{dx}=\frac{3}{2}$, if $y=\sin^{-1}(3X-4X^3)$, $x=\sec^{-1}\dfrac{1}{1-2X^2}$.

(*iii*) $x\dfrac{\partial u}{\partial x}+y\dfrac{\partial u}{\partial y}=0$, if $u=\sin^{-1}\dfrac{x}{y}+\tan^{-1}\dfrac{y}{x}$.

(*iv*) $\dfrac{\partial^2 u}{\partial x\partial y}=\dfrac{\partial^2 u}{\partial y\partial x}$, if $u=x^y$.

3. (*a*) State (without proof) Lagrange's mean value theorem and in the case when the function is representable graphically, give the geometrical interpretation of the theorem.

(*b*) State (without proof) Leibnitz's theorem.

If $y=e^{a\sin^{-1}x}$ show that

$$(1-x^2)\frac{d^{n+2}y}{dx^{n+2}}-(2n+1)x\frac{d^{n+1}y}{dx^{n+1}}-(n^2+a^2)\frac{d^ny}{dx^n}=0.$$

4. (*a*) Show that the angle of intersection of the curves

$$x^2-y^2=a^2$$
$$x^2+y^2=a^2\sqrt{2}$$

is $\pi/4$.

(*b*) Prove that the radius of curvature at the point t on the curve $x=a\cos t$, $y=b\sin t$ is

$$\frac{(a^2\sin^2 t+b^2\cos^2 t)^{\frac{3}{2}}}{ab}.$$

5. (*a*) Find the asymptotes of the curve

$$y^3-x^2y-2xy^2+2x^3-7xy+3y^2+2x^2+2x+2y+1=0.$$

(*b*) A tank, open at the top, has a given volume V, a square base and vertical sides. If the inner surface is the *least possible*, show that the ratio of the depth to the width is 1 : 2.

1968 (Pass—C. U.)

1. (*a*) Define the symbol

$$\lim_{x\to a} f(x)=b \text{ (a finite and definite number)},$$

and apply your definition to show that

$$\lim_{x\to 1} \frac{x^2-1}{x-1}=2.$$

(*b*) Define the continuity of a function $f(x)$ at $x=a$.

If $f(x)=\dfrac{2x^2}{\pi} \lim\limits_{t\to 0} \dfrac{t}{\sin t}$,

show that $f(x)$ is continuous at $x=0$.

2. (*a*) Find from first principles the differential coefficient of $\cot x$.

(*b*) Prove *any two* of the following :

(*i*) $\dfrac{dy}{dx}=\sqrt{a^2-x^2}$, if $y=\dfrac{x\sqrt{a^2-x^2}}{2}+\dfrac{a^2}{2}\sin^{-1}\dfrac{x}{a}$.

(*ii*) $\dfrac{dY}{dX}=1$, if $Y=\tan^{-1}\dfrac{2x}{1-x^2}$, $X=\sin^{-1}\dfrac{2x}{1+x^2}$.

(*iii*) $(1-x^2)\dfrac{d^2y}{dx^2}=x\dfrac{dy}{dx}-m^2y$, if $y=\sin(m\sin^{-1}x)$.

(*iv*) $\dfrac{\partial^2 u}{\partial y\partial x}=\dfrac{\partial^2 u}{\partial x\partial y}$, if $u=x\log y$.

3. (*a*) State Rolle's Theorem.

Deduce the mean value theorem of Lagrange.

(*b*) Find Lagrange's form of the remainder after n terms in the expansion of $\log_e(1+x)$ in Taylor's series $(0<x<1)$ and show that this remainder tends to zero as n tends to infinity.

4. (*a*) Find expressions for the intercepts which the tangent to the curve $y=f(x)$ at the point (x, y) cuts off from the axis of x and from the axis of y.

If for the curve $x=a\cos^3\theta$, $y=b\sin^3\theta$, x_1 and y_1 be the parts of the axes of x and y intercepted by the tangent at any point (x, y) on the curve, show that

$$\frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}=1.$$

(*b*) Prove that the envelope of the system of circles $(x-\alpha)^2+y^2=4\alpha$ is $y^2-4x-4=0$.

5. (*a*) Prove that in the curve

$$x=a\cos^3\theta, \qquad y=a\sin^3\theta$$

the radius of curvature is $3a\sin\theta\cos\theta$.

(*b*) Find the values of x for which $x(1-x^2)$ has a maximum or a minimum.

1969 (Pass—C. U.)

1. (*a*) If $f(x)=\lim\limits_{t\to 0}\left(\dfrac{2x}{\pi}\tan^{-1}\dfrac{2}{t^2}\right)$, find $f(x)$ in the interval $(-1, 1)$.

Examine the continuity and differentiability of $f(x)$ at $x=0$.

(*b*) Find the geometrical interpretation of $\dfrac{dy}{dx}$.

In the curve $y = c\,\dfrac{e^{\frac{x}{c}} + e^{-\frac{x}{c}}}{2}$

if ψ be the angle which the tangent at any point of the curve makes with the axis of x, show that $y = c \sec \psi$.

2. (*a*) Find from first principles the differential coefficient of $\operatorname{cosec} x$.

(*b*) Prove *any two* of the following :

(*i*) $\dfrac{dy}{dx} = -\dfrac{1}{2}$, if $y = \tan^{-1} \dfrac{v}{\sqrt{1-v^2}}$, $x = \sec^{-1} \dfrac{1}{2v^2 - 1}$.

(*ii*) $(1-x^2)\dfrac{d^2y}{dx^2} - x\dfrac{dy}{dx} + m^2 y = 0$, if $y = \sin(m \sin^{-1} x)$.

(*iii*) $\dfrac{dy}{dx} = -\tan t$, if $x = a\cos^3 t$, $y = a \sin^3 t$.

(*iv*) $\dfrac{\partial^2 y}{\partial y \partial x} = \dfrac{\partial^2 y}{\partial x \partial y}$, if $u = x \sin y + y \sin x$.

3. (*a*) State (*without proof*) the mean value theorem of Lagrange.

If $f(b) - f(a) = (b-a)f'(\epsilon)$, $a < \epsilon < b$ and

$f(x) = x(x-1)(x-2)$, $a = 0$, $b = \frac{1}{2}$, show that

$$\epsilon = 1 - \sqrt{\frac{7}{12}}.$$

(*b*) State (*without proof*) Leibnitz's theorem.

Differentiate n times the equation

$$(1-x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} + a^2 y = 0.$$

4. (*a*) If $p = x \cos \alpha + y \sin \alpha$ touch the curve

$$\frac{x^m}{a^m} + \frac{y^m}{b^m} = 1,$$

prove that $p^{\frac{m}{m-1}} = (a \cos \alpha)^{\frac{m}{m-1}} + (b \sin \alpha)^{\frac{m}{m-1}}$.

(*b*) Show that the asymptotes of the curve

$$y^3 - 5xy^2 + 8x^2y - 4x^3 - 3y^2 + 9xy - 6x^2 + 2y - 2x = 1$$

are $y = x$, $y = 2x + 1$, $y = 2x + 2$.

5. (*a*) Prove that the radius of curvature of the cycloid

$$x = a(\theta + \sin\theta),\ y = a(1 - \cos\theta)$$

is $\rho = 4a \cos \dfrac{\theta}{2}$.

(*b*) If $\displaystyle\lim_{x \to 0} \frac{\sin x + a_1 e^x + a_2 e^{-x} + a_3 \log(1+x)}{x^3}$

is finite, find it and determine the values of a_1, a_2, a_3.

1967 (Pass—Burdwan University)

1. (*a*) Explain the meanings of the symbols

$$\underset{x \to c}{\text{Lt}}\ f(x) = l \text{ (where } l \text{ is a definite number) and } f(c),$$

and point out the difference between them.

Evaluate $\underset{x \to 0}{\text{Lt}}\ \phi(x)$, where $\phi(x) = \sin x \cos \dfrac{1}{x}$.

Is $\phi(x)$ continuous at $x=0$? Give reasons for your answer.

(*b*) Find *any one* of the following limits :

$$(i)\ \underset{x \to \infty}{\text{Lt}}\ (\sqrt[3]{(x+1)} - \sqrt[3]{x}), \qquad (ii)\ \underset{x \to \frac{\pi}{4}}{\text{Lt}}\ \frac{1-\tan x}{1-\cot x}.$$

(*c*) Find $\dfrac{dy}{dx}$, where $y = x^{\sin x}$.

2. (*a*) State and prove Leibnitz' theorem on successive differentiation of the product of two functions.

(*b*) If $y = \dfrac{x}{2x^2+3x+1}$, find $y_n(O)$, the symbol having its usual meaning.

3. State and prove Lagrange's First Mean Value theorem of Differential Calculus.

If $f'(x) = 0$ in $1 \leqslant x \leqslant 3$ and if $f(2) = 2$, prove that $f(x) = 2$ throughout $1 \leqslant x \leqslant 3$.

4. (*a*) State and prove Euler's theorem on homogeneous functions.

(*b*) Show that the maximum value of $\left(\dfrac{1}{x}\right)^x$ is $e^{\frac{1}{e}}$.

5. (*a*) Prove that the pedal equation of the curve whose polar equation is $r^2 = a^2 \cos 2\theta$ is $r^3 = a^2 p$.

(*b*) If ρ_1 and ρ_2 be the radii of curvature at the extremities of any chord of the cardioide $r = a\ (1+\cos\theta)$ which passes through the pole, then show that

$$\rho_1^2 + \rho_2^2 = \tfrac{16}{9} a^2.$$

(*c*) Find the asymptotes of the curve

$$x^2(x^2+y^2-2xy) - 2x^2 - 2y^2 = 0,$$

parallel to the line $x = y$.

1968 (Pass—B. U.)

1. (*a*) Explain what is meant by the symbol

$$y=f(x),\ a \leqslant x \leqslant b.$$

Define the continuity and differentiability of $f(x)$ at an interior point in its interval of definition.

(*b*) If the derivative of a function $f(x)$ exists at a point, prove that it is continuous at that point.

(*c*) Find $\dfrac{dy}{dx}$ where $y=\log_{\sin x}(\cot x)$.

2. (*a*) If $y=\sin(m \sin^{-1}x)$, show that

$$(1-x^2)y_2-xy_1+m^2y=0$$

and deduce that

$$(1-x^2)y_{n+2}-(2n+1)xy_{n+1}+(m^2-n^2)y_n=0.$$

(*b*) Establish Maclaurin's series for a function $f(x)$. Apply it to find the series expansion for $\sin x$.

3. (*a*) State and explain Rolle's Theorem.

(*b*) Verify Rolle's Theorem for the function

$$x(x-1)(x-2) \text{ in } [1, 2].$$

(*c*) Show that $x-\sin x$ is steadily increasing in $0 \leqslant x \leqslant \dfrac{\pi}{2}$.

4. (*a*) Evaluate *any one* of the following :

(*i*) $\underset{x\to 0}{\text{Lt}}\ (\cos x)^{\cot^2 x}$; (*ii*) $\underset{x\to \frac{\pi}{2}}{\text{Lt}}\ (1-\sin x)\tan x.$

(*b*) If $u=\dfrac{x}{y}+\dfrac{y}{z}+\dfrac{z}{x}$, show that

$$\frac{\delta u}{\delta x}+y\frac{\delta u}{\delta y}+z\ \frac{\delta u}{\delta z}=0.$$

(*c*) Prove that $x^5-5x^4+5x^3-1$ has a *maximum* for $x=1$ and a *minimum* for $x=3$.

5. (*a*) Show that the curves

$$r=a\,(1+\cos\theta) \text{ and } r=b\,(1-\cos\theta)$$

cut *orthogonally*.

(*b*) Show that the chord of curvature parallel to the axis of y for the curve $y=a\log\sec\left(\dfrac{x}{a}\right)$ is constant.

(*c*) Find the asymptotes of the curve

$$x^2y^2=a^2(x^2+y^2).$$

1969 (Pass—B. U.)

1. (*a*) Explain the meaning of the symbols

$$\underset{x\to a+}{Lt}\ f(x)=l, \text{ and } \underset{x\to a-}{Lt}\ f(x)=m$$

where l and m are two finite numbers. What happens if $l=m$?

(*b*) Evaluate $\underset{x\to 0}{Lt}\ \phi(x)$

where $\phi(x)=\sin x\cos\frac{1}{x}$. Is $\phi(x)$ continuous at $x=0$, if $\phi(0)=1$?

(*c*) Given $y=\sqrt{x}$, find $\frac{dy}{dx}$, ab initio.

2. (*a*) State Leibnitz theorem on successive differentiation. Given $y=e^{ax}\cos(bx+c)$, a, b, c being constants, find y_n.

(*b*) State and prove the First Mean Value Theorem of Lagrange.

3. (*a*) State and prove Euler's theorem regarding homogeneous functions of two variables.

(*b*) If u is a homogeneous function of x and y of degree n, show that

$$x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2}=n(n-1)u.$$

(*c*) Find the extrema of $\sin^2 x$, $0\leqslant x\leqslant\pi$.

4. Prove Taylor's theorem for the expansion of a function with Lagrange's remainder after n terms.

Find Lagrange's remainder for the function e^x in $[0, x]$ and show that it tends to zero as $n\to\infty$.

5. (*a*) Find the radius of curvature at the point $\theta=\frac{\pi}{4}$ on the curve

$$4x=\theta+\sin\theta,\ 4y=1-\cos\theta.$$

(*b*) Find the asymptotes of the curve

$$x^2(x^2+y^2-2xy)-2x^2-2y^2=0$$

which are parallel to the line $x=y$.

1968 (Pass—Gauhati University)

1. (*a*) Explain what is meant by the statement '$f(x)$ tends to l as x tends to a'.

(*b*) Evaluate (*any two*) :—

(i) $\lim\limits_{x\to 0} \dfrac{\sin x}{x}$. (ii) $\lim\limits_{x\to 0} \dfrac{2x^3-2x^2+3x-3}{5x^2-3x-2}$.

(iii) $\lim\limits_{x\to 0} \dfrac{e^{2x}+e^{-2x}-2}{x^2}$.

(*c*) If $\lim\limits_{x\to a} f(x)=l$ and $\lim\limits_{x\to a} \phi(x)=m$,

show that $\lim\limits_{x\to a} \{f(x)+\phi(x)\}=l+m$.

2. (*a*) A function is defined as follows :

$$f(x)=\begin{cases} x & \text{when } x \geqslant 0 \\ -x & \text{when } x<0. \end{cases}$$

Sketch the curve.

Examine whether the function is continuous and differentiable at $x=0$ and at $x=2$.

(*b*) Find $\dfrac{dy}{dx}$, where

(i) $y=\tan^{-1}\dfrac{\sqrt{1+x^2}-1}{x}$. (ii) $x=a\,(t+\sin t)$, $y=a\,(1-\cos t)$.

3. (*a*) If $y=e^{ax}\cos bx$, prove that $y_2-2ay_1+(a^2+b^2)y=0$.

(*b*) State and prove Leibnitz's theorem for the nth derivative of a product of two functions.

(*c*) Find the nth derivative of $e^x \log x$.

4. (*a*) State Maclaurin's theorem, giving a form of the remainder after n terms.

Expand e^x in ascending powers of x, giving the remainder after n terms.

(*b*) If $u=\dfrac{1}{\sqrt{x^2+y^2}}$, find $\dfrac{\partial u}{\partial x}$, $\dfrac{\partial^2 u}{\partial x^2}$ and $\dfrac{\partial^2 u}{dxdy}$.

5. (a) Show that of all rectangles of given area, the square has the smallest perimeter.

(b) Show that the portion of the tangent to the curve

$$x^{2/3}+y^{2/3}=4$$

intercepted between the axes, is of constant length.

(c) Write down the expression for the radius of curvature at a point of a curve whose equation is given by

(i) $y=f(x)$. (ii) $r=f(\theta)$.

1965 (Pass—North Bengal University)

1. (a) Define a rational number and give an illustration of an irrational number.

Explain what are meant by limit and continuity of a function $f(x)$ at $x=a$.

Show that $f(x)=x^2$ is continuous at $x=1$.

(b) Evaluate $\lim\limits_{x\to 0}\left(\dfrac{\sin x}{x}\right)^{\frac{1}{x}}$.

2. (a) State Rolle's theorem and apply it to prove Cauchy's Mean Value Theorem of the Differential Calculus.

(b) If the derivative of $f(x)$ vanishes at every point of an interval, show that $f(x)$ is a constant function of x in that interval.

3. (a) If $y=2\sin x\cos x$, find $\dfrac{d^ny}{dx^n}$.

(b) Find two positive numbers such that their sum is 10 and their product is maximum.

4. (a) Expand $\log_e(1+x)$ in a Taylor's series, when $|x|<1$.

(b) Find the formula for radius of curvature of a plane curve in rectangular coordinates.

5. (a) Prove Euler's theorem that, if U be a rational homogeneous function of degree n in x and y, then

$$x\frac{\delta U}{\delta x}+y\frac{\delta U}{\delta y}=nU.$$

(b). If $U=\sin^{-1}\dfrac{x+y}{\sqrt{x}+\sqrt{y}}$, prove that

$$x\frac{\delta U}{\delta x}+y\frac{\delta U}{\delta y}=\tfrac{1}{2}\tan U.$$