RAMANUJAN

TWELVE LECTURES ON SUBJECTS SUGGESTED BY HIS LIFE AND WORK

BY

G. H. HARDY

Sadleirian Professor of Pure Mathematics in the University of Cambridge

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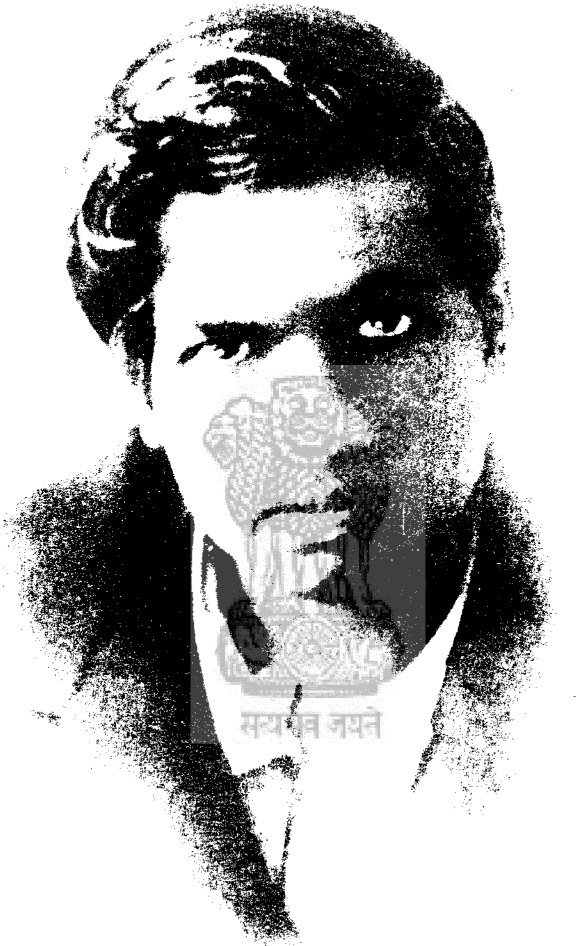

PREFACE

This book is a development of two lectures delivered at the Harvard Tercentenary Conference of Arts and Sciences in the fall of 1936. The first of these was published in Vol. 44 of the *American Mathematical Monthly*, and is reprinted here, as Lecture I, without change. The second has expanded gradually until it fills the rest of the book.

I have given many lectures on Ramanujan's work since 1936, isolated lectures to a number of universities and societies in America and England, and connected courses in Princeton and Cambridge. Lectures II–XII contain most of the substance of these courses, with the rearrangements and additions required to fit them for publication. In this sense they are genuine lectures, and they are written throughout in a lecturer's style.

The contents of the book are described quite accurately by its title. It is not a systematic account of Ramanujan's work (though most of his more important discoveries are mentioned somewhere), but a series of essays suggested by it. In each essay I have taken some part of his work as my text, and have said what occurred to me about its relations to that of earlier and later writers. But even when I digress furthest, when I am writing, for example, about Rademacher's work in Lecture VIII, or about Rankin's in Lecture XI, 'Ramanujan' is the thread which holds the whole together.

Dr R. A. Rankin has read the whole of the book both in manuscript and in proof, and has made a very large number of important suggestions and corrections. I have also to thank Dr W. N. Bailey, who helped me to revise Lecture VII; Mr F. M. Goodspeed, who read and criticised several lectures, and in particular Lecture XI; and Prof. G. N. Watson, without whose aid I could hardly have written Lecture XII. The photograph of Ramanujan was given me by Dr S. Chandrasekhar, formerly Fellow of Trinity College: I regret that I cannot state the name of the actual photographer. A considerable part of the bibliography was compiled for me by Dr V. Levin. But my first thanks are due to Harvard University, to whose invitation the book owes its existence.

G. H. H.

July 1940

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THE INDIAN MATHEMATICIAN RAMANUJAN

I have set myself a task in these lectures which is genuinely difficult and which, if I were determined to begin by making every excuse for failure, I might represent as almost impossible. I have to form myself, as I have never really formed before, and to try to help you to form, some sort of reasoned estimate of the most romantic figure in the recent history of mathematics; a man whose career seems full of paradoxes and contradictions, who defies almost all the canons by which we are accustomed to judge one another, and about whom all of us will probably agree in one judgment only, that he was in some sense a very great mathematician.

The difficulties in judging Ramanujan are obvious and formidable enough. Ramanujan was an Indian, and I suppose that it is always a little difficult for an Englishman and an Indian to understand one another properly. He was, at the best, a half-educated Indian; he never had the advantages, such as they are, of an orthodox Indian training; he never was able to pass the "First Arts Examination" of an Indian university, and never could rise even to be a "Failed B.A." He worked, for most of his life, in practically complete ignorance of modern European mathematics, and died when he was a little over thirty and when his mathematical education had in some ways hardly begun. He published abundantly—his published papers make a volume of nearly 400 pages—but he also left a mass of unpublished work which had never been analysed properly until the last few years. This work includes a great deal that is new, but much more that is rediscovery, and often imperfect rediscovery; and it is sometimes still impossible to distinguish between what he must have rediscovered and what he may somehow have learnt. I cannot imagine anybody saying with any confidence, even now, just how great a mathematician he was and still less how great a mathematician he might have been.

These are genuine difficulties, but I think that we shall find some of them less formidable than they look, and the difficulty which is the greatest for me has nothing to do with the obvious paradoxes of Ramanujan's career. The real difficulty for me is that Ramanujan was, in a way, my discovery. I did not invent him—like other great men, he invented himself—but I was the first really competent person who had the chance to see some of his work, and I can still remember with satisfaction that I could recognise at once what a treasure I had found. And I suppose that I still know more of

Ramanujan than any one else, and am still the first authority on this particular subject. There are other people in England, Professor Watson in particular, and Professor Mordell, who know parts of his work very much better than I do, but neither Watson nor Mordell knew Ramanujan himself as I did. I saw him and talked with him almost every day for several years, and above all I actually collaborated with him. I owe more to him than to anyone else in the world with one exception, and my association with him is the one romantic incident in my life. The difficulty for me then is not that I do not know enough about him, but that I know and feel too much and that I simply cannot be impartial.

I rely, for the facts of Ramanujan's life, on Seshu Aiyar and Ramachaundra Rao, whose memoir of Ramanujan is printed, along with my own, in his *Collected Papers*. He was born in 1887 in a Brahmin family at Erode near Kumbakonam, a fair-sized town in the Tanjore district of the Presidency of Madras. His father was a clerk in a cloth-merchant's office in Kumbakonam, and all his relatives, though of high caste, were very poor.

He was sent at seven to the High School of Kumbakonam, and remained there nine years. His exceptional abilities had begun to show themselves before he was ten, and by the time that he was twelve or thirteen he was recognised as a quite abnormal boy. His biographers tell some curious stories of his early years. They say, for example, that soon after he had begun the study of trigonometry, he discovered for himself "Euler's theorems for the sine and cosine" (by which I understand the relations between the circular and exponential functions), and was very disappointed when he found later, apparently from the second volume of Loney's *Trigonometry*, that they were known already. Until he was sixteen he had never seen a mathematical book of any higher class. Whittaker's *Modern analysis* had not yet spread so far, and Bromwich's *Infinite series* did not exist. There can be no doubt that either of these books would have made a tremendous difference to him if they could have come his way. It was a book of a very different kind, Carr's *Synopsis*, which first aroused Ramanujan's full powers.

Carr's book (*A synopsis of elementary results in pure and applied mathematics*, by George Shoobridge Carr, formerly Scholar of Gonville and Caius College, Cambridge, published in two volumes in 1880 and 1886) is almost unprocurable now. There is a copy in the Cambridge University Library, and there happened to be one in the library of the Government College of Kumbakonam, which was borrowed for Ramanujan by a friend. The book is not in any sense a great one, but Ramanujan has made it famous, and there is no doubt that it influenced him profoundly and that his acquaintance with it marked the real starting-point of his career. Such a book must have had its qualities, and Carr's, if not a book of any high distinction, is no mere third-rate textbook, but a book written with some real scholarship and enthusiasm and with a style and individuality of its own. Carr himself was a private coach in London, who came to Cambridge as an undergraduate when he was nearly forty, and was 12th Senior Optime in the Mathematical Tripos of 1880 (the same year in which he published the first volume of his book). He is now completely forgotten, even in his own college, except in so far as Ramanujan has kept his name alive; but he must have been in some ways rather a remarkable man.

I suppose that the book is substantially a summary of Carr's coaching notes. If you were a pupil of Carr, you worked through the appropriate sections of the *Synopsis*. It covers roughly the subjects of Schedule A of the present Tripos (as these subjects were understood in Cambridge in 1880), and is effectively the "synopsis" it professes to be. It contains the enunciations of 6165 theorems, systematically and quite scientifically arranged, with proofs which are often little more than cross-references and are decidedly the least interesting part of the book. All this is exaggerated in Ramanujan's famous notebooks (which contain practically no proofs at all), and any student of the notebooks can see that Ramanujan's ideal of presentation had been copied from Carr's.

Carr has sections on the obvious subjects, algebra, trigonometry, calculus and analytical geometry, but some sections are developed disproportionally, and particularly the formal side of the integral calculus. This seems to have been Carr's pet subject, and the treatment of it is very full and in its way definitely good. There is no theory of functions; and I very much doubt whether Ramanujan, to the end of his life, ever understood at all clearly what an analytic function is. What is more surprising, in view of Carr's own tastes and Ramanujan's later work, is that there is nothing about elliptic functions. However Ramanujan may have acquired his very peculiar knowledge of this theory, it was not from Carr.

On the whole, considered as an inspiration for a boy of such abnormal gifts, Carr was not too bad, and Ramanujan responded amazingly.

> Through the new world thus opened to him (say his Indian biographers),[1] Ramanujan went ranging with delight. It was this book which awakened his genius. He set himself to establish the formulae given therein. As he was without the aid of other books, each solution was a piece of research so far as he was concerned....Ramanujan used to say that the goddess of Namakkal inspired him with the formulae in dreams. It is a remarkable fact that frequently, on rising from bed, he would note down results and rapidly verify them, though he was not always able to supply a rigorous proof....

[1] Quotations (except those from my own memoir of Ramanujan) are from Seshu Aiyar and Ramachaundra Rao.

I have quoted the last sentences deliberately, not because I attach any importance to them—I am no more interested in the goddess of Namakkal than you are—but because we are now approaching the difficult and tragic part of Ramanujan's career, and we must try to understand what we can of his psychology and of the atmosphere surrounding him in his early years.

I am sure that Ramanujan was no mystic and that religion, except in a strictly material sense, played no important part in his life. He was an orthodox high-caste Hindu, and always adhered (indeed with a severity most unusual in Indians resident in England) to all the observances of his caste. He had promised his parents to do so, and he kept his promises to the letter. He was a vegetarian in the strictest sense—this proved a terrible difficulty later when he fell ill—and all the time he was in Cambridge he cooked all his food himself, and never cooked it without first changing into pyjamas.

Now the two memoirs of Ramanujan printed in the *Papers* (and both written by men who, in their different ways, knew him very well) contradict one another flatly about his religion. Seshu Aiyar and Ramachaundra Rao say

> Ramanujan had definite religious views. He had a special veneration for the Namakkal goddess....He believed in the existence of a Supreme Being and in the attainment of Godhead by men....He had settled convictions about the problem of life and after...;

while I say

> ...his religion was a matter of observance and not of intellectual conviction, and I remember well his telling me (much to my surprise) that all religions seemed to him more or less equally true....

Which of us is right? For my part I have no doubt at all; I am quite certain that I am.

Classical scholars have, I believe, a general principle, *difficilior lectio potior*—the more difficult reading is to be preferred—in textual criticism. If the Archbishop of Canterbury tells one man that he[1] believes in God, and another that he does not, then it is probably the second assertion which is true, since otherwise it is very difficult to understand why he should have made it, while there are many excellent reasons for his making the first whether it be true or false. Similarly, if a strict Brahmin like Ramanujan told me, as he certainly did, that he had no definite beliefs, then it is 100 to 1 that he meant what he said.

This was no sufficient reason why Ramanujan should outrage the feelings of his parents or his Indian friends. He was not a reasoned infidel, but an

[1] The Archbishop.

"agnostic" in its strict sense, who saw no particular good, and no particular harm, in Hinduism or in any other religion. Hinduism is, far more, for example, than Christianity, a religion of observance, in which belief counts for extremely little in any case, and, if Ramanujan's friends assumed that he accepted the conventional doctrines of such a religion, and he did not disillusion them, he was practising a quite harmless, and probably necessary, economy of truth.

This question of Ramanujan's religion is not itself important, but it is not altogether irrelevant, because there is one thing which I am really anxious to insist upon as strongly as I can. There is quite enough about Ramanujan that is difficult to understand, and we have no need to go out of our way to manufacture mystery. For myself, I liked and admired him enough to wish to be a rationalist about him; and I want to make it quite clear to you that Ramanujan, when he was living in Cambridge in good health and comfortable surroundings, was, in spite of his oddities, as reasonable, as sane, and in his way as shrewd a person as anyone here. The last thing which I want you to do is to throw up your hands and exclaim "here is something unintelligible, some mysterious manifestation of the immemorial wisdom of the East!" I do not believe in the immemorial wisdom of the East, and the picture which I want to present to you is that of a man who had his peculiarities like other distinguished men, but a man in whose society one could take pleasure, with whom one could drink tea and discuss politics or mathematics; the picture in short, not of a wonder from the East, or an inspired idiot, or a psychological freak, but of a rational human being who happened to be a great mathematician.

Until he was about seventeen, all went well with Ramanujan.

> In December 1903 he passed the Matriculation Examination of the University of Madras, and in the January of the succeeding year he joined the Junior First in Arts class of the Government College, Kumbakonam, and won the Subrahmanyam scholarship, which is generally awarded for proficiency in English and Mathematics...,

but after this there came a series of tragic checks.

> By this time, he was so absorbed in the study of Mathematics that in all lecture hours—whether devoted to English, History, or Physiology—he used to engage himself in some mathematical investigation, unmindful of what was happening in the class. This excessive devotion to mathematics and his consequent neglect of the other subjects resulted in his failure to secure promotion to the senior class and in the consequent discontinuance of the scholarship. Partly owing to disappointment and partly owing to the influence of a friend, he ran away northward into the Telugu country, but returned to Kumbakonam after some wandering and rejoined the college. As owing to his absence he failed to make sufficient attendances to obtain his term certificate in 1905, he entered

Pachaiyappa's College, Madras, in 1906, but falling ill returned to Kumbakonam. He appeared as a private student for the F.A. examination of December 1907 and failed....

Ramanujan does not seem to have had any definite occupation, except mathematics, until 1912. In 1909 he married, and it became necessary for him to have some regular employment, but he had great difficulty in finding any because of his unfortunate college career. About 1910 he began to find more influential Indian friends, Ramaswami Aiyar and his two biographers, but all their efforts to find a tolerable position for him failed, and in 1912 he became a clerk in the office of the Port Trust of Madras, at a salary of about £30 a year. He was then nearly twenty-five. The years between eighteen and twenty-five are the critical years in a mathematician's career, and the damage had been done. Ramanujan's genius never had again its chance of full development.

There is not much to say about the rest of Ramanujan's life. His first substantial paper had been published in 1911, and in 1912 his exceptional powers began to be understood. It is significant that, though Indians could befriend him, it was only the English who could get anything effective done. Sir Francis Spring and Sir Gilbert Walker obtained a special scholarship for him, £60 a year, sufficient for a married Indian to live in tolerable comfort. At the beginning of 1913 he wrote to me, and Professor Neville and I, after many difficulties, got him to England in 1914. Here he had three years of uninterrupted activity, the results of which you can read in the *Papers*. He fell ill in the summer of 1917, and never really recovered, though he continued to work, rather spasmodically, but with no real sign of degeneration, until his death in 1920. He became a Fellow of the Royal Society early in 1918, and a Fellow of Trinity College, Cambridge, later in the same year (and was the first Indian elected to either society). His last mathematical letter on "Mock-Theta functions", the subject of Professor Watson's presidential address to the London Mathematical Society last year, was written about two months before he died.

The real tragedy about Ramanujan was not his early death. It is of course a disaster that any great man should die young, but a mathematician is often comparatively old at thirty, and his death may be less of a catastrophe than it seems. Abel died at twenty-six and, although he would no doubt have added a great deal more to mathematics, he could hardly have become a greater man. The tragedy of Ramanujan was not that he died young, but that, during his five unfortunate years, his genius was misdirected, side-tracked, and to a certain extent distorted.

I have been looking again through what I wrote about Ramanujan sixteen years ago, and, although I know his work a good deal better now than

I did then, and can think about him more dispassionately, I do not find a great deal which I should particularly want to alter. But there is just one sentence which now seems to me indefensible. I wrote

> Opinions may differ about the importance of Ramanujan's work, the kind of standard by which it should be judged, and the influence which it is likely to have on the mathematics of the future. It has not the simplicity and the inevitableness of the very greatest work; it would be greater if it were less strange. One gift it shows which no one can deny, profound and invincible originality. He would probably have been a greater mathematician if he could have been caught and tamed a little in his youth; he would have discovered more that was new, and that, no doubt, of greater importance. On the other hand he would have been less of a Ramanujan, and more of a European professor, and the loss might have been greater than the gain...

and I stand by that except for the last sentence, which is quite ridiculous sentimentalism. There was no gain at all when the College at Kumbakonam rejected the one great man they had ever possessed, and the loss was irreparable; it is the worst instance that I know of the damage that can be done by an inefficient and inelastic educational system. So little was wanted, £60 a year for five years, occasional contact with almost anyone who had real knowledge and a little imagination, for the world to have gained another of its greatest mathematicians.

Ramanujan's letters to me, which are reprinted in full in the *Papers*, contain the bare statements of about 120 theorems, mostly formal identities extracted from his notebooks. I quote fifteen which are fairly representative. They include two theorems, (1.14) and (1.15), which are as interesting as any but of which one is false and the other, as stated, misleading. The rest have all been verified since by somebody; in particular Rogers and Watson found the proofs of the extremely difficult theorems (1.10)–(1.12).

(1.1) $$1-\frac{3!}{(1!2!)^3}x^2+\frac{6!}{(2!4!)^3}x^4-\dots$$
$$=\left(1+\frac{x}{(1!)^3}+\frac{x^2}{(2!)^3}+\dots\right)\left(1-\frac{x}{(1!)^3}+\frac{x^2}{(2!)^3}-\dots\right).$$

(1.2) $$1-5\left(\frac{1}{2}\right)^3+9\left(\frac{1.3}{2.4}\right)^3-13\left(\frac{1.3.5}{2.4.6}\right)^3+\dots=\frac{2}{\pi}.$$

(1.3) $$1+9\left(\frac{1}{4}\right)^4+17\left(\frac{1.5}{4.8}\right)^4+25\left(\frac{1.5.9}{4.8.12}\right)^4+\dots=\frac{2^{\frac{3}{2}}}{\pi^{\frac{1}{2}}\{\Gamma(\frac{3}{4})\}^2}.$$

(1.4) $$1-5\left(\frac{1}{2}\right)^5+9\left(\frac{1.3}{2.4}\right)^5-13\left(\frac{1.3.5}{2.4.6}\right)^5+\dots=\frac{2}{\{\Gamma(\frac{3}{4})\}^4}.$$

(1.5) $$\int_0^\infty \frac{1+\left(\frac{x}{b+1}\right)^2}{1+\left(\frac{x}{a}\right)^2}\cdot\frac{1+\left(\frac{x}{b+2}\right)^2}{1+\left(\frac{x}{a+1}\right)^2}\ldots dx = \tfrac{1}{2}\pi^{\frac{1}{2}}\frac{\Gamma(a+\frac{1}{2})\,\Gamma(b+1)\,\Gamma(b-a+\frac{1}{2})}{\Gamma(a)\,\Gamma(b+\frac{1}{2})\,\Gamma(b-a+1)}.$$

(1.6) $$\int_0^\infty \frac{dx}{(1+x^2)(1+r^2x^2)(1+r^4x^2)\ldots} = \frac{\pi}{2(1+r+r^3+r^6+r^{10}+\ldots)}.$$

(1.7) If $\alpha\beta = \pi^2$, then

$$\alpha^{-\frac{1}{4}}\left(1+4\alpha\int_0^\infty \frac{xe^{-\alpha x^2}}{e^{2\pi x}-1}\,dx\right) = \beta^{-\frac{1}{4}}\left(1+4\beta\int_0^\infty \frac{xe^{-\beta x^2}}{e^{2\pi x}-1}\,dx\right).$$

(1.8) $$\int_0^a e^{-x^2}dx = \tfrac{1}{2}\pi^{\frac{1}{2}} - \frac{e^{-a^2}}{2a+}\,\frac{1}{a+}\,\frac{2}{2a+}\,\frac{3}{a+}\,\frac{4}{2a+\ldots}.$$

(1.9) $$4\int_0^\infty \frac{xe^{-x\sqrt{5}}}{\cosh x}\,dx = \frac{1}{1+}\,\frac{1^2}{1+}\,\frac{1^2}{1+}\,\frac{2^2}{1+}\,\frac{2^2}{1+}\,\frac{3^2}{1+}\,\frac{3^2}{1+\ldots}.$$

(1.10) If $$u = \frac{x}{1+}\,\frac{x^5}{1+}\,\frac{x^{10}}{1+}\,\frac{x^{15}}{1+\ldots},\quad v = \frac{x^{\frac{1}{5}}}{1+}\,\frac{x}{1+}\,\frac{x^2}{1+}\,\frac{x^3}{1+\ldots},$$

then $$v^5 = u\,\frac{1-2u+4u^2-3u^3+u^4}{1+3u+4u^2+2u^3+u^4}.$$

(1.11) $$\frac{1}{1+}\,\frac{e^{-2\pi}}{1+}\,\frac{e^{-4\pi}}{1+\ldots} = \left\{\sqrt{\left(\frac{5+\sqrt{5}}{2}\right)} - \frac{\sqrt{5}+1}{2}\right\}e^{\frac{2}{5}\pi}.$$

(1.12) $$\frac{1}{1+}\,\frac{e^{-2\pi\sqrt{5}}}{1+}\,\frac{e^{-4\pi\sqrt{5}}}{1+\ldots} = \left[\frac{\sqrt{5}}{1+\sqrt[5]{\left\{5^{\frac{3}{4}}\left(\frac{\sqrt{5}-1}{2}\right)^{\frac{5}{2}}-1\right\}}} - \frac{\sqrt{5}+1}{2}\right]e^{2\pi/\sqrt{5}}.$$

(1.13) If $F(k) = 1+\left(\frac{1}{2}\right)^2 k + \left(\frac{1.3}{2.4}\right)^2 k^2 + \ldots$ and $F(1-k) = \sqrt{(210)}\,F(k)$, then

$$k = (\sqrt{2}-1)^4(2-\sqrt{3})^2(\sqrt{7}-\sqrt{6})^4(8-3\sqrt{7})^2(\sqrt{10}-3)^4 \times (4-\sqrt{15})^4(\sqrt{15}-\sqrt{14})^2(6-\sqrt{35})^2.$$

(1.14) The coefficient of x^n in $(1-2x+2x^4-2x^9+\ldots)^{-1}$ is the integer nearest to

$$\frac{1}{4n}\left(\cosh\pi\sqrt{n} - \frac{\sinh\pi\sqrt{n}}{\pi\sqrt{n}}\right).$$

(1.15) The number of numbers between A and x which are either squares or sums of two squares is

$$K\int_A^x \frac{dt}{\sqrt{(\log t)}} + \theta(x),$$

where $K = 0{\cdot}764\ldots$ and $\theta(x)$ is very small compared with the previous integral.

I should like you to begin by trying to reconstruct the immediate reactions of an ordinary professional mathematician who receives a letter like this from an unknown Hindu clerk.

The first question was whether I could recognise anything. I had proved things rather like (1.7) myself, and seemed vaguely familiar with (1.8). Actually (1.8) is classical; it is a formula of Laplace first proved properly by Jacobi; and (1.9) occurs in a paper published by Rogers in 1907. I thought that, as an expert in definite integrals, I could probably prove (1.5) and (1.6), and did so, though with a good deal more trouble than I had expected. On the whole the integral formulae seemed the least impressive.

The series formulae (1.1)–(1.4) I found much more intriguing, and it soon became obvious that Ramanujan must possess much more general theorems and was keeping a great deal up his sleeve. The second is a formula of Bauer well known in the theory of Legendre series, but the others are much harder than they look. The theorems required in proving them can all be found now in Bailey's Cambridge Tract on hypergeometric functions.

The formulae (1.10)–(1.13) are on a different level and obviously both difficult and deep. An expert in elliptic functions can see at once that (1.13) is derived somehow from the theory of "complex multiplication", but (1.10)–(1.12) defeated me completely; I had never seen anything in the least like them before. A single look at them is enough to show that they could only be written down by a mathematician of the highest class. They must be true because, if they were not true, no one would have had the imagination to invent them. Finally (you must remember that I knew nothing whatever about Ramanujan, and had to think of every possibility), the writer must be completely honest, because great mathematicians are commoner than thieves or humbugs of such incredible skill.

The last two formulae stand apart because they are not right and show Ramanujan's limitations, but that does not prevent them from being additional evidence of his extraordinary powers. The function in (1.14) is a genuine approximation to the coefficient, though not at all so close as Ramanujan imagined, and Ramanujan's false statement was one of the most fruitful he ever made, since it ended by leading us to all our joint work on partitions. Finally (1.15), though literally "true", is definitely misleading (and Ramanujan was under a real misapprehension). The integral has no advantage, as an approximation, over the simpler function

$$\frac{Kx}{\sqrt{(\log x)}}, \tag{1.16}$$

found in 1908 by Landau. Ramanujan was deceived by a false analogy with the problem of the distribution of primes. I must postpone till later what

I have to say about Ramanujan's work on this side of the theory of numbers.

It was inevitable that a very large part of Ramanujan's work should prove on examination to have been anticipated. He had been carrying an impossible handicap, a poor and solitary Hindu pitting his brains against the accumulated wisdom of Europe. He had had no real teaching at all; there was no one in India from whom he had anything to learn. He can have seen at the outside three or four books of good quality, all of them English. There had been periods in his life when he had access to the library in Madras, but it was not a very good one; it contained very few French or German books; and in any case Ramanujan did not know a word of either language. I should estimate that about two-thirds of Ramanujan's best Indian work was rediscovery, and comparatively little of it was published in his lifetime, though Watson, who has worked systematically through his notebooks, has since disinterred a good deal more.

The great bulk of Ramanujan's published work was done in England. His mind had hardened to some extent, and he never became at all an "orthodox" mathematician, but he could still learn to do new things, and do them extremely well. It was impossible to teach him systematically, but he gradually absorbed new points of view. In particular he learnt what was meant by proof, and his later papers, while in some ways as odd and individual as ever, read like the works of a well-informed mathematician. His methods and his weapons, however, remained essentially the same. One would have thought that such a formalist as Ramanujan would have revelled in Cauchy's Theorem, but he practically never used it,[1] and the most astonishing testimony to his formal genius is that he never seemed to feel the want of it in the least.

It is easy to compile an imposing list of theorems which Ramanujan rediscovered. Such a list naturally cannot be quite sharp, since sometimes he found a part only of a theorem, and sometimes, though he found the whole theorem, he was without the proof which is essential if the theorem is to be properly understood. For example, in the analytic theory of numbers he had, in a sense, discovered a great deal, but he was a very long way from understanding the real difficulties of the subject. And there is some of his work, mostly in the theory of elliptic functions, about which some mystery still remains; it is not possible, after all the work of Watson and Mordell, to draw the line between what he may have picked up somehow and what he must have found for himself. I will take only cases in which the evidence seems to me tolerably clear.

[1] Perhaps never. There is a reference to "the theory of residues" on p. 129 of the *Papers*, but I believe that I supplied this myself.

Here I must admit that I am to blame, since there is a good deal which we should like to know now and which I could have discovered quite easily. I saw Ramanujan almost every day, and could have cleared up most of the obscurity by a little cross-examination. Ramanujan was quite able and willing to give a straight answer to a question, and not in the least disposed to make a mystery of his achievements. I hardly asked him a single question of this kind; I never even asked him whether (as I think he must have done) he had seen Cayley's or Greenhill's *Elliptic functions*.

I am sorry about this now, but it does not really matter very much, and it was entirely natural. In the first place, I did not know that Ramanujan was going to die. He was not particularly interested in his own history or psychology; he was a mathematician anxious to get on with the job. And after all I too was a mathematician, and a mathematician meeting Ramanujan had more interesting things to think about than historical research. It seemed ridiculous to worry him about how he had found this or that known theorem, when he was showing me half a dozen new ones almost every day.

I do not think that Ramanujan discovered much in the classical theory of numbers, or indeed that he ever knew a great deal. He had no knowledge at all, at any time, of the general theory of arithmetical forms. I doubt whether he knew the law of quadratic reciprocity before he came here. Diophantine equations should have suited him, but he did comparatively little with them, and what he did do was not his best. Thus he gave solutions of Euler's equation

$$x^3+y^3+z^3=w^3, \qquad (1.17)$$

such as

$$\begin{cases} x=3a^2+5ab-5b^2, & y=4a^2-4ab+6b^2,\\ z=5a^2-5ab-3b^2, & w=6a^2-4ab+4b^2; \end{cases} \qquad (1.18)$$

and

$$\begin{cases} x=m^7-3m^4(1+p)+m(2+6p+3p^2),\\ y=2m^6-3m^3(1+2p)+1+3p+3p^2,\\ z=m^6-1-3p-3p^2, \quad w=m^7-3m^4p+m(3p^2-1); \end{cases} \qquad (1.19)$$

but neither of these is the general solution.

He rediscovered the famous theorem of von Staudt about the Bernoullian numbers:

$$(-1)^n B_n = G_n+\frac{1}{2}+\frac{1}{p}+\frac{1}{q}+\ldots+\frac{1}{r}, \qquad (1.20)$$

where $p, q, \ldots$ are those odd primes such that $p-1, q-1, \ldots$ are divisors of $2n$, and G_n is an integer. In what sense he had proved it it is difficult to say, since he found it at a time of his life when he had hardly formed any definite concept of proof. As Littlewood says, "the clear-cut idea of what is *meant* by a proof, nowadays so familiar as to be taken for granted, he perhaps did not possess at all; if a significant piece of reasoning occurred somewhere, and the total mixture of evidence and intuition gave him certainty, he looked no further". I shall have something to say later about this question of proof, but I postpone it to another context in which it is much more important. In this case there is nothing in the proof that was not obviously within Ramanujan's powers.

There is a considerable chapter of the theory of numbers, in particular the theory of the representation of integers by sums of squares, which is closely bound up with the theory of elliptic functions. Thus the number of representations of n by two squares is

$$r(n) = 4\{d_1(n) - d_3(n)\}, \tag{1.21}$$

where $d_1(n)$ is the number of divisors of n of the form $4k+1$ and $d_3(n)$ the number of divisors of the form $4k+3$. Jacobi gave similar formulae for 4, 6 and 8 squares. Ramanujan found all these, and much more of the same kind.

He also found Legendre's theorem that n is the sum of 3 squares except when it is of the form

$$4^a(8k+7), \tag{1.22}$$

but I do not attach much importance to this. The theorem is quite easy to guess and difficult to prove. All known proofs depend upon the general theory of ternary forms, of which Ramanujan knew nothing, and I agree with Professor Dickson in thinking it very unlikely that he possessed one. In any case he knew nothing about the number of representations.

Ramanujan, then, before he came to England, had added comparatively little to the theory of numbers; but no one can understand him who does not understand his passion for numbers in themselves. I wrote before

> He could remember the idiosyncrasies of numbers in an almost uncanny way. It was Littlewood who said that every positive integer was one of Ramanujan's personal friends. I remember going to see him once when he was lying ill in Putney. I had ridden in taxi-cab No. 1729, and remarked that the number seemed to me rather a dull one, and that I hoped that it was not an unfavourable omen. "No," he replied, "it is a very interesting number; it is the smallest number expressible as a sum of two cubes in two different ways."[1] I asked him, naturally, whether he could tell me the solution of the corresponding problem for fourth powers; and he replied, after a moment's thought, that he knew no obvious example, and supposed that the first such number must be very large.

In algebra, Ramanujan's main work was concerned with hypergeometric series and continued fractions (I use the word algebra, of course, in its old-

[1] $1729 = 12^3 + 1^3 = 10^3 + 9^3$.

fashioned sense). These subjects suited him exactly, and here he was unquestionably one of the great masters. There are three now famous identities, the "Dougall-Ramanujan identity"

$$(1.23)\quad \sum_{n=0}^{\infty}(-1)^n(s+2n)\frac{s^{(n)}}{1^{(n)}}\frac{(x+y+z+u+2s+1)^{(n)}}{(x+y+z+u+s)_{(n)}}\prod_{x,y,z,u}\frac{x_{(n)}}{(x+s+1)^{(n)}}$$

$$=\frac{s}{\Gamma(s+1)\,\Gamma(x+y+z+u+s+1)}\prod_{x,y,z,u}\frac{\Gamma(x+s+1)\,\Gamma(y+z+u+s+1)}{\Gamma(z+u+s+1)},$$

where $\quad a^{(n)}=a(a+1)\dots(a+n-1),\quad a_{(n)}=a(a-1)\dots(a-n+1),$

and the "Rogers-Ramanujan identities"

$$(1.24)\quad \begin{cases} 1+\dfrac{q}{1-q}+\dfrac{q^4}{(1-q)(1-q^2)}+\dfrac{q^9}{(1-q)(1-q^2)(1-q^3)}+\dots \\ \qquad\qquad =\dfrac{1}{(1-q)(1-q^6)\dots(1-q^4)(1-q^9)\dots}, \\ 1+\dfrac{q^2}{1-q}+\dfrac{q^6}{(1-q)(1-q^2)}+\dfrac{q^{12}}{(1-q)(1-q^2)(1-q^3)}+\dots \\ \qquad\qquad =\dfrac{1}{(1-q^2)(1-q^7)\dots(1-q^3)(1-q^8)\dots}, \end{cases}$$

in which he had been anticipated by British mathematicians, and about which I shall speak in other lectures.[1] As regards hypergeometric series one may say, roughly, that he rediscovered the formal theory, set out in Bailey's tract, as it was known up to 1920. There is something about it in Carr, and more in Chrystal's *Algebra*, and no doubt he got his start from that. The four formulae (1.1)–(1.4) are highly specialised examples of this work.

His masterpiece in continued fractions was his work on

$$(1.25)\qquad \frac{1}{1+}\,\frac{x}{1+}\,\frac{x^2}{1+\dots},$$

which includes the theorems (1.10)–(1.12). The theory of this fraction depends upon the Rogers-Ramanujan identities, in which he had been anticipated by Rogers, but he had gone beyond Rogers in other ways and the theorems which I have quoted are his own. He had many other very general and very beautiful formulae, of which formulae like Laguerre's

$$(1.26)\qquad \frac{(x+1)^n-(x-1)^n}{(x+1)^n+(x-1)^n}=\frac{n}{x+}\,\frac{n^2-1}{3x+}\,\frac{n^2-2^2}{5x+\dots}$$

are extremely special cases. Watson has recently published a proof of the most imposing of them.

[1] See Lectures VI and VII.

It is perhaps in his work in these fields that Ramanujan shows at his very best. I wrote

> It was his insight into algebraical formulae, transformation of infinite series, and so forth, that was most amazing. On this side most certainly I have never met his equal, and I can compare him only with Euler or Jacobi. He worked, far more than the majority of modern mathematicians, by induction from numerical examples; all his congruence properties of partitions, for example, were discovered in this way. But with his memory, his patience, and his power of calculation he combined a power of generalisation, a feeling for form, and a capacity for rapid modification of his hypotheses, that were often really startling, and made him, in his own peculiar field, without a rival in his day.

I do not think now that this extremely strong language is extravagant. It is possible that the great days of formulae are finished, and that Ramanujan ought to have been born 100 years ago; but he was by far the greatest formalist of his time. There have been a good many more important, and I suppose one must say greater, mathematicians than Ramanujan during the last fifty years, but not one who could stand up to him on his own ground. Playing the game of which he knew the rules, he could give any mathematician in the world fifteen.

In analysis proper Ramanujan's work is inevitably less impressive, since he knew no theory of functions, and you cannot do real analysis without it, and since the formal side of the integral calculus, which was all that he could learn from Carr or any other book, has been worked over so repeatedly and so intensively. Still, Ramanujan rediscovered an astonishing number of the most beautiful analytic identities. Thus the functional equation for the Riemann Zeta-function

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s},$$

namely

$$\zeta(1-s) = 2(2\pi)^{-s} \cos \tfrac{1}{2}s\pi\, \Gamma(s)\, \zeta(s), \tag{1.27}$$

stands (in an almost unrecognisable notation) in the notebooks. So does Poisson's summation formula

$$\alpha^{\frac{1}{2}}\{\tfrac{1}{2}\phi(0)+\phi(\alpha)+\phi(2\alpha)+\ldots\} = \beta^{\frac{1}{2}}\{\tfrac{1}{2}\psi(0)+\psi(\beta)+\psi(2\beta)+\ldots\}, \tag{1.28}$$

where
$$\psi(x) = \sqrt{\left(\frac{2}{\pi}\right)} \int_0^{\infty} \phi(t) \cos xt\, dt$$

and $\alpha\beta = 2\pi$; and so also does Abel's functional equation

$$L(x)+L(y)+L(xy)+L\left\{\frac{x(1-y)}{1-xy}\right\}+L\left\{\frac{y(1-x)}{1-xy}\right\} = 3L(1) \tag{1.29}$$

for
$$L(x) = \frac{x}{1^2}+\frac{x^2}{2^2}+\frac{x^3}{3^2}+\ldots.$$

He had most of the formal ideas which underlie the recent work of Watson and of Titchmarsh and myself on "Fourier kernels" and "reciprocal functions"; and he could of course evaluate any evaluable definite integral. There is one particularly interesting formula, viz.

$$(1.30) \qquad \int_0^\infty x^{s-1}\{\phi(0)-x\phi(1)+x^2\phi(2)-\ldots\}\,dx = \frac{\pi\phi(-s)}{\sin s\pi},$$

of which he was especially fond and made continual use. This is really an "interpolation formula", which enables us to say, for example, that, under certain conditions, a function which vanishes for all positive integral values of its argument must vanish identically. I have never seen this formula stated explicitly by anyone else, though it is closely connected with the work of Mellin and others.

I have left till last the two most intriguing sides of Ramanujan's early work, his work on elliptic functions and in the analytic theory of numbers. The first is probably too specialised and intricate for anyone but an expert to understand, and I shall say nothing about it now.[1] The second subject is still more difficult (as anyone who has read Landau's book on primes or Ingham's tract will know), but anyone can understand roughly what the problems of the subject are, and any decent mathematician can understand roughly why they defeated Ramanujan. For this was Ramanujan's one real failure; he showed, as always, astonishing imaginative power, but he proved next to nothing, and a great deal even of what he imagined was false.

Here I am obliged to interpolate some remarks on a very difficult subject: *proof* and its importance in mathematics. All physicists, and a good many quite respectable mathematicians, are contemptuous about proof. I have heard Professor Eddington, for example, maintain that proof, as pure mathematicians understand it, is really quite uninteresting and unimportant, and that no one who is really certain that he has found something good should waste his time looking for a proof. It is true that Eddington is inconsistent, and has sometimes even descended to proof himself. It is not enough for him to have direct knowledge that there are exactly

$$136 . 2^{256}$$

protons in the universe; he cannot resist the temptation of proving it; and I cannot help thinking that the proof, whatever it may be worth, gives him a certain amount of intellectual satisfaction. His apology would no doubt be that "proof" means something quite different for him from what it means for a pure mathematician, and in any case we need not take him too

[1] See Lecture XII.

literally. But the opinion which I have attributed to him, and with which I am sure that almost all physicists agree at the bottom of their hearts, is one to which a mathematician ought to have some reply.

I am not going to get entangled in the analysis of a particularly prickly concept, but I think that there are a few points about proof where nearly all mathematicians are agreed. In the first place, even if we do not understand exactly what proof is, we can, in ordinary analysis at any rate, recognise a proof when we see one. Secondly, there are two different motives in any presentation of a proof. The first motive is simply to secure conviction. The second is to exhibit the conclusion as the climax of a conventional pattern of propositions, a sequence of propositions whose truth is admitted and which are arranged in accordance with rules. These are the two ideals, and experience shows that, except in the simplest mathematics, we can hardly ever satisfy the first ideal without also satisfying the second. We may be able to recognise directly that 5, or even 17, is prime, but nobody can convince himself that

$$2^{127}-1$$

is prime except by studying a proof. No one has ever had an imagination so vivid and comprehensive as that.

A mathematician usually discovers a theorem by an effort of intuition; the conclusion strikes him as plausible, and he sets to work to manufacture a proof. Sometimes this is a matter of routine, and any well-trained professional could supply what is wanted, but more often imagination is a very unreliable guide. In particular this is so in the analytic theory of numbers, where even Ramanujan's imagination led him very seriously astray.

There is a striking example, which I have very often quoted, of a false conjecture which seems to have been endorsed even by Gauss and which took about 100 years to refute. The central problem of the analytic theory of numbers is that of the distribution of the primes. The number $\pi(x)$ of primes less than a large number x is approximately

$$\frac{x}{\log x}; \tag{1.31}$$

this is the "Prime Number Theorem", which had been conjectured for a very long time, but was never established properly until Hadamard and de la Vallée-Poussin proved it in 1896. The approximation errs by defect, and a much better one is

$$\operatorname{li} x = \int_0^x \frac{dt}{\log t}.^{1} \tag{1.32}$$

In some ways a still better one is

$$\operatorname{li} x - \tfrac{1}{2}\operatorname{li} x^{\frac{1}{2}} - \tfrac{1}{3}\operatorname{li} x^{\frac{1}{3}} - \tfrac{1}{5}\operatorname{li} x^{\frac{1}{5}} + \tfrac{1}{6}\operatorname{li} x^{\frac{1}{6}} - \tfrac{1}{7}\operatorname{li} x^{\frac{1}{7}} + \dots \tag{1.33}$$

[1] The integral is a 'principal value'. See § 2.2.

(we need not trouble now about the law of formation of the series). It is extremely natural to infer that

(1.34) $$\pi(x) < \operatorname{li} x,$$

at any rate for large x, and Gauss and other mathematicians commented on the high probability of this conjecture. The conjecture is not only plausible but is supported by *all* the evidence of the facts. The primes are known up to 10,000,000, and their number at intervals up to 1,000,000,000, and (1.34) is true for every value of x for which data exist.

In 1912 Littlewood proved that the conjecture is false, and that there are an infinity of values of x for which the sign of inequality in (1.34) must be reversed. In particular, there is a number X such that (1.34) is false for some x less than X. Littlewood proved the existence of X, but his method did not give any particular value, and it is only very recently that an admissible value, viz.

$$X = 10^{10^{10^{34}}},$$

was found by Skewes. I think that this is the largest number which has ever served any definite purpose in mathematics.

The number of protons in the universe is about

$$10^{80}.$$

The number of possible games of chess is much larger, perhaps

$$10^{10^{50}}$$

(in any case a second-order exponential). If the universe were the chessboard, the protons the chessmen, and any interchange in the position of two protons a move, then the number of possible games would be something like the Skewes number. However much the number may be reduced by refinements on Skewes's argument, it does not seem at all likely that we shall ever know a single instance of the truth of Littlewood's theorem.

This is an example in which the truth has defeated not only all the evidence of the facts and of common sense but even a mathematical imagination so powerful and profound as that of Gauss; but of course it is taken from the most difficult parts of the theory. No part of the theory of primes is really easy, but up to a point simple arguments, although they will prove very little, do not actually mislead us. For example, there are simple arguments which might lead any good mathematician to the conclusion

(1.35) $$\pi(x) \sim \frac{x}{\log x}$$

of the Prime Number Theorem,[1] or, what is the same thing, to the conclusion

[1] $f(x) \sim g(x)$ means that the ratio f/g tends to unity.

that

$$(1.36) \qquad p_n \sim n \log n,$$

where p_n is the n-th prime number.

In the first place, we may start from Euler's identity

$$(1.37) \quad \prod_p \frac{1}{1-p^{-s}} = \frac{1}{(1-2^{-s})(1-3^{-s})(1-5^{-s})\ldots} = \frac{1}{1^s}+\frac{1}{2^s}+\frac{1}{3^s}+\ldots = \sum_n \frac{1}{n^s}.$$

This is true for $s > 1$, but both series and product become infinite for $s = 1$. It is natural to argue that, when $s = 1$, the series and the product should diverge in the same sort of way. Also

$$(1.38) \quad \log \Pi \frac{1}{1-p^{-s}} = \Sigma \log \frac{1}{1-p^{-s}} = \Sigma \frac{1}{p^s} + \Sigma\left(\frac{1}{2p^{2s}}+\frac{1}{3p^{3s}}+\ldots\right),$$

and the last series remains finite for $s = 1$. It is natural to infer that

$$\Sigma \frac{1}{p}$$

diverges like
$$\log\left(\Sigma \frac{1}{n}\right),$$

or, more precisely, that

$$(1.39) \qquad \sum_{p \leq x} \frac{1}{p} \sim \log\left(\sum_{n \leq x} \frac{1}{n}\right) \sim \log\log x$$

for large x. Since also

$$\sum_{n \leq x} \frac{1}{n \log n} \sim \log\log x,$$

formula (1.39) indicates that p_n is about $n \log n$.

There is a slightly more sophisticated argument which is really simpler. It is easy to see that the highest power of a prime p which divides $x!$ is

$$\left[\frac{x}{p}\right]+\left[\frac{x}{p^2}\right]+\left[\frac{x}{p^3}\right]+\ldots,$$

where $[y]$ denotes the integral part of y. Hence

$$x! = \prod_{p \leq x} p^{[x/p]+[x/p]^2+\ldots},$$

$$(1.40) \qquad \log x! = \sum_{p \leq x}\left(\left[\frac{x}{p}\right]+\left[\frac{x}{p^2}\right]+\ldots\right)\log p.$$

The left-hand side of (1.40) is practically $x \log x$, by Stirling's Theorem. As regards the right-hand side, one may argue; squares, cubes, ... of primes are comparatively rare, and the terms involving them should be unimportant, and it should also make comparatively little difference if we replace $[x/p]$ by x/p. We thus infer that

$$x \sum_{p \leq x} \frac{\log p}{p} \sim x \log x, \quad \sum_{p \leq x} \frac{\log p}{p} \sim \log x,$$

and this again just fits the view that p_n is approximately $n \log n$.

This is broadly the argument used, naturally in a less naïve form, by Tchebychef, who was the first to make substantial progress in the theory of primes, and I imagine that Ramanujan began by arguing in the same sort of way, though there is nothing in the notebooks to show. All that is plain is that Ramanujan found the form of the Prime Number Theorem for himself. This was a considerable achievement; for the men who had found the form of the theorem before him, like Legendre, Gauss, and Dirichlet, had all been very great mathematicians; and Ramanujan found other formulae which lie still further below the surface. Perhaps the best instance is (1.15). The integral is better replaced by the simpler function (1.16), but what Ramanujan says is correct as it stands and was proved by Landau in 1909; and there is nothing obvious to suggest its truth.

The fact remains that hardly any of Ramanujan's work in this field had any permanent value. The analytic theory of numbers is one of those exceptional branches of mathematics in which proof really is everything and nothing short of absolute rigour counts. The achievement of the mathematicians who found the Prime Number Theorem was quite a small thing compared with that of those who found the proof. It is not merely that in this theory (as Littlewood's theorem shows) you can never be sure of the facts without the proof, though this is important enough. The whole history of the Prime Number Theorem, and the other big theorems of the subject, shows that you cannot reach any real understanding of the structure and meaning of the theory, or have any sound instincts to guide you in further research, until you have mastered the proofs. It is comparatively easy to make clever guesses; indeed there are theorems, like "Goldbach's Theorem",[1] which have never been proved and which any fool could have guessed.

The theory of primes depends upon the properties of Riemann's function $\zeta(s)$, considered as an analytic function of the complex variable s, and in particular on the distribution of its zeros; and Ramanujan knew nothing at all about the theory of analytic functions. I wrote before

> Ramanujan's theory of primes was vitiated by his ignorance of the theory of functions of a complex variable. It was (so to say) what the theory might be if the Zeta-function had no complex zeros. His method depended upon a wholesale

[1] "Any even number greater than 2 is the sum of two primes."

use of divergent series....That his proofs should have been invalid was only to be expected. But the mistakes went deeper than that, and many of the actual results were false. He had obtained the dominant terms of the classical formulae, although by invalid methods; but none of them are such close approximations as he supposed.

This may be said to have been Ramanujan's one great failure...

and if I had stopped there I should have had nothing to add, but I allowed myself again to be led away by sentimentalism. I went on to argue that "his failure was more wonderful than any of his triumphs", and that is an absurd exaggeration. It is no use trying to pretend that failure is something else. This much perhaps we may say, that his failure is one which, on the balance, should increase and not diminish our admiration for his gifts, since it gives us additional, and surprising, evidence of his imagination and versatility.

But the reputation of a mathematician cannot be made by failures or by rediscoveries; it must rest primarily, and rightly, on actual and original achievement. I have to justify Ramanujan on this ground, and that I hope to do in my later lectures.

NOTES ON LECTURE I

p. 1. This lecture is a reprint of one delivered at the Harvard Tercentenary Conference of Arts and Sciences on August 31, 1936, and published in the *American Math. Monthly*, 44 (1937), 137–155.

pp. 7–8. For (1.1), see Preece (2); for (1.2) and (1.3), Hardy (**4**, **8**); for (1.4), Hardy (**4**), Whipple (**1**) and Watson (**7**). All these formulae are special cases of much more general formulae discussed in Bailey's Cambridge Tract *Generalised hypergeometric series* (no. 32, 1935). See also Lecture VII.

For (1.5) and (1.6), see no. 11 of the *Papers*, and Hardy (**5**, **6**).

There are formulae of the same type as (1.7) in a paper by Hardy in the *Quarterly Journal of Math.* 35 (1904), 193–207. The formula (1.7) itself is proved by Preece (**2**). See also no. 11 of the *Papers*.

For (1.8), see Watson (**2**); for (1.9), Preece (**4**); for (1.10)–(1.12), Watson (**4**, **6**); and for (1.13), Watson (**8**).

As regards (1.15), see Landau, *Archiv der Math. und Physik* (3), 13 (1908), 305–315, and Stanley (**1**). Miss Stanley shows just how Ramanujan's statement is misleading. See also Lecture IV (B).

p. 10. The Librarian of the University of Madras has very kindly sent me a copy of the catalogue published in 1914, which makes it plain that the library was better equipped than I had supposed. For example it possessed two standard French treatises on elliptic functions (Appell and Lacour, Tannery and Molk) as well as the books of Cayley and Greenhill. It seems plain from other evidence that Ramanujan knew something of the English books but nothing of the French ones.

p. 11. Euler found the general *rational* solution of (1.17), and his solution was afterwards simplified by Binet and other writers. See, for example, Hardy and Wright, 198–202. A number of special solutions similar to (1.18) will be found in

Dickson's *History*, ii, 500 *et seq.* The solution (1.19) is substantially the same as one found by J. R. Young (Dickson, *History*, ii, 554).

The simplest known solution of

$$x^4 + y^4 = z^4 + t^4$$

is Euler's $$158^4 + 59^4 = 134^4 + 133^4 = 635318657.$$

See Dickson, *Introduction*, 60–62, and *History*, ii, 644–647. Euler gave a solution involving two parameters, but no 'general' solution is known.

There is a proof of von Staudt's theorem, due to R. Rado, in Hardy and Wright, 89–92.

The quotation from Littlewood is from his review of the *Papers*.

p. 12. The general theory of the representation of numbers as sums of an even number of squares is discussed in Lecture IX. For Legendre's 'three square' theorem see Landau, *Vorlesungen*, i, 114–122.

p. 13. Laguerre's formula (1.26) is formula (18) of Ch. XII of the 'second edition' of Ramanujan's notebooks. See Watson (**10**), 146.

For 'the most imposing' formula see Watson (**14**).

p. 14. The equation (1.29) was rediscovered by Rogers, *Proc. London Math. Soc.* (2), 4 (1907), 169–189, and is attributed to him in the *Papers*, 337; but it is to be found in a posthumous fragment of Abel (*Œuvres*, ii, 193).

p. 15. For (1.30) see Lecture XI.

p. 17. Skewes, *Journal London Math. Soc.* 8 (1933), 277–283. Skewes assumes the truth of the Riemann Hypothesis, but he has since found a (much larger) value for X independent of the hypothesis. This work is still unpublished.

p. 18. For (1.40) see, for example, Hardy and Wright, 342; Ingham, 20; Landau, *Handbuch*, 75–76.

II

RAMANUJAN AND THE THEORY OF PRIME NUMBERS

2.1. I shall begin by discussing Ramanujan's work in the "analytic" theory of numbers, and particularly on its most famous problem, the problem of the distribution of the primes. I told you that his work in this field has little permanent value, but I am not afraid that you will find what I have to say the less interesting for that. The problem is one of the most fascinating in the whole of mathematics, and Ramanujan's attack on it was conceived in a thoroughly interesting way; and I have unpublished manuscripts to refer to, and can explain to you for the first time just where and how he failed.

Ramanujan wrote about the problem in both of his first two letters. He did not write at length, and I can quote what he says in full.

16 *Jan.* 1913

In p. 36[1] it is stated that "the number of prime numbers less than x is

$$\int_2^x \frac{dt}{\log t} - \rho(x),^2$$

where the precise order of $\rho(x)$ has not been determined".

I. I have found a function which exactly represents the number of prime numbers less than x, "exactly" in the sense that the difference between the function and the actual number of primes is generally 0 or some small finite value even when x becomes infinite. I have got the function in the form of infinite series and have expressed it in two ways.

(1) In terms of Bernoullian numbers. From this we can easily calculate the number of prime numbers up to 100 millions, with generally no error and in some cases with an error of 1 or 2.

(2) As a definite integral from which we can calculate for all values.

I have observed that $\rho(e^{2\pi x})$ is of such a nature that its value is very small when x lies between 0 and 3 (its value is less than a few hundreds when $x = 3$) and rapidly increases when x is greater than 3.

II. I have also got expressions to find the actual number of prime numbers of the form $An + B$.

[1] Of my Cambridge Track *Orders of infinity* (no. 12, ed. 2, 1924).

[2] I have altered the sign of the second term so as to agree with Ramanujan's later assertions.

The difference between the number of prime numbers of the form $4n-1$ and which are less than x, and those of the form $4n+1$ less than x, is infinite when x becomes infinite....

29 *Feb.* 1913

1. The number of prime numbers less than x is

$$\int_0^\infty \frac{y^t}{t\zeta(t+1)\Gamma(t+1)}dt, \tag{2.1.1}$$

where

$$\zeta(t+1) = \frac{1}{1^{t+1}} + \frac{1}{2^{t+1}} + \dots \tag{2.1.2}$$

and $y = \log x$.

2. The number of prime numbers less than x is

$$\frac{2}{\pi}\left\{\frac{2}{B_2}\left(\frac{\log x}{2\pi}\right) + \frac{4}{3B_4}\left(\frac{\log x}{2\pi}\right)^3 + \frac{6}{5B_6}\left(\frac{\log x}{2\pi}\right)^5 + \dots\right\}, \tag{2.1.3}$$

where $B_2 = \frac{1}{6}$, $B_4 = \frac{1}{30}$, ..., the Bernoullian numbers.

3. The number of prime numbers less than x is

$$\int_c^x \frac{dt}{\log t} - \frac{1}{2}\int_c^{\sqrt{x}} \frac{dt}{\log t} - \frac{1}{3}\int_c^{\sqrt[3]{x}} \frac{dt}{\log t} - \frac{1}{5}\int_c^{\sqrt[5]{x}} \frac{dt}{\log t} + \frac{1}{6}\int_c^{\sqrt[6]{x}} \frac{dt}{\log t} - \dots, \tag{2.1.4}$$

where $c = 1{\cdot}45136380$ nearly ....[1]

I have also found expressions for the number of prime numbers of a given form (say of the form $24n+17$) less than any given number.

Primes of the form $4n+1$ = primes of the form $6n+1$,
.... $4n-1$ = ... $6n-1$,
.... $8n+1$ = ... $12n+1$.

Those of the forms $8n+3$, $8n+5$, $8n+7$, $12n+5$, $12n+7$, and $12n+11$ are all equal.

But

(primes of the form $4n-1$) − (those of the form $4n+1$) $\to \infty$,
(...................... $6n-1$) − (...................... $6n+1$) $\to \infty$,
(...................... $8n+3$) − (...................... $8n+1$) $\to \infty$,
(...................... $12n+5$) − (...................... $12n+1$) $\to \infty$.

[1] Ramanujan proceeds to explain the law of formation of the series. It is of course

$$\Sigma \frac{\mu(m)}{m}\int_c^{x^{1/m}} \frac{dt}{\log t}, \tag{2.1.5}$$

where $\mu(m)$ is the Möbius function which is $(-1)^\rho$ when m is "quadratfrei" (a product of ρ different primes) and 0 otherwise.

He continues with instructions concerning computation from the series (see *Papers*, 351).

I have not merely shown that the difference tends to infinity, but found out expressions (like those for prime numbers) for the difference, within any given number..........

We must read the second letter in the light of the first; the formulae (2.1.1), (2.1.3) and (2.1.4) are naturally not exact.[1] Ramanujan defines three functions, the integral (2.1.1), which I will call $J(x)$, and the two series (2.1.3) and (2.1.4), which I will call $G(x)$ and $R(x)$ respectively, and asserts that they differ boundedly from $\pi(x)$: if $F(x)$ is any one of the three functions, then

(2.1.6) $$\pi(x) = F(x) + O(1).$$

The series $R(x)$ is a famous series which occurs in Riemann's work, and a series much like $G(x)$ was found by Gram. The integral $J(x)$ had never, so far as I know, appeared before; and in any case I am sure (for reasons which I will state later) that Ramanujan had found all three functions for himself.

Ramanujan's series and integral

2.2. The series used by Gram is not $G(x)$ but

(2.2.1) $$g(x) = 1 + \sum_{1}^{\infty} \frac{(\log x)^n}{n\zeta(n+1)\Gamma(n+1)},$$

and $G(x)$ is twice the sum of its odd terms.[2] This series can (as no doubt Ramanujan knew) be shown to be identical with $R(x)$.

We require some preliminary observations about $R(x)$, which I will write in the form (2.1.5).[3] It is known that

(2.2.2) $$\sum \frac{\mu(m)}{m} = 0$$

and

(2.2.3) $$\sum \frac{\mu(m)}{m} \log m = -1$$

(though even the first of these equations is as "deep" as the Prime Number Theorem). It follows from (2.2.2) that the value of c in (2.1.4) is immaterial (so long as it is not 1). In particular, if we define the logarithm integral $\operatorname{li} x$, when $x > 1$, by

$$\operatorname{li} x = \int_0^x \frac{dt}{\log t} = \lim_{\epsilon \to 0}\left(\int_0^{1-\epsilon} + \int_{1+\epsilon}^{x}\right)\frac{dt}{\log t}$$

[1] Since $\pi(x)$ is discontinuous.

[2] Since $$\zeta(2k) = \frac{2^{2k-1}\pi^{2k}}{2k!} B_{2k}$$

(I follow Ramanujan's notation for the Bernoullian numbers, writing B_{2k} where B_k is more usual.)

[3] See the footnote to p. 23.

(a "principal value" in Cauchy's sense), then we may write (2.1.5) in the more usual form

(2.2.4) $$R(x) = \Sigma \frac{\mu(m)}{m} \operatorname{li} x^{1/m}$$

It is known that

(2.2.5) $$\operatorname{li} x = \gamma + \log\log x + \sum_1^\infty \frac{(\log x)^n}{n.n!},$$

where γ is Euler's constant. Hence

(2.2.6) $$\operatorname{li} x^{1/m} = \gamma + \log\log x - \log m + \sum_1^\infty \frac{(\log x)^n}{n.n!m^n}$$

$$= -\log m + \gamma + \log\log x + O\left(\frac{1}{m}\right)$$

when $m \to \infty$, and the convergence of (2.2.4) depends upon that of (2.2.2) and (2.2.3).

If we substitute from (2.2.6) into $R(x)$, and remember the formulae (2.2.2) and (2.2.3), we obtain

$$R(x) = (\gamma + \log\log x)\Sigma\frac{\mu(m)}{m} - \Sigma\frac{\mu(m)}{m}\log m + \sum_m \frac{\mu(m)}{m}\sum_n \frac{(\log x)^n}{n.n!m^n}$$

$$= 1 + \sum_n \frac{(\log x)^n}{n.n!}\sum_m \frac{\mu(m)}{m^{n+1}} = 1 + \Sigma \frac{(\log x)^n}{n\zeta(n+1)\,\Gamma(n+1)} = g(x),^{[1]}$$

so that the sums of Riemann's and Gram's series are the same. Also, if we write $\log x = y$ and

$$g(x) = h(\log x) = h(y) = 1 + \sum_1^\infty \frac{y^n}{n.n!\,\zeta(n+1)},$$

then $$G(x) = h(y) - h(-y),$$

and it is not difficult to show that

(2.2.7) $$h(-y) \to 0$$

when $y \to \infty$.[2] Hence $G(x) = R(x) + o(1)$,

and Ramanujan's series is equivalent to Gram's.

We can also prove that

(2.2.8) $$J(x) = G(x) + o(1);$$

but the proof, which depends upon the formula

$$\int_0^\infty \frac{a^{x+1}}{\Gamma(x+1)}dx = e^a - \int_0^\infty \frac{e^{-ax}\,dx}{x\{\pi^2 + (\log x)^2\}},$$

is a little more difficult. If we may take this for granted, we can conclude that *Ramanujan's three approximations are equivalent.* I may therefore confine myself, in what follows, to the most familiar function, $R(x)$.

[1] All summations are from 1 to ∞.

[2] See the notes at the end of the lecture.

The series $R(x)$

2.3. Ramanujan's assertions amount to this, that

$$\pi(x) - R(x) = O(1), \tag{2.3.1}$$

i.e. that $\pi(x) - R(x)$ is bounded. In this case

$$\pi(x) - R(x) = O(x^{\delta}) \tag{2.3.2}$$

for every positive δ, so that the error is less important than any term of $R(x)$. It would be an easy deduction that

$$\pi(x) = \operatorname{li} x + O(x^{\frac{1}{2}+\delta}), \tag{2.3.3}$$

$$\pi(x) = \operatorname{li} x - \tfrac{1}{2} \operatorname{li} x^{\frac{1}{2}} + O(x^{\frac{1}{3}+\delta}), \tag{2.3.4}$$

and so on, for every positive δ. In particular it would follow that

$$\pi(x) - \operatorname{li} x \to -\infty \tag{2.3.5}$$

when $x \to \infty$. I may as well say at once that, of these assertions, (2.3.1), (2.3.2), (2.3.4) and (2.3.5) are certainly false, while (2.3.3) stands or falls with the Riemann Hypothesis.

Ramanujan's main thesis is false, but it is astonishing how well it fits the facts. The primes are tabulated up to 10,000,000: and the value of $\pi(x)$ has been found for a number of much larger values of x. The table shows some of the values of $\pi(x)$, and the errors of $\operatorname{li} x$ and $R(x)$ by excess or default.

x	$\pi(x)$	$\operatorname{li} x$	$R(x)$
100,000	9,592	+ 38	− 5
1,000,000	78,498	+ 130	+ 30
2,000,000	148,933	+ 122	− 9
3,000,000	216,816	+ 155	0
4,000,000	283,146	+ 206	+ 33
5,000,000	348,513	+ 125	− 64
6,000,000	412,849	+ 228	+ 24
7,000,000	476,648	+ 179	− 38
8,000,000	539,777	+ 223	− 6
9,000,000	602,489	+ 187	− 53
10,000,000	664,579	+ 339	+ 88
100,000,000	5,761,455	+ 755	+ 97
1,000,000,000	50,847,478	+ 1758	− 23

The largest error known in $R(x)$ is $+228$ for $x = 90{,}000{,}000$. The agreement is very striking and (though Ramanujan had not facts on this scale to mislead him[1]) it is not at all surprising that he should have been misled.

[1] He quotes only results for quite small x; errors $-0{\cdot}1$, $-0{\cdot}1$, $+0{\cdot}2$ for $x = 50, 300, 1000$ (*Papers*, 351).

The early history of the theory of primes

2.4. Ramanujan, of course, had not merely "guessed" his theorems; no flight of pure imagination could carry any man so far. He had a "proof", a definite, and very ingenious, train of reasoning. This I intend to explain, but it is essential that I should give first a rapid sketch of the history and structure of the "classical" theory.

The Prime Number Theorem,

$$\pi(x) \sim \frac{x}{\log x}, \tag{2.4.1}$$

was conjectured independently by Legendre and by Gauss. Gauss gave the approximation $\operatorname{li} x$, now known to be more accurate, but neither author is very explicit about the degree of accuracy which they claimed for their formulae. As I said in my first lecture, (2.4.1) is equivalent to $p_n \sim n \log n$.

The first definite progress was made by Tchebychef. Tchebychef proved that

$$\frac{Cx}{\log x} < \pi(x) < \frac{Cx}{\log x}, \tag{2.4.2}$$ [1]

or (what is the same thing) that

$$Cn \log n < p_n < Cn \log n. \tag{2.4.3}$$

He also introduced the functions

$$\vartheta(x) = \sum_{p \leq x} \log p, \quad \psi(x) = \sum_{p^m \leq x} \log p. \tag{2.4.4}$$ [2]

These functions are in some ways more *natural* than $\pi(x)$. Thus

$$\vartheta(x) = \log\left(\prod_{p \leq x} p\right)$$

(and the most natural operation to perform on a set of primes is to *multiply* them); and $\psi(x)$ is the logarithm of the least common multiple of the numbers up to x. It is the more complex function $\psi(x)$, as we shall see, which presents itself most naturally in the analytic theory. The number of squares, cubes, ... of primes up to x does not exceed

$$x^{\frac{1}{2}} + x^{\frac{1}{3}} + x^{\frac{1}{4}} + \ldots = O(x^{\frac{1}{2}} \log x),$$ [3]

[1] I use C generally for an "absolute constant" (a number such as 7 or π). The various C's are naturally not equal.

[2] Count $\log p$ for every p, p^2, ... up to x. Thus

$$\begin{aligned}\psi(10) &= \log 2 + \log 3 + \log 2 + \log 5 + \log 7 + \log 2 + \log 3 \\ &= 3 \log 2 + 2 \log 3 + \log 5 + \log 7\end{aligned}$$

($\log 2$, for example, arising from 2, $2^2 = 4$, $2^3 = 8$).

[3] Since $p^m > x$ if $m > \log x / \log 2$, we need take only $O(\log x)$ terms of the series.

which is insignificant compared with functions whose order is about x. Hence powers of primes are comparatively unimportant in the theory, and we may think of $\vartheta(x)$ and $\psi(x)$ as, for our present purpose, substantially the same.

It is natural to expect that

$$\log x \, . \, \pi(x) \sim \vartheta(x) \sim \psi(x), \tag{2.4.5}$$

since (i) $\vartheta(x)$ and $\psi(x)$ are "much the same", and (ii) $\vartheta(x)$ contains $\pi(x)$ terms in most of which $\log p$ is nearly $\log x$. Tchebychef gave an accurate proof of (2.4.5), and inferred that

$$\lim \frac{\log x \, . \, \pi(x)}{x} = \lim \frac{\vartheta(x)}{x} = \lim \frac{\psi(x)}{x} \tag{2.4.6}$$

if any one of the three limits exists, and that the Prime Number Theorem is equivalent to either of

$$\vartheta(x) \sim x, \quad \psi(x) \sim x. \tag{2.4.7}$$

Finally, he proved that the only possible value of the limits is 1; but he could not overcome the fundamental difficulty, the proof of the existence of the limits.

The proof of the Prime Number Theorem

2.5. The Prime Number Theorem was finally proved by Hadamard and de la Vallée-Poussin in 1896. The analytic attack upon the problem depends on Euler's identity

$$\zeta(s) = \sum_n n^{-s} = \prod_p \frac{1}{1-p^{-s}}, \tag{2.5.1}$$

where $\Re s > 1$. It follows from (2.5.1) that

$$\log \zeta(s) = \sum_p \log \frac{1}{1-p^{-s}} = \sum_{p,m} \frac{1}{mp^{ms}},$$

the summation being over all primes p and all positive integers m. Hence

$$-\frac{\zeta'(s)}{\zeta(s)} = \sum_{p,m} \frac{\log p}{p^{ms}} = \sum a_n n^{-s}, \tag{2.5.2}$$

where

$$a_n = \log p \qquad (n = p^m),$$

$$a_n = 0 \qquad (\textit{otherwise}).$$

Here a_n is the arithmetical function usually denoted by $\Lambda(n)$, and the "sum function"

$$A(x) = \sum_{n \leq x} a_n$$

of a_n is $\psi(x)$.

We now require a general theorem in the theory of Dirichlet's series. Suppose that $s = \sigma + it$, and that

$$f(s) = \Sigma a_n n^{-s}$$

is absolutely convergent for $\sigma > 1$. Then

$$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{y^s}{s}\, ds$$

is $$0,\ \tfrac{1}{2},\ 1$$

in the three cases $0 < y < 1$, $y = 1$, $y > 1$;[1] and it is easily proved that

$$\frac{1}{2\pi i}\int f(s)\frac{x^s}{s}\,ds = \Sigma\, a_n \int \left(\frac{x}{n}\right)^s \frac{ds}{s} = \sum_{n\leq x}{}' a_n = A^*(x),$$

where $A^*(x)$ differs from $A(x)$ only in that, when x is an integer, the last term a_n in $A(x)$ is to be multiplied by $\frac{1}{2}$. In particular

$$\psi^*(x) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} f(s)\frac{x^s}{s}\,ds, \tag{2.5.3}$$

where $c > 1$ and $$f(s) = -\frac{\zeta'(s)}{\zeta(s)}.$$

This formula contains the "analytic set-up" of the prime number problem. In what follows I shall give two approximations to the solution, the first so rough that, in the present state of knowledge, it cannot be carried through.

It is known that $\zeta(s)$ is an analytic function of s, which has a simple pole of the type

$$\frac{1}{s-1} + \ldots$$

at $s = 1$, but is otherwise regular. It satisfies Riemann's equation

$$\zeta(1-s) = 2(2\pi)^{-s}\cos\tfrac{1}{2}s\pi\,\Gamma(s)\,\zeta(s).$$

It has simple zeros (the "trivial" zeros) at

$$s = -2,\ -4,\ -6,\ \ldots$$

and an infinity of complex zeros ρ for all of which $0 < \sigma < 1$. The "Riemann Hypothesis" is the hypothesis that all the ρ have the real part $\frac{1}{2}$.

The function (2.5.2) has also a pole of type

$$\frac{1}{s-1} + \ldots$$

at $s = 1$, and poles of types

$$-\frac{1}{s-2n} + \ldots, \qquad -\frac{1}{s-\rho} + \ldots$$

[1] The integral is a principal value at infinity (i.e. the limit of an integral from $c - iT$ to $c + iT$) when $x = 1$.

at the zeros of $\zeta(s)$. Let us assume that the Riemann Hypothesis is true, and that the behaviour of $f(s)$, when $\sigma > \frac{1}{2}$ and $|t| \to \infty$, is "not too bad". Then it is natural to suppose that we can move the contour of integration in (2.5.3) across the pole at $s = 1$, and deduce

(2.5.4) $$\psi^*(x) = x + \frac{1}{2\pi i}\int_{\gamma - i\infty}^{\gamma + i\infty} f(s)\frac{x^s}{s}\,ds,$$

where $\frac{1}{2} < \gamma < 1$. The x is the residue arising from the pole at $s = 1$. Further, since every element of the integrand is of order x^γ in x, it is natural to suppose that the integral is of much the same order, and that

(2.5.5) $$\psi^*(x) = x + O(x^\beta)$$

for every $\beta > \frac{1}{2}$. Since the difference between $\psi^*(x)$ and $\psi(x)$ is at most $\log x$, this gives the Prime Number Theorem (and a good deal more). The argument cannot be developed accurately on quite the simple lines which I have sketched, but (2.5.5) is actually a correct conclusion from the Riemann Hypothesis.

It is easy to deduce from (2.5.5) that

(2.5.6) $$\pi(x) = \operatorname{li} x + O(x^\beta),$$

again for any $\beta > \frac{1}{2}$. For (2.5.5) is equivalent to

(2.5.7) $$\vartheta(x) = x + O(x^\beta).$$

Also $$\pi(x) = 1 + \int_2^x \frac{d\vartheta(t)}{\log t}\text{[1]}$$

and so
$$\begin{aligned}
\pi(x) - \operatorname{li} x &= \int_2^x \frac{d\vartheta(t)}{\log t} - \int_2^x \frac{dt}{\log t} + O(1) \\
&= \int_2^x \frac{d\{\vartheta(t) - t\}}{\log t} + O(1) \\
&= \frac{\vartheta(x) - x}{\log x} - \frac{\log 2 - 2}{\log 2} + \int_2^x \frac{\vartheta(t) - t}{t(\log t)^2}\,dt + O(1) \\
&= O\left(\frac{x^\beta}{\log x}\right) + O\left\{\int_2^x \frac{t^{\beta-1}}{(\log t)^2}\right\} + O(1) \\
&= O(x^\beta),
\end{aligned}$$

which is (2.5.6).

The function

(2.5.8) $$\Pi(x) = \pi(x) + \tfrac{1}{2}\pi(x^{\frac{1}{2}}) + \tfrac{1}{3}\pi(x^{\frac{1}{3}}) + \ldots$$

[1] It is convenient to use the notation of the Stieltjes integral: $\vartheta(x)$ is the step-function with jumps $\log p$ at the points $x = p$.

is related to $\psi(x)$ much as is $\pi(x)$ to $\vartheta(x)$. Since

$$\tfrac{1}{2}\pi(x^{\frac{1}{2}})+\tfrac{1}{3}\pi(x^{\frac{1}{3}})+\ldots \leqq x^{\frac{1}{2}}+x^{\frac{1}{3}}+\ldots = O(x^{\frac{1}{2}}\log x),$$

we have also

(2.5.9) $$\Pi(x) = \operatorname{li} x + O(x^{\beta}).$$

Second approximation to the proof

2.6. This is the "first approximation" to the proof of the Prime Number Theorem, but it is essential that we should get a little closer to the truth if we are to understand Ramanujan's attempt. In my second approximation I shall discard the unproved Riemann Hypothesis; but that will not be the most important improvement.

The main difficulty in my first "proof" lay, not in the assumption of the Riemann Hypothesis, but in the naïve argument about the "order" of the integral in (2.5.4). The fundamental difficulty is that the integral is not absolutely convergent, and that we cannot be sure that its order is not much greater than that of any of its elements. We avoid this difficulty by considering not $\psi^*(x)$ itself but *an average of* $\psi^*(x)$ or $\psi(x)$, and finding an asymptotic formula for this average. We have then to infer back from the average to the function itself, and this introduces a new element, a "Tauberian" element, into the proof.

It is most convenient to state the argument in terms of general Dirichlet's series, and I shall prove the following general theorem. Suppose

(i) that

(2.6.1) $$f(s) = \Sigma a_n n^{-s}$$

is absolutely convergent for $\sigma > 1$;

(ii) that $f(s)$ is regular on the line $\sigma = 1$, except for a simple pole, with residue 1, at $s = 1$

(iii) that

(2.6.2) $$f(\sigma+it) = O(|t|^{\alpha}),$$

where $\alpha < 1$, for $\sigma \geqq 1$ and large $|t|$;

(iv) that

(2.6.3) $$a_n > -K,$$

where K is a constant. Then

(2.6.4) $$A(x) \sim x.$$

We write

(2.6.5) $$A_1(x) = \int_0^x A(y)\,dy = \int_0^x A^*(y)\,dy.$$[1]

[1] $A(y)$ and $A^*(y)$ differ only at isolated points.

From
$$A^*(x) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} f(s)\frac{x^s}{s}\,ds \qquad (c>1)$$
it follows by integration that

(2.6.6)
$$A_1(x) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} f(s)\frac{x^{s+1}}{s(s+1)}\,ds.$$

We have also
$$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \zeta(s)\frac{x^s}{s}\,ds = x+O(1),[1]$$

(2.6.7)
$$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \zeta(s)\frac{x^{s+1}}{s(s+1)}\,ds = \tfrac{1}{2}x^2+O(x).$$

Subtracting (2.6.7) from (2.6.6), we obtain

(2.6.8)
$$A_1(x)-\tfrac{1}{2}x^2+O(x) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} g(s)\frac{x^{s+1}}{s\,(s+1)}\,ds,$$
where

(2.6.9)
$$g(s) = f(s)-\zeta(s);$$
and a simple continuity argument enables us to take $c = 1$. Thus

(2.6.10)
$$A_1(x)-\tfrac{1}{2}x^2+O(x) = \frac{1}{2\pi i}\int_{1-i\infty}^{1+i\infty} g(s)\frac{x^{s+1}}{s(s+1)}\,ds.[2]$$

Now it is easy to prove that
$$\zeta(s) = O(|t|^\alpha),[3]$$
where $0<\alpha<1$, i.e. that $\zeta(s)$, and therefore $g(s)$, satisfies condition (iii) of the theorem. Hence the integral in (2.6.10) is of the form
$$x^2\int_{-\infty}^{\infty} O\left(\frac{|t|^\alpha}{1+t^2}\right)x^{it}\,dt,$$
the product of x^2 by an integral of the type
$$\int_{-\infty}^{\infty} H(t)\,e^{it\log x}\,dt = \int_{-\infty}^{\infty} H(t)\,e^{i\xi t}\,dt,$$
where $\int |H(t)|\,dt<\infty$. Such an integral tends to 0 when $\xi = \log x\to\infty$,[4] and therefore
$$A_1(x)-\tfrac{1}{2}x^2 = o(x^2),$$
or

(2.6.11)
$$A_1(x)\sim\tfrac{1}{2}x^2.$$

[1] Exactly, $[x]$ if x is not an integer, $x-\frac{1}{2}$ if it is.

[2] We must subtract (2.6.7) from (2.6.6) before putting $c = 1$, since $f(s)$ and $\zeta(s)$ each have a pole at $s = 1$.

[3] In fact $\zeta(s) = O(\log|t|)$: see Ingham, 27, or Landau, *Handbuch*, 169.

[4] By the "Riemann-Lebesgue theorem" for trigonometrical integrals: see, for example, Titchmarsh, *Fourier integrals*, 11 (Theorem 1).

2.7. We have now to pass from (2.6.11) to (2.6.4). This requires a "Tauberian" argument, here of a very simple kind. If

(2.7.1) $$b_n = a_n - 1,$$

then $$B(x) = \sum_{n \leq x} (a_n - 1) = A(x) - x + O(1),$$

$$B_1(x) = \int_0^x B(y)\,dy = A_1(x) - \tfrac{1}{2}x^2 + O(x),$$

and so

(2.7.2) $$B_1(x) = o(x^2);$$

and

(2.7.3) $$b_n > -K - 1 = -L,$$

say, by (2.6.3) and (2.7.1). We have to prove that

(2.7.4) $$B(x) = o(x).$$

Suppose that (2.7.4) is false. Then there is a positive δ such that one or other of

(2.7.5) $$B(x) > \delta x, \quad B(x) < -\delta x$$

is true for arbitrarily large values of x.

Take, for example, the first hypothesis, and suppose that $B(\xi) > \delta\xi$. Then

$$B(x) - B(\xi) = \sum_{\xi < n \leq x} b_n > -L(x - \xi)$$

and $$B(x) > \delta\xi - L(x - \xi) > \tfrac{1}{2}\delta\xi$$

for $$\xi < x < \xi' = \left(1 + \frac{\delta}{2L}\right)\xi.$$

Hence $$B_1\left(\xi + \frac{\delta\xi}{2L}\right) - B_1(\xi) > \int_\xi^{\xi'} \tfrac{1}{2}\delta\xi\,dx = \frac{\delta^2}{4L}\xi^2.$$

But this contradicts (2.7.2), since each term on the left is $o(\xi^2)$.

We can deduce a contradiction similarly[1] from the second hypothesis (2.7.5), and the two contradictions prove (2.7.4).

In order to deduce the Prime Number Theorem we must show that the function (2.5.2) satisfies the conditions (ii), (iii) and (iv) of § 2.6. The last condition is satisfied because $A(n) \geqq 0$. Condition (ii) is equivalent to

(2.7.6) $$\zeta(1 + it) \neq 0,$$

and (iii) asserts rather more. It is the verification of these conditions that is the main difficulty of the proof.

[1] But now using an interval (ξ', ξ) to the left of ξ.

Later developments

2.8. There are two chapters in this proof (and in all other proofs) of the Prime Number Theorem. The first part of the proof is properly function-theoretic. We show that $\zeta(s)$ has certain properties, and deduce an asymptotic formula for

$$\frac{\psi_1(x)}{x} = \frac{1}{x}\int_0^x \psi(t)\,dt$$

or for some other average of $\psi(x)$. The second part is "Tauberian". The difficulties are divided differently, in different proofs, between the two chapters. In the "classical" proof, which I have sketched to you, the first chapter is difficult and the second easy, only a very simple Tauberian theorem being required. Later proofs simplify the first chapter at the expense of the second.

The most important of these more recent developments is due, in essentials, to Wiener. Wiener and his followers have shown that we may simply strike out the condition (iii) in the general theorem of § 2.6. We still need (ii); in particular, in the application to primes, we must still prove (2.7.6); but we do not require to know anything at all about the behaviour of $\zeta(s)$ at infinity. It had for long been said, a little vaguely, that "the Prime Number Theorem is equivalent to the assertion that $\zeta(s)$ has no zeros on $\sigma = 1$", but Wiener has enabled us now to interpret this statement literally, and this is naturally a very important contribution to the logic of prime number theory.

What is relevant now, however, is not Wiener's generalisation, the "true" generalisation of the theorem, but a half-way generalisation made by Littlewood and myself in 1915. This half-way theorem has lost its significance since its supersession by Wiener's, but it is just what is wanted for the comprehension of Ramanujan's work.

Littlewood and I found ourselves in possession of a powerful Tauberian theorem: *if*

(2.8.1) $$\Sigma a_n e^{-ny} \sim \frac{1}{y}$$

when $y \to 0$, *and*

(2.8.2) $$a_n \geqq 0,$$

then $$A(x) \sim x.$$

This has an immediate application to the Prime Number Theorem, since, if we can prove that

(2.8.3) $$\sum_2^\infty \Lambda(n)\, e^{-ny} \sim \frac{1}{y},$$

it will follow that $\psi(x) \sim x$. Since

$$(1-e^{-y})\,\Sigma a_n e^{-ny} = (1-e^{-y})^2\,\Sigma(a_0+a_1+\ldots+a_n)\,e^{-ny}$$

$$= \frac{A(0)+A(1)\,e^{-y}+\ldots+A(n)\,e^{-ny}+\ldots}{1+2e^{-y}+\ldots+(n+1)\,e^{-ny}+\ldots}$$

and $1-e^{-y} \sim y$, we can regard

$$y\,\Sigma a_n e^{-ny}$$

as a kind of average of $A(n)/n$. Hence a function-theoretic proof of (2.8.3) will lead to a proof of the Prime Number Theorem conforming to my general description.

Littlewood and I, then, proceeded as follows. In the first place, we proved an extension of the general theorem; we showed that condition (iii) of § 2.6 may be replaced by

$$f(\sigma+it) = O(e^{A|t|}), \tag{2.8.4}$$

for any positive constant A. This of course has no importance now (since in fact *no* condition of this kind is wanted). Secondly, we deduced (2.8.3); and finally we applied our Tauberian theorem. It is on (2.8.3) that I want you to fix your attention, since this was also Ramanujan's first objective.

Ramanujan's argument

2.9. I can now pass to Ramanujan's actual "proof", which he showed me some time after his arrival in England. I shall be able to make the points of it clearer if I allow myself to misinterpret his object a little.

Ramanujan was trying to prove, not merely the Prime Number Theorem, not even merely a result like (2.5.6), which is true on the Riemann Hypothesis, but a much more precise result which we know to be false. He committed a definite, and a very curious, fallacy: his argument is not merely "unrigorous" but in a more drastic sense "unsound", and I want to make his mistake quite plain. I can do this more easily, and without doing him the slightest injustice, by talking as if his aims had been limited to the proof of the Prime Number Theorem.

Ramanujan writes

$$\phi(y) = \sum_p \log p \sum_{m=1}^{\infty} e^{-p^m y} - \log 2 \sum_{m=1}^{\infty} 2^m e^{-2^m y} \tag{2.9.1}$$

$$= \phi_1(y) - \phi_2(y),$$

and sets out to prove, first, that

$$\phi_1(y) \sim \frac{1}{y}. \tag{2.9.2}$$

He does not use the notation $\Lambda(n)$, but actually

(2.9.3) $$\phi_1(y) = \sum_2^\infty \Lambda(n)\, e^{-ny},$$

so that the Prime Number Theorem would really follow from (2.9.2).

Next, he writes

(2.9.4) $$\Phi(y) = \phi(y) - \phi(2y) + \phi(3y) - \ldots = \Phi_1(y) - \Phi_2(y),$$

where Φ_1 and Φ_2 are the functions related to ϕ_1 and ϕ_2 as Φ is to ϕ. Then

(2.9.5) $$\Phi_1(y) = \sum_p \log p \sum_{m=1}^\infty \frac{e^{-p^m y}}{1+e^{-p^m y}} = \sum_2^\infty \Lambda(n) \frac{e^{-ny}}{1+e^{-ny}};$$

and

(2.9.6) $$\Phi_2(y) = \log 2 \sum_{m=1}^\infty \frac{2^m e^{-2^m y}}{1+e^{-2^m y}} = 2\log 2 \frac{e^{-2y}}{1-e^{-2y}},$$

by the elementary identity $\dfrac{2}{e^{2y}+1} + \dfrac{4}{e^{4y}+1} + \dfrac{8}{e^{8y}+1} + \ldots = \dfrac{2}{e^{2y}-1}$.

2.10. Ramanujan now makes a transformation of $\Phi_1(y)$. We have

(2.10.1) $$\Phi_1(y) = \Psi_1(y) - 2\Psi_1(2y),$$

where

(2.10.2) $$\Psi_1(y) = \Sigma \Lambda(n) \frac{e^{-ny}}{1-e^{-ny}}.$$

But

(2.10.3) $$\Psi_1(y) = \sum_{n=2}^\infty \Lambda(n) \sum_{m=1}^\infty e^{-mny} = \sum_2^\infty c_k e^{-ky},$$

where

(2.10.4) $$c_k = \sum_{n|k} \Lambda(n).$$

Also, if $k = \Pi p^a$,

$$\sum_{n|k} \Lambda(n) = \sum_{p|k} a \log p = \sum_{p|k} \log p^a = \log k.^{1}$$

Hence $$\Psi_1(y) = \sum_2^\infty \log k\, e^{-ky}$$

and

(2.10.5) $$\begin{aligned}\Phi_1(y) &= \sum_2^\infty \log k\, e^{-ky} - 2\sum_2^\infty \log k\, e^{-2ky} \\ &= \sum_2^\infty \log k\, e^{-ky} - 2\sum_2^\infty \log 2k\, e^{-2ky} + 2\log 2 \sum_2^\infty e^{-2ky} \\ &= e^{-y}\log 1 - e^{-2y}\log 2 + e^{-3y}\log 3 - \ldots + 2\log 2 \frac{e^{-2y}}{1-e^{-2y}}.\end{aligned}$$

[1] Since we must count $\log p$ for each of the divisors $p, p^2, \ldots, p^a$.

Combining (2.9.6) and (2.10.5), we obtain

(2.10.6) $$\Phi(y) = e^{-y}\log 1 - e^{-2y}\log 2 + e^{-3y}\log 3 - \ldots.$$

Ramanujan now infers that

(2.10.7) $$\Phi(y) \to l,$$ [1]

or

(2.10.8) $$\phi(y) - \phi(2y) + \phi(3y) - \ldots \to l,$$

for some l. He gives no reason, but the conclusion is correct and easily proved.[2] Up to this point his argument, though expressed in a less convenient notation than that which I have used, is quite sound.

Next, Ramanujan infers from (2.10.8) that

(2.10.9) $$\phi(y) \to l.$$

All that would be necessary, if he were aiming at the Prime Number Theorem only, would be the milder conclusion that

(2.10.10) $$\phi(y) = o\left(\frac{1}{y}\right),$$

and we may continue his argument as if he asserted no more than this. He then states that

(2.10.11) $$\phi_2(y) = \log 2 \sum_1^\infty 2^m e^{-2^m y} \sim \frac{1}{y},$$

and from (2.10.10) and (2.10.11) he deduces that

(2.10.12) $$\phi_1(y) = \phi(y) + \phi_2(y) \sim \frac{1}{y},$$

which is (2.8.3). What he actually says and professes to derive from (2.10.9) is that

(2.10.13) $$\phi_1(y) = \frac{1}{y} + O(1),$$

or at any rate

(2.10.14) $$\phi_1(y) = \frac{1}{y} + O(y^{-\delta})$$

for every positive δ

Now (2.8.3) is true; it is, as I said, the half-way stage in the "Hardy-Littlewood" proof; and from (2.8.3) we can deduce the Prime Number Theorem in an "elementary" manner, that is to say by arguments which make no use of the notion of an analytic function of the complex variable.

[1] I use l for "a limit" (not necessarily the same in different contexts).

[2] For example, the series
$$\log 1 - \log 2 + \log 3 - \ldots$$
is summable $(C, 1)$. The sum is $-\frac{1}{2}\log\frac{1}{2}\pi$.

It follows that, if Ramanujan really had proved (2.10.12), he would have found an elementary proof of the Prime Number Theorem, a proof involving no function-theory at all. In particular, he would never have needed (2.7.6); and this is of course enough to convince any reader who knows the subject that the proof cannot possibly be correct. And in fact Ramanujan has deduced the true conclusion from two false propositions, the proposition (2.10.11), and the proposition that (2.10.8) implies (2.10.10).

2.11. I had better show the falsity of these propositions at once. In the first place, (2.10.8) does not imply (2.10.10), and still less (2.10.9). Suppose, for example, that

$$\chi(y) = y^{-1-ai}.$$

Then
$$\chi(y)-\chi(2y)+\chi(3y)-\ldots = y^{-1-ai}(1-2^{-1-ai}+3^{-1-ai}-\ldots)$$
$$= (1-2^{-ai})\,\zeta(1+ai)\,y^{-1-ai},$$

which is 0 if
$$a = \frac{2k\pi}{\log 2};$$

but $y\chi(y)$ oscillates, in contradiction to Ramanujan's statement. It is true that $\chi(y)$ is not a power-series in e^{-y}, as is Ramanujan's $\phi_2(y)$, but we can find such series which mimic the behaviour of $\chi(y)$ as closely as we please, and the statement cannot be rehabilitated by any such reservation.

It is only natural that Ramanujan's argument should contain flaws like this, where his instincts misled him about the validity of difficult general theorems. There are true Tauberian theorems which have some superficial resemblance to the one which I have just refuted, and a good deal of experience and subtlety is needed to distinguish the true from the false. His second error is much more surprising, since one would have expected him to be right about the behaviour of a special function like $\phi_2(y)$.

He seems to have been deceived by an "integral analogy". The integral analogue of the series (2.10.11) is

$$\log 2 \int_0^\infty 2^x e^{-2^x y}\,dx, \tag{2.11.1}$$

and
$$\int_0^\infty 2^x e^{-2^x y}\,dx = \frac{1}{\log 2}\int_1^\infty e^{-yz}\,dz = \frac{e^{-y}}{y\log 2} \sim \frac{1}{y\log 2},$$

so that (2.11.1) behaves in the manner which he attributes to (2.10.11). But (2.10.11) itself behaves differently, having "wobbles" of order $1/y$.

We can refute Ramanujan's assertion in numerous ways. In the first place, if (2.10.11) were true it would follow (by the Hardy-Littlewood Tauberian theorem) that

$$\sum_{2^n \leq x} 2^n \sim \frac{x}{\log 2}.$$

This is plainly false, since the series is practically doubled when x passes through a value 2^m.

A more direct argument is as follows. The function $\phi_2(y)$ satisfies the equation

$$\phi_2(y)-2\phi_2(2y) = 2e^{-2y}\log 2.$$

It may also be verified at once that

$$\psi_2(y) = -\log 2\sum_0^\infty \frac{(-1)^r y^r}{r!}\frac{2^{r+1}}{2^{r+1}-1}$$

satisfies
$$\psi_2(y)-2\psi_2(2y) = 2e^{-2y}\log 2,$$

and therefore
$$h(y) = \phi_2(y)-\psi_2(y)$$

satisfies
$$h(y)-2h(2y) = 0.$$

Also $yh(y)$ is not a constant.[1]

If now we write
$$yh(y) = H(\log y),$$

then
$$H(\log y) = H(\log y+\log 2),$$

so that H is periodic and not constant. Hence $yh(y)$ does not tend to a limit, nor does $y\phi_2(y)$.

Finally we can, if we please, exhibit the "wobbles" in a formula. We can prove that

$$(2.11.2)\quad \phi_2(y) = \frac{1}{y}-\log 2\sum_0^\infty \frac{(-1)^r y^r}{r!}\frac{2^{r+1}}{2^{r+1}-1}-\frac{1}{y}\sum_{-\infty}^{\infty}{}' \Gamma\left(\frac{1+2k\pi i}{\log 2}\right) y^{-2k\pi i/\log 2},$$

where the dash excludes the value $k = 0$; and the last series shows the wobbles, of order $1/y$, explicitly. It converges rapidly, and the wobbles are small compared with the dominant term.

The genesis of Riemann's series

2.12. It will be plain by now that Ramanujan's proof of the Prime Number Theorem was quite wrong. His errors were fundamental; he was wrong not merely because he could not supply the necessary "rigour", but because the path which he followed did not follow the facts. I should like to say that "rigour apart, he found the Hardy-Littlewood proof", but I cannot.

Ramanujan could not prove the true Prime Number Theorem, and naturally he could not prove the false (2.3.1) or (2.3.2). I shall say something later about the way in which he professes to derive them from (2.10.13), but I must first make a few remarks about the status of the series $R(x)$ in the orthodox theory.

[1] $\psi_2(y)$ is a meromorphic function, while $\phi_2(y)$ has a barrier along the imaginary axis. The point could also be settled by calculation.

There are two goals in the theory. One is to prove the Prime Number Theorem, or some refinement upon it. The other is to find an *exact* analytical expression for $\pi(x)$, or one of its associated functions; an expression which may incidentally provide an approximation for $\pi(x)$, but is sought as an end in itself. Riemann (who never even mentions the Prime Number Theorem) attacked the second problem.

It is easy to see how such identities may be derived from the integral (2.5.3). It is natural to suppose that the integral is equal to the sum of the residues at all the poles to the left of the line of integration. These are easily calculated, and we find the formula

$$\psi^*(x) = x - \sum_{\rho} \frac{x^{\rho}}{\rho} - \frac{\zeta'(0)}{\zeta(0)} - \tfrac{1}{2}\log\left(1 - \frac{1}{x^2}\right). \tag{2.12.1}$$

Here the first term arises, as in (2.5.4), from $s = 1$, and the last from the "trivial" zeros. It is fairly easy to deduce a formula for the function $\Pi(x)$ defined by (2.5.8). If $\Pi^*(x)$ is related to $\Pi(x)$ as $\psi^*(x)$ is to $\psi(x)$, then

$$\Pi^*(x) = \operatorname{li} x - \sum_{\rho} \operatorname{li} x^{\rho} + \int_x^{\infty} \frac{du}{(u^2-1)\,u\log u} - \log 2. \tag{2.12.2}$$

The definition of $\operatorname{li} z$ must be extended appropriately to cover complex z.

We pass from $\Pi(x)$ to $\pi(x)$ by one of the "inversion formulae" associated with the Möbius function. If

$$g(\xi) = \sum_{n=1}^{\infty} f\left(\frac{\xi}{n}\right),$$

then (subject to certain reservations about convergence)

$$f(\xi) = \sum_{n=1}^{\infty} \mu(n)\, g\left(\frac{\xi}{n}\right).$$

It follows from this and (2.5.8)[1] that

$$\pi(x) = \sum_{1}^{\infty} \frac{\mu(n)}{n} \Pi(x^{1/n}), \tag{2.12.3}$$

and, if we ignore the difference between Π and Π^*, and replace $\Pi(x^{1/n})$ in every term of (2.12.3) by $\operatorname{li} x^{1/n}$, we obtain $R(x)$. This involves neglecting every term in (2.12.2) except the leading term, that is to say, substantially, ignoring the complex zeros of $\zeta(s)$.

The exact proof of (2.12.1) and (2.12.2) is given in the textbooks. Riemann (whose proof is not exact) argued rather differently. He observed first that

$$\int_0^{\infty} \Pi(x)\, x^{-s-1}\, dx = \frac{1}{s}\int_0^{\infty} x^{-s}\, d\Pi(x) = \frac{1}{s}\sum_{p,m} \frac{1}{mp^{ms}} = \frac{\log \zeta(s)}{s} \tag{2.12.4}$$

[1] Take $f(\xi) = \xi\pi(e^{\xi})$, $g(\xi) = \xi\Pi(e^{\xi})$.

for $\sigma > 1$. From this it follows, by "Mellin's inversion formula", that

$$\Pi(x) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{\log \zeta(s)}{s} x^s\,ds \tag{2.12.5}$$

if $c > 1$.[1]

If

$$s = \tfrac{1}{2} + iz, \tag{2.12.6}$$

then

$$\xi(z) = \tfrac{1}{2}s(s-1)\,\pi^{-\frac{1}{2}s}\,\Gamma(\tfrac{1}{2}s)\,\zeta(s) \tag{2.12.7}$$

is an integral function of z. Its zeros are given by

$$s = \rho, \quad z = -i(\rho - \tfrac{1}{2}) = \tau. \tag{2.12.8}$$

They lie symmetrically about the origin, and are real if the Riemann Hypothesis is true; and

$$\xi(z) = \xi(0)\prod_{\tau}\left(1 - \frac{z^2}{\tau^2}\right). \tag{2.12.9}$$

From (2.12.7) and (2.12.9) we obtain

$$\log \zeta(s) = -\log(s-1) - \log\Gamma(\tfrac{1}{2}s+1) + \tfrac{1}{2}s\log\pi + \log\xi(0) + \Sigma\log\left\{1 + \frac{(s-\frac{1}{2})^2}{\tau^2}\right\}; \tag{2.12.10}$$

and Riemann substitutes from (2.12.10) into (2.12.5) and evaluates the terms of the resulting series separately. The dominating term of (2.12.2) results from the term $-\log(s-1)$.

There are gaps in the argument, of which the most important is the lack of any sufficient proof of (2.12.9). This was only made possible much later by Hadamard's work on integral functions. But what I want you to notice particularly is that there is nowhere any explicit use of Cauchy's Theorem. Riemann's formal machinery, his term by term integrations of series, his use of Fourier's Theorem in the proof of (2.12.5), and his evaluations of particular definite integrals, would all have been intelligible, and highly sympathetic, to Ramanujan.

2.13. I return to Ramanujan's argument. We can write (2.8.3) or (2.9.2) as

$$\int_0^\infty e^{-yz}\,d\psi(z) \sim \frac{1}{y} \tag{2.13.1}$$

[1] Riemann put $s = c + it$, and expressed (2.12.4) as a Fourier integral with a factor

$$e^{-it\log x},$$

then appealing to the Fourier formulae. The two arguments are formally equivalent.

(though Ramanujan does not use this notation), and from these propositions we can deduce the Prime Number Theorem. Ramanujan thought that he could prove much more, viz. (2.10.13) or (2.10.14), which may be written as

$$(2.13.2) \qquad \int_0^\infty e^{-yz}\,d\psi(z) = \frac{1}{y} + O(1),$$

or

$$(2.13.3) \qquad \int_0^\infty e^{-yz}\,d\psi(z) = \frac{1}{y} + O(y^{-\delta}).$$

From these he deduced

$$(2.13.4) \qquad \psi(x) - x = O(1) \qquad (\text{or } O(x^\delta))$$

and the passage to

$$(2.13.5) \qquad \pi(x) - R(x) = O(1) \qquad (\text{or } O(x^\delta))$$

was easy. But his argument here is unsound beyond any possibility of restoration. Not only are all of (2.13.2)–(2.13.5) demonstrably false, but the passage from (2.13.2) or (2.13.3) to (2.13.4) is also fallacious. There are no Tauberian theorems which permit us to make inferences like this. And I do not think that it is worth while to probe more closely into the details of Ramanujan's reasoning. There is one reproach at any rate that cannot be made against it. Unsound as it is, it is not "dim"; Ramanujan was never dim. It contains a very interesting idea, and one which, when properly pruned, fits into its place in the theory.

The question of Ramanujan's originality

2.14. Whatever you may think of Ramanujan's argument, you will agree that his formal ideas were fine, and you are sure to wonder whether they were all his own. In particular, did he really discover the Riemann series himself?

My own opinion is that he did. There is, however, just one book in which he might conceivably have seen the series. There was a copy of Mathews's *Theory of numbers* in the Madras library, and this book contains a (rather uncritical) reproduction of Riemann's analysis. It seems worth while to consider for a moment what books, of importance to him and accessible in Madras, Ramanujan may have consulted.

There were five books which would have been particularly important to Ramanujan: Whittaker's *Modern analysis* (published in 1902), Bromwich's *Infinite series* (1908), Mathews (1892), and Cayley's and Greenhill's treatises on elliptic functions. My own opinion is that he had seen one at least, and possibly both, of the last two books, but none of the other three. I have no idea how this may have happened, and my opinion is based on internal evidence only, but no other hypothesis seems to fit.

In the first place, Ramanujan *cannot* have seen Whittaker, since he did not know Cauchy's Theorem. For the same reason, of course, he cannot have seen Forsyth's *Theory of functions*. This is important because it shows that there were "obvious" books which Ramanujan had never seen although they were certainly accessible in Madras.

The evidence about Bromwich is not so conclusive, but I cannot believe that Ramanujan had seen the book. He was keenly interested in divergent series, about which he had a "theory" of his own. Bromwich has a long and very interesting chapter on the subject, which would have fascinated Ramanujan; but Ramanujan never showed any knowledge of Cesàro or Borel summability, or of any of the standard work. It seems plain indeed from passages in his letters that he had no idea that any scientific theory of divergent series existed.

I do not think, then, that Ramanujan had seen either Whittaker or Bromwich; but it seems plain that he had read some book on elliptic functions, and I agree with Littlewood in thinking that it was probably Greenhill's. He never refers to books, but he never refers to any of the standard theorems of the subject as though he thought them his own. He claims to have extended the theory in different directions, as he had done, but not to have invented elliptic integrals, theta functions, or modular equations. All these things he treats as parts of common knowledge. His own knowledge was remarkable both for its extent and for its limitations, and both the extent and the limitations fit excellently with the hypothesis that it was based on Greenhill's stimulating but eccentric book.[1]

These theorems about prime numbers, on the other hand, Ramanujan claimed quite definitely as his own (though of course he recognised his mistakes later, and learnt the outlines of the established theory). This, to anyone who knew Ramanujan well, is conclusive; but the hypothesis of Ramanujan's complete independence is also the only one which seems to me to fit the facts.

In the first place, if Ramanujan had ever seen Mathews, how could he have been as ignorant as he was of the classical theory of quadratic forms, which Mathews discusses elaborately and which fills nearly half his book? The series for the class-number, in particular, would have fascinated Ramanujan, and it is certain that he would have studied them intensively.

But the most conclusive evidence is Mathews's chapter on primes itself. This, whatever its defects, contains a fairly adequate description of Riemann's memoir. It lays all the proper emphasis on Riemann's great discovery, that the theory of primes depends upon the "complex" properties of $\zeta(s)$, and in particular on the location of its zeros. The complex zeros of

[1] In which we learn first on p. 258 that the elliptic functions are doubly periodic.

$\zeta(s)$ dominate the analysis, as they must in any account of Riemann's work. Ramanujan had no accurate ideas about analytic functions, but he knew quite well that an equation could have an infinity of complex roots, and he could have followed the argument without difficulty. It is incredible that, after seeing this chapter, he should have proceeded to construct a theory in which "all the zeros of $\zeta(s)$ were real".

My conclusion is, therefore, that all this work of Ramanujan, with its flashes of inspiration and its crude mistakes, was an individual and unassisted achievement. There is no other hypothesis which seems to me to be tenable, or to make any sort of mathematical or psychological sense.

In conclusion I may say this. No one before Riemann, so far as I know, had written down $R(x)$, and if Ramanujan found the series himself it may seem a very astonishing performance. Even Gauss stopped at $\mathrm{li}\,x$.

There is, however, a certain danger of exaggeration. If we see that

$$\Pi(x) = \pi(x) + \tfrac{1}{2}\pi(x^{\frac{1}{2}}) + \tfrac{1}{3}\pi(x^{\frac{1}{3}}) + \dots$$

plays a more "natural" part in the theory than $\pi(x)$ itself, as $\psi(x)$ plays a more natural part than $\vartheta(x)$; if we realise that it is $\Pi(x)$ which "corresponds naturally" to $\mathrm{li}\,x$; and if we are familiar with the inversion formula of Möbius; then, when we pass back from $\Pi(x)$ to $\pi(x)$ by this formula, the series $R(x)$ presents itself inevitably. All these ideas were familiar to Tchebychef, and there was no reason at all why he should not have written down $R(x)$, though he never seems to have done so. Ramanujan knew all these things too, as the argument which I have presented to you shows quite clearly, and everything that he did, however we may judge it, is thoroughly intelligible.

NOTES ON LECTURE II

This lecture is a revised and enlarged edition of one given to the London Mathematical Society on February 18, 1937.

§ 2.1. The letters are those printed in full in the *Papers*, xxiii–xxix and 349–352. About the time when the letters were written, Narayana Aiyar (1) published Ramanujan's principal assertions in the *Journal Indian Math. Soc.*

I have altered Ramanujan's notation for the sake of convenience of reference: he has e^a for x and a for y in (2.1.1), n for x in (2.1.3) and (2.1.4), and μ for c in (2.1.4). He also writes S_{t+1} for $\zeta(t+1)$.

Ramanujan implies that $\rho(x) \to \infty$ when $x \to \infty$, which is false, as Littlewood proved in 1914. Littlewood's proof, which is difficult, will be found in Ingham, ch. 5 or Landau, *Vorlesungen*, ii, Kap. 11. The similar assertions about primes in arithmetical progressions, at the end of the passage quoted from the second letter, are also false.

§ 2.2. For the series (2.2.1) see Gram, *Skrifter d. K. Danske Videnskabernes Selskab* (6), 2 (1884), 185–308 (212, 295).

For the history of (2.2.2) and (2.2.3) see Landau, *Handbuch*, 567–574. That the Prime Number Theorem could be deduced from (2.2.2) by 'elementary' reasoning (that is to say by reasoning independent of the theory of functions of a complex variable) was first proved by Landau, *Wiener Sitzungsberichte*, 120 (1911), 973–988; the *Handbuch* contains only a deduction from (2.2.3).

According to Soldner, li $x = 0$ for $x = 1{\cdot}4513692346\ldots$, and it seems probable that this is the real value of Ramanujan's c.

For (2.2.5) see Bromwich, *Infinite series*, ed. 2, 334.

To prove (2.2.7) we observe that

$$(1)\qquad h(-y) = 1+\Sigma\frac{(-1)^n y^n}{n.n!\,\zeta(n+1)} = 1+\sum_m\frac{\mu(m)}{m}\sum_n\frac{(-1)^n}{n.n!}\left(\frac{y}{m}\right)^n$$

$$= 1-\Sigma\frac{\mu(m)}{m}\int_0^{y/m}\frac{1-e^{-u}}{u}\,du$$

$$= -\Sigma\frac{\mu(m)}{m}\left(\int_0^{y/m}\frac{1-e^{-u}}{u}\,du+\log m\right)$$

$$= \Sigma\frac{\mu(m)}{m}\,\chi(y,m),$$

say, by (2.2.3). Now

$$\chi(y,m)-\chi(y,m+1) = -\int_{y/(m+1)}^{y/m}\frac{1-e^{-u}}{u}\,du-\log\frac{m}{m+1}$$

$$= \int_{y/(m+1)}^{y/m}\frac{e^{-u}}{u}\,du = \omega(y,m),$$

say. If we write

$$\sum_1^m\frac{\mu(k)}{k} = g(m),$$

and transform the last series in (1) by partial summation, we obtain

$$h(-y) = \Sigma\, g(m)\,\omega(y,m).$$

Now it is known that

$$(2)\qquad g(m) = O\left\{\frac{1}{(\log m)^2}\right\};$$

and

$$0<\omega(y,m)\leqq\frac{y}{m(m+1)}\,\frac{m+1}{y}\,e^{-y/(m+1)}\leqq\frac{1}{m}.$$

Hence the series is majorised by a multiple of

$$\Sigma\frac{1}{m(\log m)^2},$$

and is uniformly convergent for all y. Finally, all of its terms tend to 0.

For (2), and much stronger results, see Landau, *Handbuch*, 594–597.

For the proof of (2.2.8) see Hardy (**3**).

§ 2.3. For (2.3.3), which is true on the Riemann Hypothesis, see Ingham, 83 (Theorem 30) or Landau, *Handbuch*, 378–388. For the falsity of (2.3.4), which is much easier to prove than that of (2.3.5) or the other false assertions of § 2.3, see Ingham, 90 (Theorem 32) or Landau, *Handbuch*, 711–719.

To deduce (2.3.4) from (2.3.2), observe that

$$R(x)-\mathrm{li}\,x+\tfrac{1}{2}\,\mathrm{li}\,x^{\frac{1}{2}} = O\left\{\sum_{m=3}^{\infty}\frac{1}{m}\sum_{p=1}^{\infty}\frac{(\log x)^p}{p.p!\,m^p}\right\}$$

$$= O\left\{\log x\sum_{m=3}^{\infty}\frac{1}{m^2}\sum_{p=0}^{\infty}\frac{1}{p!}\left(\frac{\log x}{3}\right)^p\right\} = O(x^{\frac{1}{3}}\log x).$$

The table is extracted from the much more comprehensive one in D. N. Lehmer's *List of prime numbers from* 1 *to* 10,006,721 (Washington, 1914), except for the values of li 10^8 and li 10^9, which I have taken from Ingham. The values of $\pi(x)$ are less by 1 than Lehmer's, since he counts 1 as a prime. Those beyond the range of the factor tables were found by Meissel in a series of papers in the *Math. Annalen* of which the latest appeared in vol. 25 (1885), 251–257, and afterwards confirmed by Bertelsen. Meissel's method is an elaboration of the 'sieve' of Eratosthenes: see Mathews, ch. 10. There is an account of Bertelsen's work in a paper by Gram, *Acta Math.* 17 (1893), 301–314.

There is a long and interesting essay on these topics in the introduction to J. Glaisher's *Factor table for the sixth million* (London 1883). Torelli's monograph *Sulla totalità dei numeri primi fino ad un limite assegnato* (Naples 1901) also contains much interesting information.

§ 2.4. There are proofs of Tchebychef's theorems in Ingham, in both of Landau's books, and in Hardy and Wright, ch. 22.

§§ 2.5–2.7. There is no proof of the Prime Number Theorem anywhere in this book, though §§ 2.6–2.7 contain a sketch of Landau's proof, which is simpler than the original proofs of Hadamard and de la Vallée Poussin. To complete it, we must prove that

$$\left|\frac{\zeta'(1+it)}{\zeta(1+it)}\right| = O(|t|^{\alpha}) \tag{1}$$

for an α less than 1. This naturally presupposes (2.7.6), which is proved, in two different ways, in Lecture IV (A). The stronger proposition (1) can be proved by a development of the first proof (Hadamard's) of (2.7.6).

Both Ingham's tract and Landau's *Handbuch* contain (*a*) the simplest proofs of the Prime Number Theorem (apart from Wiener's proof referred to in § 2.8) and (*b*) more elaborate proofs of much stronger theorems.

A 'Tauberian' theorem may be defined as the corrected form of the false converse of an 'Abelian' theorem. An 'Abelian' theorem asserts that, if a sequence or function behaves regularly, then some average of it behaves regularly. Thus

$$A(x) \sim x$$

implies

$$A_1(x) = \int_0^x A(t)\,dt \sim \tfrac{1}{2}x^2:$$

this Abelian theorem is true for any $A(x)$ and in particular for the $A(x)$ of the text. The converse is false, but becomes true when we subject $A(x)$ to an appropriate additional condition, here implied by (2.6.3).

The first Tauberian theorem was Tauber's converse of Abel's theorem on the continuity of power series: see, for example, Bromwich, *Infinite series*, ed. 2, 256. Since

$$a_0 + a_1 x + a_2 x^2 + \dots = \frac{a_0 + (a_0 + a_1)\,x + (a_0 + a_1 + a_2)\,x^2 + \dots}{1 + x + x^2 + \dots},$$

the limit of the series is the limit of a certain average of a_0, $a_0 + a_1$, $a_0 + a_1 + a_2$, ...

§ 2.8. What is here called Wiener's proof of the Prime Number Theorem is the 'Wiener-Ikehara' proof, contained in §§ 19 *et seq.* of Wiener's *The Fourier integral* (Cambridge 1933). There is a quite different proof in §§ 17–18.

The proof has been much simplified by Bochner, *Math. Zeitschrft*, 37 (1933), 1–9 and Landau, *Berliner Sitzungsberichte* (1932), 514–521. Landau's version embodies the shortest extant proof of the Prime Number Theorem, but has not yet appeared in any book. The version in my lithographed lectures on 'Ramanujan's work' (Institute for advanced study, 1936) is substantially Bochner's.

For the 'Hardy-Littlewood' proof see *Quarterly Journal of Math.* 46 (1915), 215–219 or *Acta Math.* 41 (1918), 119–196 (127–134). The simplest proof of our Tauberian theorem is that given by Karamata in *Math. Zeitschrift*, 32 (1930), 319–320.

§ 2.10. For the summability of log 2 − log 3 + ... see Bromwich, *Infinite series*, ed. 1, 351.

§ 2.11. The argument towards the end of the section was suggested to me many years ago by Prof. Maclagan Wedderburn. See Hardy, *Quarterly Journal of Math.* 38 (1907), 269–288 (277).

The formula (2.11.2) may be deduced by differentiation from the last formula on p. 283 of this paper (in which the sign of the last term should be changed).

Ramanujan, when I disputed the truth of his statement, produced the amended formula

$$`\phi_2(y) + \log 2\left(1 - \frac{y}{3.1!} + \frac{y^2}{7.2!} - \frac{y^3}{15.3!} + \dots\right) = \frac{1}{y} + F(y),$$

where
$$yF(y) = \cdot 0000098844 \cos\left(\frac{2\pi \log y}{\log 2} + \cdot 872811\right)$$

correct to 10 places of decimals'. This takes account explicitly of the terms in which $k = \pm 1$.

§ 2.12. The proofs of the 'explicit formulae' are given in Ingham, ch. 4 and Landau, *Handbuch*, Kap. 19. Riemann's own argument is reproduced in Mathews, ch. 10.

For the Möbius inversion formulae see Hardy and Wright, 234–237, or Landau, *Handbuch*, 577–580.

§ 2.14. Ramanujan seems to have started his own theory of divergent series with series of positive terms, such as

$$1^{-s} + 2^{-s} + 3^{-s} + \dots \quad (s < 1)$$

and to have 'defined' the sum of such a series as the constant of the Euler-Maclaurin sum formula. But he was not in the habit of giving strict 'definitions'.

III

ROUND NUMBERS

3.1. A number is described in popular language as *round* if it is the product of a considerable number of comparatively small factors. Thus $1200 = 2^4.3.5^2$ would certainly be called round. The number $2187 = 3^7$ is even rounder, but this is obscured by the decimal notation.

It is a matter of common observation that *round numbers are very rare*; the fact may be verified by anyone who will make a habit of factorising numbers, such as numbers of motor-cars or railway carriages, which are presented to his attention in a random manner. Both Ramanujan and I had observed this phenomenon, which seems at first a little paradoxical,[1] and were curious about its mathematical explanation.

We therefore proposed to ourselves the problem of determining the "normal degree of compositeness" of a number n. *How many prime factors may one expect to occur in a random large number n?*

3.2. It is natural to measure the "compositeness" of a number by the number of its prime factors; and we may count this in two different ways, according as we count multiple factors multiply or not. Suppose that

(3.2.1) $$n = p_1^{a_1} p_2^{a_2} \dots p_\nu^{a_\nu} = \prod_1^\nu p_r^{a_r},$$

where $$p_1 < p_2 < \dots < p_\nu,$$

is the standard expression of n as a product of primes. Then

(3.2.2) $$f(n) = \nu$$

is the number of different prime factors of n, and

(3.2.3) $$F(n) = \sum_1^\nu a_r$$

is the total number of prime factors when multiple factors are counted multiply; and either of these functions may be adopted as a measure of the compositeness of n. We shall find that it makes no important difference which measure we adopt, and for the moment I consider $f(n)$.

[1] "Half the numbers are divisible by 2, one-third by 3, one-sixth by both 2 and 3, and so on. Surely then we may expect most numbers to have a *large* number of factors? But the facts seem to show the opposite."

3.3. It is clear that $f(n)$ cannot be "very large". The worst, or roundest, numbers, those for which $f(n)$ is largest compared with n, are the numbers

$$n = 2.3.5 \dots p_\nu,$$

which are products of the first ν primes. For these numbers

$$f(n) = \nu = \pi(p_\nu),$$

and
$$\log n = \sum_{r \le \nu} \log p_r = \vartheta(p_\nu).$$

Now

(3.3.1) $$A\nu \log \nu < p_\nu < B\nu \log \nu,$$

for constant A and B, and so

$$\log p_\nu \sim \log \nu.$$

Hence

(3.3.2) $$\nu = f(n) = \log n \frac{\pi(p_\nu)}{\vartheta(p_\nu)} \sim \frac{\log n}{\log p_\nu} \sim \frac{\log n}{\log \nu},$$

and
$$\nu \sim \frac{\log n}{\log \log n}.$$

Thus a large number n cannot have more than about this number of prime factors. A number about 10^7, the limit of the tables, cannot have more than about 6 or 7, and a number about 10^{80}, i.e. about the Eddington number, cannot have more than about 30.

The *total* number of prime factors may be a good deal larger; thus 10^{80} has 160, and $n = 2^k$ has

$$\frac{\log n}{\log 2}.$$

Here there is a big difference between $f(n)$ and $F(n)$, but we shall see that this is exceptional.

3.4. The theorem just proved about $f(n)$ is a theorem about *all* numbers, and we are concerned with (in some sense) "almost all". The function

$$\frac{\log n}{\log \log n}$$

increases slowly, but by no means so slowly as to explain the facts. We find, if we try numbers at random from near the end of the factor tables, that $f(n)$ is usually not 7 or 8 but 3 or 4; and we should find, if the tables could be extended to the limits of the Eddington number, that it was usually not about 30 but about 5 or 6. A number like the Skewes number[1] would generally have about the Eddington number of factors (and that may help us to form some image of its size).

[1] See Lecture I, p. 17.

We need a definition of "almost all". Suppose that P is a property of a number n expressed by a proposition $P(n)$; that $N(x)$ is the number of numbers, up to x, for which $P(n)$ is false; and that

$$N(x) = o(x).$$

Then we say that "almost all numbers possess the property P". Roughly, the proportion of exceptional n is infinitesimal.

The answer to our problem is that *almost all numbers n have about* $\log\log n$ *prime factors*. More exactly, given ϵ, then *almost all numbers have between* $(1-\epsilon)\log\log n$ *and* $(1+\epsilon)\log\log n$ *prime factors*. The theorem is true whichever way we measure the number of factors, and, as we shall see, it can be stated still more precisely.

3.5. The function $\log\log n$ is suggested in another way. We have

$$\sum_{n\leq x} f(n) = \sum_{n\leq x}\sum_{p|n} 1 = \sum_{pm\leq x} 1,\text{[1]}$$

the summation being extended over all primes p and positive integers m satisfying the inequality; and when we sum with respect to m we obtain

(3.5.1) $$\sum_{n\leq x} f(n) = \sum_{p\leq x}\left[\frac{x}{p}\right],$$

or

(3.5.2) $$\sum_{n\leq x} f(n) = x\sum_{p\leq x}\frac{1}{p} + O(x),$$

since the removal of a square bracket involves an error of 1 at most. But

(3.5.3) $$\sum_{p\leq x}\frac{1}{p} = \log\log x + O(1),$$

and therefore

(3.5.4) $$\sum_{n\leq x} f(n) = x\log\log x + O(x).$$

Also $$\sum_{n\leq x} F(n) = \sum_{n\leq x}\sum_{p^\mu|n} 1 = \sum_{p^\mu m\leq x} 1$$

(the summation extending over primes p and positive integers μ and m), and so

(3.5.5) $$\sum_{n\leq x} F(n) = \sum_{p^\mu\leq x}\left[\frac{x}{p^\mu}\right] = \sum_{p\leq x}\left(\left[\frac{x}{p}\right]+\left[\frac{x}{p^2}\right]+\left[\frac{x}{p^3}\right]+\dots\right);$$

and

(3.5.6) $$\sum_{p\leq x}\left(\left[\frac{x}{p^2}\right]+\left[\frac{x}{p^3}\right]+\dots\right) < x\sum_{p}\left(\frac{1}{p^2}+\frac{1}{p^3}+\dots\right) = x\sum_{p}\frac{1}{p(p-1)} = O(x).$$

[1] $p|n$ means 'p divides n', so that $\sum_{p|n} 1$ implies counting 1 for every prime divisor of n. Similarly $\sum_{p^\mu|n} 1$ implies counting 1 for every prime or power of a prime which divides n.

Hence we have also

(3.5.7) $$\sum_{n \leq x} F(n) = x \log\log x + O(x).$$

In particular

(3.5.8) $$\sum_{n \leq x} f(n) \sim x \log\log x, \quad \sum_{n \leq x} F(n) \sim x \log\log x$$

Now if $\phi(n)$ is some simple increasing function, and

$$g(2) + g(3) + \ldots + g(n) \sim \phi(2) + \phi(3) + \ldots + \phi(n),$$[1]

then it is natural to say that "the average order of $g(n)$ is $\phi(n)$". And since

$$\log\log 2 + \ldots + \log\log n \sim n \log\log n,$$

the average order of both $f(n)$ and $F(n)$ is $\log\log n$.

3.6. This is an interesting theorem, but quite different from the one we want to prove. We want to prove that $f(n)$ and $F(n)$ are *usually* about $\log\log n$. If $g(n)$ is $\log p \log\log p$ when n is a prime p, and 0 otherwise, then

$$\sum_{n \leq x} g(n) = \sum_{p \leq x} \log p \log\log p \sim x \log\log x,$$[2]

and the average order of $g(n)$ is still $\log\log n$; but $g(n)$ is *usually* 0 (since "almost all" numbers are composite), so that the *normal* order of $g(n)$ is 0. In our problem the average and the normal orders happen to be the same, but that is a peculiarity of the problem.

3.7. There are two proofs of our theorem, our original proof and another given much later by Turan. Turan's proof is very simple and elegant, and I will give it later; but I insert first a sketch of the original proof, which is in some ways more suggestive.

We need only consider one of the two functions, say $f(n)$. For $F(n) \geqq f(n)$ and, by (3.5.6),

$$\sum_{n \leq x} \{F(n) - f(n)\} = \sum_{p \leq x} \left(\left[\frac{x}{p^2}\right] + \left[\frac{x}{p^3}\right] + \ldots \right) < Cx,$$

for some C. If $N(x)$ is the number of numbers up to x for which

$$F(n) - f(n) > G,$$

then $$\frac{N(x)}{x} < \frac{C}{G},$$

[1] Or $g(1) + \ldots + g(n) \sim \phi(1) + \ldots + \phi(n)$. We start from 2 here because $\log\log n$ is not defined for $n = 1$.

[2] Since $$\vartheta(x) = \sum_{p \leq x} \log p \sim x$$
and the factor $\log\log p$ adds an additional $\log\log x$.

which is small when G is large; and the number of numbers for which $F(n)-f(n)>\chi(n)$, where $\chi(n)$ is any function of n which increases to infinity, is $o(x)$. In this sense "$F(n)-f(n)$ is almost always bounded".

We confine our attention, then, to $f(n)$. Suppose that $\omega_r(x)$ is the number of numbers, not exceeding x, for which $f(n)=r$. Then

$$\omega_1(x) \sim \pi(x) \sim \frac{x}{\log x}. \tag{3.7.1}$$

More generally, it is known that

$$\omega_r(x) \sim \frac{x}{\log x}\frac{(\log\log x)^{r-1}}{(r-1)!}. \tag{3.7.2}$$

Now

$$[x] = \omega_1(x)+\omega_2(x)+\ldots+\omega_r(x)+\ldots \tag{3.7.3}$$

and

$$x = \frac{x}{\log x}e^{\log\log x} = \frac{x}{\log x}\left(1+\xi+\frac{\xi^2}{2!}+\ldots+\frac{\xi^{r-1}}{(r-1)!}+\ldots\right), \tag{3.7.4}$$

where

$$\xi = \log\log x; \tag{3.7.5}$$

and (3.7.2) indicates a certain similarity between corresponding terms of the two series. The similarity cannot, of course, be too close, since the first series terminates; but corresponding terms of fixed rank are asymptotically equivalent when $x\to\infty$.

The largest term in the series

$$1+\xi+\frac{\xi^2}{2!}+\ldots = \sum_1^\infty \frac{\xi^{r-1}}{(r-1)!}$$

is that for which $r=[\xi]+1$.[1] We write (3.7.4) as

$$x = \frac{x}{\log x}\sum_r \frac{\xi^{r-1}}{(r-1)!} = \frac{x}{\log x}\sum_\mu \frac{\xi^{[\xi]+\mu-1}}{([\xi]+\mu-1)!}, \tag{3.7.6}$$

μ assuming both positive and negative values. By Stirling's formula

$$\frac{\xi^{[\xi]+\mu-1}}{([\xi]+\mu-1)!} \sim \frac{e^\xi}{\sqrt{(2\pi\xi)}}e^{-\mu^2/2\xi}$$

if μ is fairly small compared with ξ. Hence the right-hand side of (3.7.4) may be compared with

$$\frac{x}{\log x}.\log x.\frac{1}{\sqrt{(2\pi\xi)}}\Sigma e^{-\mu^2/2\xi} \tag{3.7.7}$$

or with

$$\frac{x}{\sqrt{(2\pi\xi)}}\int_{-\infty}^{\infty} e^{-t^2/2\xi}\,dt = x;$$

[1] There are two equal terms when ξ is an integer.

and the part of this integral for which t is of higher order than $\sqrt{\xi}$ is negligible. Thus practically all of the sum of (3.7.4) is contributed by the terms for which μ is $O(\sqrt{\xi})$. It is natural to suppose that the same must be true of the series (3.7.3); that practically all of its sum comes from the terms in which

$$|r-\xi| = |r-\log\log x|$$

is

$$O(\sqrt{\xi}) = O\{\sqrt{(\log\log x)}\}.$$

This, as we shall see in a moment, would prove our theorem and a good deal more.

The argument is, as it stands, rough and inconclusive. We have to prove that we can neglect most of the terms $\omega_r(x)$ in the series (3.7.3), and for this we need inequalities instead of asymptotic equalities. But these can be found without much difficulty, and the conclusion is that, *if $\chi(x)$ is any function of x such that*

$$\frac{\chi(x)}{\sqrt{(\log\log x)}} \to \infty,$$

then almost all numbers not exceeding x have between

$$\log\log x \pm \chi(x)$$

prime factors. Since $\log\log x$ and $\log\log n$ are practically indistinguishable over most of the range $(1, x)$,[1] it is the same thing to say that *almost all numbers n have between*

$$\log\log n \pm \chi(n)$$

prime factors.

3.8. There is one remark to add before I pass to Turan's proof of the theorem. The asymptotic formulae (3.7.2) are corollaries of the Prime Number Theorem but the final proof is "elementary". We have to show that the "tails" of the series (3.7.3) are negligible, and for this we use an inequality

$$\omega_r(x) < A\frac{x}{\log x}\frac{(\log\log x + C)^{r-1}}{(r-1)!}, \tag{3.8.1}$$

where A and C are independent of both x and r. The proof of this depends on Tchebychef's inequality

$$\pi(x) < A\frac{x}{\log x},$$

and does not require the Prime Number Theorem.

[1] If $0 < c < 1$ and $x^c < n < x$, then $\log\log n$ lies between

$$\log(c\log x) = \log\log x - \log(1/c)$$

and $\log\log x$.

3.9. Turan's proof of the theorem depends on the identities (3.5.1) and

$$\sum_{n\leqq x}\{f(n)\}^2 = \sum_{pp'\leqq x,\, p'\neq p}\left[\frac{x}{pp'}\right]+\sum_{p\leqq x}\left[\frac{x}{p}\right]. \tag{3.9.1}$$

To prove (3.9.1), we observe that the left-hand side is

$$\sum_{n\leqq x}\left(\sum_{p|n}1\sum_{p'|n}1\right) = \sum_{pm=p'm'\leqq x}1 = \sum_{p\neq p',\, pp'\mu\leqq x}1+\sum_{pm\leqq x}1.$$

Here p, p' are primes, m and μ any positive integers, and the ranges of summation are indicated by the subscripts. Summing the last sums first with respect to μ and m, we obtain the result.

Now we have seen in § 3.5 that

$$\sum_{n\leqq x}f(n) = \sum_{p\leqq x}\left[\frac{x}{p}\right] = x\log\log x+O(x). \tag{3.9.2}$$

Also

$$\sum_{pp'\leqq x,\, p\neq p'}\left[\frac{x}{pp'}\right] = x\sum_{pp'\leqq x}\frac{1}{pp'}+O(x);$$

we may drop the restriction $p\neq p'$ because Σp^{-2} is convergent. But

$$\left(\sum_{p\leqq\sqrt{x}}\frac{1}{p}\right)^2\leqq\sum_{pp'\leqq x}\frac{1}{pp'}\leqq\left(\sum_{p\leqq x}\frac{1}{p}\right)^2, \tag{3.9.3}$$

since $p\leqq\sqrt{x}$ and $p'\leqq\sqrt{x}$ imply $pp'\leqq x$, and $pp'\leqq x$ implies $p\leqq x$ and $p'\leqq x$; and each of the two extreme terms in (3.9.3) is

$$\{\log\log x+O(1)\}^2 = (\log\log x)^2+O(\log\log x).$$

Hence

$$\sum_{pp'\leqq x,\, p\neq p'}\left[\frac{x}{pp'}\right] = x(\log\log x)^2+O(x\log\log x); \tag{3.9.4}$$

and it follows from (3.9.1), (3.9.2) and (3.9.4) that

$$\sum_{n\leqq x}\{f(n)\}^2 = x(\log\log x)^2+O(x\log\log x). \tag{3.9.5}$$

Finally, writing ξ for $\log\log x$, as in (3.7.5), we have

$$\begin{aligned}\sum_{n\leqq x}\{f(n)-\xi\}^2 &= \sum_{n\leqq x}\{f(n)\}^2-2\xi\sum_{n\leqq x}\{f(n)\}+\xi^2\sum_{n\leqq x}1\\ &= x\{\xi^2+O(\xi)\}-2\xi x\{\xi+O(1)\}+\xi^2\{x+O(1)\}\\ &= O(x\xi),\end{aligned} \tag{3.9.6}$$

by (3.9.2) and (3.9.5). But if

$$|f(n)-\xi|>\chi(x)$$

for more than δx of the n less than x, then

$$\sum_{n\leqq x}\{f(n)-\xi\}^2>\delta x\chi^2,$$

which contradicts (3.9.6) if χ is of higher order than $\sqrt{\xi}$; and this proves the theorem.

3.10. There is a curious corollary about $d(n)$, the number of divisors of n. It is familiar that

$$d(1)+d(2)+\ldots+d(n) \sim n \log n,$$

so that the average order of $d(n)$ is $\log n$. What is its normal order?

If $n = p_1^{a_1} p_2^{a_2} \ldots p_\nu^{a_\nu}$, then

(3.10.1) $$f(n) = \nu, \quad F(n) = \Sigma a_r, \quad d(n) = \Pi(1+a_r).$$

Also $$2 \leqq 1+a \leqq 2^a.$$

Hence

(3.10.2) $$2^\nu \leqq \Pi(1+a_r) \leqq 2^{\Sigma a_r},$$

or

(3.10.3) $$2^{f(n)} \leqq d(n) \leqq 2^{F(n)}.$$

Since $f(n)$ and $F(n)$ are both usually about $\log\log n$, it follows that $d(n)$ is usually about

(3.10.4) $$2^{\log\log n} = (\log n)^{\log 2} = (\log n)^{0\cdot 6\ldots}.$$

We cannot quite say that "the normal order of $d(n)$ is $2^{\log\log n}$", since the inequalities which we prove for $d(n)$ are of a much less precise type than those which we prove for $f(n)$,[1] but we can say, more roughly, that the normal order of $d(n)$ is "about $2^{\log\log n}$".

In this case the normal and average orders do not agree; $d(n)$ is *usually* much below its average order. The explanation is simple; $d(n)$ is too irregular. Most numbers have about $2^{\log\log n}$ divisors, but some have a very much larger number, so much larger that these abnormal numbers dominate the average of $d(n)$. The irregularities of $f(n)$ and $F(n)$ are not strong enough to produce a similar effect.

It is natural to put the same question about $r(n)$, the number of representations of n as a sum of two squares; but in this case the answer is immediate. Since

$$r(1)+r(2)+\ldots+r(n) \sim \pi n,$$

the average order of $r(n)$ is π. The normal order, on the other hand, is 0, since most numbers are not representable.[2]

[1] Of the type $$2^{\log\log n - \chi(n)} < d(n) < 2^{\log\log n + \chi(n)}.$$

The normal order of $\log d(n)$ is

$$\log 2 \log\log n.$$

[2] Only about $$\frac{Ax}{\sqrt{(\log x)}}$$

of the first x numbers are representable. See Lecture IV (B). For the average order, see Lecture V, § 5.1.

3.11. I conclude by mentioning a conjecture which I made in 1936 and which has since been proved, independently, by Erdös and Pillai.

The asymptotic formula (3.7.2) is true for every fixed r, and it is natural to suppose that it is also true when r is a function of x which tends to infinity sufficiently slowly. If it held for

$$r = [\log\log x],$$

then a simple application of Stirling's theorem would give

$$\omega_r(x) \sim \frac{1}{\sqrt{(2\pi)}} \frac{x}{\sqrt{(\log\log x)}};$$

and at any rate we might hope to prove that

$$\omega_r(x) > \frac{Ax}{\sqrt{(\log\log x)}} \tag{3.11.1}$$

for some positive A. The opposite inequality

$$\omega_r(x) < \frac{Ax}{\sqrt{(\log\log x)}}$$

(for some A) is a trivial corollary of (3.8.1). All that I could prove, however, when I made the conjecture, was that

$$\omega_r(x) > \frac{Ax}{(\log\log x)^{\frac{3}{4}}}.$$

Erdös and Pillai have now proved (3.11.1), and indeed that the inequality is true for

$$\log\log x - B\sqrt{(\log\log x)} < r < \log\log x + B\sqrt{(\log\log x)};$$

and Pillai has proved a good deal more, viz. that, if $0 < k < e$, then

$$\omega_r(x) > \frac{x}{\log x} \frac{(\log\log x - C)^{r-1}}{(r-1)!}$$

for

$$r \leqq k(\log\log x - C),$$

where C depends only on k. The truth of (3.11.1), for the range of values of r stated, is a simple corollary.

NOTES ON LECTURE III

The main theorem of the lecture was proved by Hardy and Ramanujan, *Quarterly Journal of Math.* 48 (1917), 76–92. This paper is no. 35 of the *Papers*, no. 32 being a preliminary account.

Turan's proof, given in § 3.9, was published in his paper 1. We have incorporated a simplification suggested by Mr Marshall Hall. The proof is also reproduced in Hardy and Wright, § 22.13. Turan (2) has proved several generalisations of the theorem.

§ 3.3. The inequalities (3.3.1) and the asymptotic relation

$$\pi(x) \sim \frac{\vartheta(x)}{\log x}$$

can be proved by 'elementary' reasoning. See for example Hardy and Wright, ch. 22, and Lecture II.

For (3.5.3), and sharper results, see Hardy and Wright, ch. 22; Ingham, 22–24; Landau, *Handbuch*, 100–102. The slightly sharper result

$$\sum_{p \leq x} \frac{1}{p} = \log\log x + A + o(1)$$

(with the proper value of A) is equivalent to Mertens's theorem

$$\prod_{p \leq x} \left(1 - \frac{1}{p}\right) \sim \frac{e^{-\gamma}}{\log x},$$

where γ is Euler's constant.

It is almost as easy to prove the more precise equations

$$\sum_{n \leq x} f(n) = x \log\log x + Ax + O\left(\frac{x}{\log x}\right), \quad \sum_{n \leq x} F(n) = x \log\log x + Bx + O\left(\frac{x}{\log x}\right),$$

where
$$B = A + \Sigma \frac{1}{p(p-1)}.$$

These equations are stated in § 1.2(3) of the joint paper referred to in the first of these notes.

§ 3.7. For (3.7.2) see Landau, *Handbuch*, 203–213.

§ 3.10. The maximum order of $d(n)$ is roughly

$$2^{\frac{\log n}{\log\log n}}.$$

This was first proved by Wigert: see Landau, *Handbuch*, 219–222. Wigert and Landau use the Prime Number Theorem in the proof, but Ramanujan (*Papers*, 85–86) showed that this was unnecessary.

§ 3.11. See p. 5 of my lithographed lectures referred to in the note on § 2.8 (p. 47). The proofs of Erdös and Pillai have not yet been published.

IV

SOME MORE PROBLEMS OF THE ANALYTIC THEORY OF NUMBERS

4.1. In this lecture I return to the classical problems of the analytic theory of numbers. Its contents are miscellaneous and rather disconnected, but have a thread of unity because most of them are suggested by those of Ramanujan's letters. I begin by a digression on a topic about which Ramanujan said nothing, but to which I referred in Lecture II.

A

The proof that $\zeta(s)$ has no zeros on $\sigma = 1$

4.2. The prime number theorem is "equivalent" to the theorem that

(4.2.1) $$\zeta(1+it) \neq 0,$$

in the sense that no deeper properties of $\zeta(s)$ are required for the proof. The strict equivalence appears only when we use Wiener's method, but (4.2.1) is essential in any version. The standard proof of (4.2.1) is Hadamard's, but there is an alternative proof, due to Ingham, which is interesting in itself and relevant here because it depends on one of Ramanujan's formulae.

Hadamard's proof depends on a simple trigonometrical inequality, viz.

(4.2.2) $$3+4\cos\theta+\cos 2\theta = 2(1+\cos\theta)^2 \geqq 0$$

(for all real θ). We use this as follows. By Euler's formula

$$\log\zeta(s) = \Sigma\log\frac{1}{1-p^{-s}} = \Sigma p^{-s}+f(s),$$

where $f(s)$ is regular for $\sigma \geqq 1$ (and indeed for $\sigma > \frac{1}{2}$). Hence

$$\log|\zeta(\sigma+it)| = \Re\,\Sigma p^{-\sigma-it}+g(\sigma,t)$$
$$= \Sigma p^{-\sigma}\cos(p\log t)+g(\sigma,t),$$

for $\sigma > 1$, $g(\sigma,t)$ being bounded, for any fixed t, when $\sigma \to 1$.

It follows that, if we write

$$\chi(\sigma,t) = \log|\zeta^3(\sigma)\,\zeta^4(\sigma+it)\,\zeta(\sigma+2it)|,$$

and continue to use $g(\sigma,t)$ in the same sense, then

$$\chi(\sigma,t) = \Sigma p^{-\sigma}\{3+4\cos(t\log p)+\cos(2t\log p)\}+g(\sigma,t);$$

and therefore (since the terms of the series are positive)

$$(4.2.3) \qquad \chi(\sigma, t) > -A(t),$$

where $A(t)$ is independent of σ, when $\sigma \to 1$.

We now fix t. If $\zeta(1+it) = 0,$

then $\zeta(\sigma+it) = \zeta(1+it+\sigma-1) = (\sigma-1)^k h(\sigma, t),$

where k is a positive integer and $\log h(\sigma, t)$ is bounded when $\sigma \to 1$; and therefore

$$\log|\zeta(\sigma+it)| < -k\log\frac{1}{\sigma-1} + A(t)$$

when $\sigma \to 1$. Also $\log|\zeta(\sigma+2it)| < A(t)$

and
$$\log|\zeta(\sigma)| < \log\frac{1}{\sigma-1} + A,$$

where A is constant. Thus

$$\chi(\sigma, t) < (3-4k)\log\frac{1}{\sigma-1} + A(t) \to -\infty,$$

in contradiction to (4.2.3).

4.3. Ingham's proof depends on a formula published by Ramanujan in 1915, viz.

$$(4.3.1) \qquad f(s) = \Sigma\frac{\sigma_a(n)\,\sigma_b(n)}{n^s} = \frac{\zeta(s)\,\zeta(s-a)\,\zeta(s-b)\,\zeta(s-a-b)}{\zeta(2s-a-b)},$$

where $\sigma_a(n)$ is the sum of the a-th powers of the divisors of n, and

$$\Re s, \quad \Re(s-a), \quad \Re(s-b), \quad \Re(s-a-b)$$

are all greater than 1. In particular

$$(4.3.2) \qquad \Sigma\frac{d^2(n)}{n^s} = \frac{\zeta^4(s)}{\zeta(2s)}$$

for $\sigma > 1$.

To prove (4.3.1) we observe that

$$\chi(n) = \sigma_a(n)\,\sigma_b(n)$$

is "multiplicative", i.e. that

$$\chi(nn') = \chi(n)\,\chi(n')$$

for coprime n and n'. Hence

$$(4.3.3) \qquad f(s) = \prod_p f_p(s),$$

where p runs through the primes,

$$f_p(s) = 1 + \sum_{\lambda=1}^{\infty}\frac{\sigma_a(p^\lambda)\,\sigma_b(p^\lambda)}{p^{\lambda s}},$$

and s has any value for which the product is absolutely convergent. But

$$f_p(s) = \sum_{\lambda=0}^{\infty} \frac{p^{(\lambda+1)a}-1}{p^a-1}\frac{p^{(\lambda+1)b}-1}{p^b-1} p^{-\lambda s}$$

$$= \frac{1}{(p^a-1)(p^b-1)}\left\{\frac{p^{a+b}}{1-p^{a+b-s}} - \frac{p^a}{1-p^{a-s}} - \frac{p^b}{1-p^{b-s}} + \frac{1}{1-p^{-s}}\right\}$$

$$= \frac{1-p^{a+b-2s}}{(1-p^{-s})(1-p^{a-s})(1-p^{b-s})(1-p^{a+b-s})};$$

so that
$$f(s) = \prod_p \frac{1-p^{a+b-2s}}{(1-p^{-s})(1-p^{a-s})(1-p^{b-s})(1-p^{a+b-s})}$$

$$= \frac{\zeta(s)\zeta(s-a)\zeta(s-b)\zeta(s-a-b)}{\zeta(2s-a-b)}.$$

Let us now suppose that

$$\zeta(1+ic) = 0,$$

for a positive c. If we take $a = ic$ and $b = -ic$ in (4.3.1), we obtain

(4.3.4)
$$f(s) = \Sigma \frac{|\sigma_{ic}(n)|^2}{n^s} = \frac{\zeta^2(s)\zeta(s-ic)\zeta(s+ic)}{\zeta(2s)}.$$

This formula is valid, in the first instance, for $\sigma > 1$. But $f(s)$ is regular for $\sigma > \frac{1}{2}$, since the double pole of $\zeta^2(s)$, for $s = 1$, is cancelled by the zeros of $\zeta(s-ic)$ and $\zeta(s+ic)$; and the coefficients in the series are *positive*. Hence, by a well-known theorem of Landau, the series is convergent, and represents $f(s)$, for $\sigma > \frac{1}{2}$, and in particular when $s = \frac{1}{2}+\delta > \frac{1}{2}$. Thus

$$f(\tfrac{1}{2}+\delta) = \Sigma \frac{|\sigma_{ic}(n)|^2}{n^{\frac{1}{2}+\delta}} > 1$$

for $\delta > 0$. On the other hand

$$\zeta(2s) = \zeta(1+2\delta) \to \infty$$

when $\delta \to 0$, and all three factors in the numerator of (4.3.4) are bounded; so that

$$f(\tfrac{1}{2}+\delta) \to 0.$$

The contradiction shows that $\zeta(1+ic) \neq 0$ for any positive c.

B

The number of numbers which are sums of two squares

4.4. The assertion which I numbered (1.15) in my introductory lecture was "the number of numbers between A and x which are either squares or sums of two squares is

$$K\int_A^x \frac{dt}{\sqrt{(\log t)}} + \theta(x),$$

where $K = 0{\cdot}764\ldots$ and $\theta(x)$ is very small compared with the integral".

It is immaterial whether we include actual squares or not. Ramanujan later (i) gave the exact value of K, viz.

$$\left\{\tfrac{1}{2}\Pi\left(\frac{1}{1-r^{-2}}\right)\right\}^{\frac{1}{2}},$$

where r runs through the primes $4m+3$, and (ii) stated that $\theta(x)$ is of order

$$\sqrt{\left(\frac{x}{\log x}\right)}.$$

We shall see that this last assertion is false.

4.5. This problem was solved by Landau in 1908. The solution is very interesting because it depends on the application of the classical methods of prime number theory to a function with an algebraical singularity.

We denote primes $4m+1$ and $4m+3$ by q and r respectively. In order that n should be a sum of two squares it is necessary and sufficient that

$$n = 2^a\mu\nu^2,$$

where μ is a product of primes q and ν a product of primes r. If we define b_n as 1 when n is a sum of two squares, and 0 otherwise, then b_n is 1 when

$$n = 2^a\Pi q^b \Pi r^{2c},$$

and

$$f(s) = \sum\frac{b_n}{n^s} = \frac{1}{1-2^{-s}}\prod_q\frac{1}{1-q^{-s}}\prod_r\frac{1}{1-r^{-2s}}.$$

Also

$$\zeta(s) = \frac{1}{1-2^{-s}}\prod_q\frac{1}{1-q^{-s}}\prod_r\frac{1}{1-r^{-s}}$$

and

$$L(s) = 1-\frac{1}{3^s}+\frac{1}{5^s}-\ldots = \Pi\frac{1}{1-q^{-s}}\Pi\frac{1}{1+r^{-s}};$$

so that

$$\{f(s)\}^2 = \psi(s)\,\zeta(s)\,L(s), \tag{4.5.1}$$

where

$$\psi(s) = \frac{1}{1-2^{-s}}\Pi\frac{1}{1-r^{-2s}}. \tag{4.5.2}$$

It is plain that $\psi(s)$ is regular, and has no zeros, for $\sigma > \frac{1}{2}$. As regards the other factors in (4.5.1), $L(s)$ is an integral function, whose value for $s = 1$ is $\frac{1}{4}\pi$, while $\zeta(s)$ is regular except for its pole at $s = 1$. It is also known that neither $\zeta(s)$ nor $L(s)$ vanishes in a region D, stretching to the left of $\sigma = 1$, of type

$$\sigma > 1 - \frac{A}{\{\log(|t|+2)\}^A}.$$

Finally, $\zeta(s)$ and $L(s)$ are $O\{(\log|t|)^A\}$ for large t in D.

It follows that $$f(s) = (s-1)^{-\frac{1}{2}} g(s),$$
where $g(s)$ is regular in D and
$$g(1) = \{\tfrac{1}{2}\pi\psi(1)\}^{\frac{1}{2}} = \left\{\tfrac{1}{2}\pi\Pi\left(\frac{1}{1-r^{-2}}\right)\right\}^{\frac{1}{2}} = K\sqrt{\pi}.$$

4.6. If $$B(x) = \sum_{n \leq x} b_n,$$

then $B(x)$ is the number of representable numbers up to x; and

$$B^*(x) = \sum_{n \leq x}' b_n = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} f(s)\frac{x^s}{s}\,ds \tag{4.6.1}$$

for $c > 1$.[1] We have to treat this integral (or some derived integral) as we treated the integral for $\psi^*(x)$ in Lecture II.

I give a rough sketch of a proof, which may be compared with the "first approximation" (§ 2.5) to the proof of the Prime Number Theorem. There are two important differences. The integral (4.6.1) is much simpler in one way, since there is no zeta-function in the denominator; but the integrand has an algebraic singularity instead of a pole. Hence $B^*(x)$ will be approximated not by a residue but by a loop integral round $s = 1$. We must transform the path of integration into a path of the type C (Fig. 1), and our approximating function will be

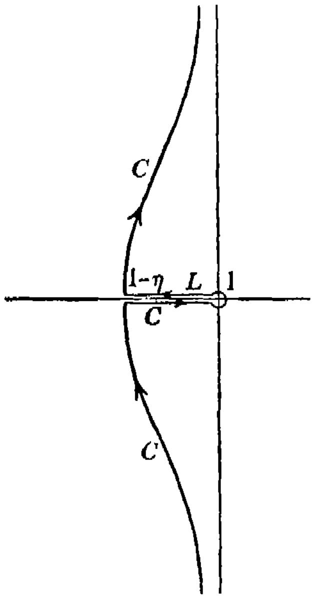


Fig. 1

$$\frac{1}{2\pi i}\int_L f(s)\frac{x^s}{s}\,ds \tag{4.6.2}$$
$$= \frac{1}{2\pi i}\int_L \frac{x^s}{(s-1)^{\frac{1}{2}} s} g(s)\,ds = \frac{K\sqrt{\pi}}{2\pi i}\int_L \frac{x^s}{(s-1)^{\frac{1}{2}}} h(s)\,ds,$$

where

$$h(s) = 1 + a_1(s-1) + a_2(s-1)^2 + \ldots \tag{4.6.3}$$

near $s = 1$, and L is the lacet from $1-\eta$ round 1 shown in the figure.

If we treat $h(s)$ as 1, we obtain

$$\frac{K}{\sqrt{\pi}}\int_{1-\eta}^{1} \frac{x^s}{(1-s)^{\frac{1}{2}}}\,ds = \frac{Kx}{\sqrt{\pi}}\int_0^{\eta} \frac{e^{-u\log x}}{\sqrt{u}}\,du,$$

which is practically $$\frac{Kx}{\sqrt{(\log x)}}.$$

[1] The star and dash have the same meanings as in § 2.5.

And the argument, when properly developed, will show that

$$(4.6.4) \qquad B(x) = \frac{Kx}{\sqrt{(\log x)}}\left\{1+\frac{\alpha_1}{\log x}+\frac{\alpha_2}{(\log x)^2}+\dots\right\},$$

the series being an asymptotic series in Poincaré's sense. This is effectively Landau's result, though he does not push the analysis so far.

4.7. Now

$$(4.7.1) \qquad K\int_A^x \frac{dt}{\sqrt{(\log t)}} = \frac{Kx}{\sqrt{(\log x)}}\left\{1+\frac{\beta_1}{\log x}+\frac{\beta_2}{(\log x)^2}+\dots\right\},$$

the series being again asymptotic. Thus Ramanujan's result is, apart from the $\theta(x)$, of the same form as (4.6.4); but the coefficients in the two series are not the same.[1] Those in (4.6.4) depend, in a rather intricate way, on the coefficients $a_1, a_2, \dots$ in (4.6.3). The integral (4.7.1) has no advantage as an approximation over the first term of the series (4.6.4), and Ramanujan's assertion (15), though true as it stands, is definitely misleading. He was misled, not unnaturally, by the analogy with the Prime Number Theorem, in which the logarithm integral $\operatorname{li} x$ is a significant and a particularly good approximation.

It remains true that

$$B(x) \sim \frac{Kx}{\sqrt{(\log x)}},$$

and it would be very interesting to know just how Ramanujan came to this conclusion. It seems clear, from the form of K, that he used the formula (4.5.1). The rest of his argument was no doubt highly speculative.

4.8. Ramanujan made a similar mistake later in an unpublished manuscript which was examined after his death by Miss Stanley, Watson, and myself. Ramanujan's function $\tau(n)$[2] is "almost always" divisible by 5. More precisely, if $t_n = 0$ when $\tau(n)$ is divisible by 5, $t_n = 1$ otherwise, and

$$T(x) = \sum_{n \leq x} t_n,$$

then

$$T(x) \sim C\frac{x}{(\log x)^{\frac{1}{4}}}$$

for a certain C. Here again Ramanujan was under the misapprehension that

$$C\int_A^x \frac{dt}{(\log t)^{\frac{1}{4}}}$$

was a much better approximation.

[1] In fact $\beta_1 \neq \alpha_1$.

[2] See Lecture X.

C

A note on the Möbius function $\mu(n)$

4.9. It is well known that certain theorems concerning Möbius's function $\mu(n)$ are "of the same depth" as the Prime Number Theorem; they are equivalent to it in the sense that they can be deduced from it, or it from them, in an "elementary" manner. In particular this is true of the theorems

(4.9.1) $$\Sigma \frac{\mu(n)}{n} = 0$$

and

(4.9.2) $$M(x) = \sum_{n \leq x} \mu(n) = o(x);$$

and these two theorems are naturally equivalent in the same sense. That (4.9.2) follows from (4.9.1), indeed, is trivial, since the convergence of

$$\Sigma \frac{a_n}{n}$$

implies $$A_n = a_1 + a_2 + \ldots + a_n = o(n),$$

whatever a_n. The deduction of (4.9.1) from (4.9.2), while technically "elementary", is much less immediate, and depends upon a rather delicate theorem of Axer and the special properties of $\mu(n)$.

There are assertions in Ramanujan's first letter to me which imply his familiarity with (4.9.1) and (4.9.2). There is of course no question of his possessing a real proof. It is immediate that

$$\lim_{s \to 1} \Sigma \frac{\mu(n)}{n^s} = \lim_{s \to 1} \frac{1}{\zeta(s)} = 0,$$

and he would hardly have distinguished between this and (4.9.1).

The assertions are those numbered (2) on p. xxiv of the *Papers*. If u is a number which is the product of an odd number of different prime factors, and $U(x)$ the number of such numbers up to x, then

(4.9.3) $$U(x) \sim \frac{3x}{\pi^2},$$

(4.9.4) $$\Sigma \frac{1}{u^2} = \frac{9}{2\pi^2},$$

(4.9.5) $$\Sigma \frac{1}{u^4} = \frac{15}{2\pi^4}.$$

The last two theorems are easy. We write q for a quadratfrei number, u in the sense just explained, and v for a q which is not a u. Then

$$\mu(u) = -1, \quad \mu(v) = 1$$

and $$\Sigma\frac{1}{u^s} = \tfrac{1}{2}\Sigma\frac{|\mu(n)|-\mu(n)}{n^s}.$$

But $$\Sigma\frac{\mu(n)}{n^s} = \frac{1}{\zeta(s)}$$

and $$\Sigma\frac{|\mu(n)|}{n^s} = \Pi\left(1+\frac{1}{p^s}\right) = \frac{\zeta(s)}{\zeta(2s)}.$$

Hence $$\Sigma\frac{1}{u^s} = \frac{1}{2}\left\{\frac{\zeta(s)}{\zeta(2s)} - \zeta(s)\right\}$$

for $s > 1$, and (4.9.4) and (4.9.5) are particular cases.

The theorem (4.9.3) lies deeper, and depends on (4.9.2). If we define $Q(x)$ and $V(x)$ as we defined $U(x)$, then

$$Q(x) = U(x) + V(x)$$

and $$M(x) = V(x) - U(x).$$

It is quite easy to prove that

$$Q(x) \sim \frac{6x}{\pi^2},$$

and (4.9.3) then follows from (4.9.2). Thus Ramanujan's assertion here is just of the same "depth" as the Prime Number Theorem.

NOTES ON LECTURE IV

§ 4.2. Hadamard's proof can be developed so as to show that $\zeta(s)$ has no zeros, and indeed

$$|\zeta(s)| > \frac{A}{\{\log(|t|+2)\}^A},$$

in a region $$\sigma > 1 - \frac{A}{\{\log(|t|+2)\}^A}.$$

Here the A are appropriate positive constants. See Landau, *Handbuch*, 169–180.

Ingham's proof was published in his paper **2**. It can be applied to all Dirichlet's "L-functions", and in particular to

$$L(s) = 1^{-s} - 3^{-s} + 5^{-s} - \ldots;$$

but it cannot be developed in the same way as Hadamard's.

§ 4.3. The formula (4.3.1) is stated in the *Papers*, 135 (15). There is a proof in B. M. Wilson (**1**).

For (4.3.3) see, for example, Hardy and Wright, 247–248.

Landau's theorem was published in *Math. Annalen*, 61 (1905), 527–550. The proof is also given in the *Handbuch*, 697–698.

§ 4.4. See the notes on p. 10 (printed on p. 20).

For Ramanujan's actual assertions see *Papers*, xxiv and xxviii. It is plain that, if his statement about the order of $\theta(x)$ were correct, it would be important whether we counted actual squares or not.

§ 4.5. The properties of $\zeta(s)$ assumed here are included among those proved by Landau in the part of the *Handbuch* referred to in the note on § 4.2. For the extensions to L-functions, and in particular $L(s) = 1^{-s} - 3^{-s} + \ldots$, see *Handbuch*, 459–464.

§ 4.6. The argument here is a little less unsophisticated than that of § 2.5, since there I assumed the truth of the Riemann Hypothesis, wheras here I assume nothing about the zeros of $\zeta(s)$ and $L(s)$ which has not been proved (and have for that reason to use a curvilinear contour). In other respects the argument is as rough as in § 2.5.

§ 4.8. See Stanley (**1**). The function $\tau(n)$ has many curious congruence properties: for example $\tau(n) \equiv 0 \pmod{691}$ for almost all n. For these, see Mordell (**1**), Watson (**23**), and Lecture X.

§ 4.9. See Landau, *Prac. Matematyczno-Fizycznych*, 21 (1910), 97–177 (130–137), and the note on § 2.2.

For the asymptotic formula for $Q(x)$ see Hardy and Wright, 267–268; Landau, *Handbuch*, 604–609. The theorem is due to Gegenbauer.

V

A LATTICE-POINT PROBLEM

5.1. Suppose that D is a bounded region in the plane of u and v, including the origin O inside it; and that $D(x)$ is the result of magnifying D about O in the linear ratio $x^{\frac{1}{2}}:1$ or the areal ratio $x:1$. About how many lattice-points (i.e. points with integral coordinates) are there, inside or on the boundary of $D(x)$, when x is large?

The most familiar of such "lattice-point problems" is Gauss's "circle problem". Here D is the unit circle and $D(x)$ is the circle

$$u^2+v^2 \leqq x.$$

It is easy to show that, if $N(x)$ is the number of lattice-points in $D(x)$, then

$$N(x) = \pi x + O(x^{\frac{1}{2}}) \tag{5.1.1}$$

(the area of the circle, with an error of the order of the circumference). This theorem is almost intuitive, but is very far from expressing the final truth, more profound analysis of the problem having shown that the $\frac{1}{2}$ in (5.1.1) can be replaced, first by $\frac{1}{3}$, and then by various smaller numbers.

We can also state the problem in a less geometrical form. If $r(n)$ is the number of representations of an integer n as the sum of two squares (representations which differ in the order or sign of the bases of the squares being reckoned separately), then $r(n)$ is the number of lattice-points on the circle $u^2+v^2 = n$, and

$$N(x) = r(1)+r(2)+\ldots+r([x]) = \sum_{n\leqq x} r(n);$$

so that (5.1.1) may be written as

$$\sum_{n\leqq x} r(n) = \pi x + O(x^{\frac{1}{2}}), \tag{5.1.2}$$

and becomes a theorem about the "average order" of the arithmetical function $r(n)$.

There are many generalisations of the problem, to ellipses, and to hyperspheres and hyperellipsoids in space of any number of dimensions.

5.2. Another famous lattice-point problem is "Dirichlet's divisor problem". It is convenient to state this in a slightly different form. The circle of § 5.1 was symmetrical in all four quadrants, and we might have stated the problem for one quadrant only. If $D(x)$ is now defined by

$$u \geqq 0, \quad v \geqq 0, \quad u^2+v^2 \leqq x,$$

then

$$N(x) = \tfrac{1}{4}\pi x + O(x^{\frac{1}{2}}).$$

We could equally well define $D(x)$ by

$$u > 0, \quad v > 0, \quad u^2 + v^2 \leqq x,$$

discarding the points on the axes; and we shall see that this modification is essential in the divisor problem.

In the divisor problem $D(x)$ is defined by

$$u > 0, \quad v > 0, \quad uv \leqq x,$$

so that $N(x)$ is the number of lattice-points between the axes and the rectangular hyperbola $uv = x$, counting those, if any, on the hyperbola but not those on the axes. There are an infinity of points on the axes, so that it is essential to exclude them.

It was proved by Dirichlet that in this case

$$N(x) = x \log x + (2\gamma - 1)x + O(x^{\frac{1}{2}}), \tag{5.2.1}$$

where γ is Euler's constant. This theorem corresponds to (5.1.1), though its proof is not quite so trivial. As in the circle problem, the theorem has been refined by modern writers, the results being much the same. Alternative statements are (i) that

$$\sum_{n \leqq x} d(n) = x \log x + (2\gamma - 1)x + O(x^{\frac{1}{2}}), \tag{5.2.2}$$

$d(n)$ being the number of divisors of n, and (ii) that

$$[x] + \left[\frac{x}{2}\right] + \left[\frac{x}{3}\right] + \ldots = x \log x + (2\gamma - 1)x + O(x^{\frac{1}{2}}). \tag{5.2.3}$$

The last form is that used in Dirichlet's proof.

5.3. I do not know how much Ramanujan had thought about these problems in his early days. He was familiar with the dominant terms

$$x \log x + (2\gamma - 1)x$$

of Dirichlet's approximation, and had probably found them by an argument of the same kind. In this case he should have known (5.2.1). But he was unlikely to make any substantial contribution to the subject, since the sting of all these problems lies in estimates of error, about which his early ideas were quite vague.

Thus in his first letter to me he says

$$\text{“}d(1) + d(2) + \ldots + d(n) = n \log n + (2\gamma - 1)n + \tfrac{1}{2}d(n)\text{”}.$$

We should infer (if we took this statement at all strictly) that the order of the error in Dirichlet's formula does not exceed that of $d([x])$, and in particular that

$$N(x) = x \log x + (2\gamma - 1)x + O(x^{\epsilon})$$

for every positive ϵ. This is false (indeed with $\epsilon = \frac{1}{4}$), though not very easy to disprove. It is however much more likely that Ramanujan inserted the term $\frac{1}{2}d(n)$ merely as a recognition of a quite sound formal principle. When we are investigating the "sum-function"

$$A(x) = \sum_{n \leq x} a_n$$

of an arithmetical function a_n, it is usually not $A(x)$ but

$$A^*(x) = \sum_{n \leq x}{}' a_n,$$

with the last term $a_{[x]}$ multiplied by $\frac{1}{2}$ when x is an integer, which presents itself naturally in the analysis.[1]

Later in his life, of course, Ramanujan was interested in these problems in a more sophisticated way, though his contributions were not important.

5.4. Here, however, I am not concerned with either of these classical problems, but with one connected with another of Ramanujan's assertions. This also is in his first letter to me, and runs

"the number of numbers of the form $2^u 3^v$ less than n is

$$\frac{\log 2n \log 3n}{2 \log 2 \log 3}\text{''}.$$

The formula is of course intended as an approximation, and there is no evidence to show how accurate Ramanujan supposed it to be.

It will be convenient to write

$$\eta = \log n, \quad \omega = \log 2, \quad \omega' = \log 3.$$

Then Ramanujan's assertion is that the number of solutions of the inequalities

(5.4.1) $$u \geqq 0, \quad v \geqq 0, \quad \omega u + \omega' v \leqq \eta$$

is "approximately" $$\frac{\eta^2}{2\omega\omega'} + \frac{\eta}{2\omega} + \frac{\eta}{2\omega'} + \frac{1}{2}.$$

The $\frac{1}{2}$ at the end, of course, is not to be taken too seriously; and we shall see that, if any constant has a right to stand there, it is not $\frac{1}{2}$.

I propose in this lecture to give a short account of some of the very curious and interesting analysis which can be made to hang upon Ramanujan's assertion.

[1] As in Perron's formula

$$A^*(x) = \sum_{\lambda_n \leq x}{}' a_n = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} f(s)\frac{e^{xs}}{s}\,ds,$$

where $f(s) = \Sigma a_n e^{-\lambda_n s}$. See Hardy and Riesz, *The general theory of Dirichlet's series*, 12. Here the last term is to be multiplied by $\frac{1}{2}$ if x is a λ_n.

5.5. Let us suppose then that ω and ω' are any positive numbers, that

$$N(\eta) = N(\eta, \omega, \omega')$$

is the number of solutions of (5.4.1), i.e. the number of lattice-points in a certain right-angled triangle, that

$$\Omega(\eta) = \frac{\eta^2}{2\omega\omega'} + \frac{\eta}{2\omega} + \frac{\eta}{2\omega'}, \tag{5.5.1}$$

and that

$$N(\eta) = \Omega(\eta) + R(\eta). \tag{5.5.2}$$

The problem is that of finding the best bounds that we can for $R(\eta)$. It has been considered, in different forms, by a number of writers, and in particular by Hardy and Littlewood and by Ostrowski. Hardy and Littlewood, in two papers published in 1921 and 1922, consider it in the form in which it is stated here. Ostrowski considers a slightly different problem, superficially more special but substantially equivalent, and obtains, by different methods, very much the same results.

The problem is not changed materially if we disregard lattice-points on the axes. The horizontal and vertical sides of the triangle are η/ω and η/ω', and the number of lattice-points on them is

$$1 + \left[\frac{\eta}{\omega}\right] + \left[\frac{\eta}{\omega'}\right] = \frac{\eta}{\omega} + \frac{\eta}{\omega'} + j,$$

where $|j| \leqq 1$. Hence, if we denote by $M(\eta)$ the number of lattice-points inside the triangle or on its hypotenuse, then

$$M(\eta) = \frac{\eta^2}{2\omega\omega'} - \frac{\eta}{2\omega} - \frac{\eta}{2\omega'} + r(\eta),$$

where $r(\eta)$ differs from $R(\eta)$ by 1 at most.

It is plain first that

$$N(\eta) = \frac{\eta^2}{2\omega\omega'} + O(\eta) \tag{5.5.3}$$

(the area of the triangle, with an error at most of the order of its perimeter). Beyond this everything depends upon the arithmetic nature of

$$\theta = \omega/\omega'.$$

Since $$N(\eta, \omega, \omega') = N(k\eta, k\omega, k\omega'),$$

it is only the ratio θ that is really relevant to the problem.

Rational θ

5.6. The simplest case is that in which θ is rational. Then (since only the ratio of ω and ω' is relevant) we may suppose that

$$\omega = a, \quad \omega' = b,$$

where a and b are positive coprime integers.

The number of solutions of

(5.6.1) $$au+bv = n$$

is $$\left[\frac{n}{ab}\right]+\zeta,$$

where ζ is 0 or 1.[1] It follows that $N(\eta)$ has a jump of order η when η passes through the value n, so that the equation

$$R(\eta) = o(\eta)$$

is false, $R(\eta)$ being effectively of order η.

5.7. We can calculate $N(\eta)$ explicitly as follows. If λ_ν is the sequence of numbers

$$au+bv \qquad (u, v = 0, 1, 2, \ldots)$$

[1] See for example Bachmann, *Niedere Zahlentheorie*, ii, 129. The proof is simple. Suppose that

$$n = mab+r \qquad (0 \leqq r < ab),$$

and write $$u = bU+\beta, \quad v = aV+\alpha \qquad (0 \leqq \alpha < a,\ 0 \leqq \beta < b);$$

so that (5.6.1) becomes

(5.6.2) $$mab+r = (U+V)ab+a\beta+b\alpha.$$

Consider the set of ab numbers

$$a\beta+b\alpha \qquad (0 \leqq \alpha < a,\ 0 \leqq \beta < b).$$

All these numbers are less than $2ab$ and incongruent $(\mathrm{mod}\, ab)$. Hence they can be written as

$$\rho, \quad ab+\rho',$$

where ρ and ρ' together run through

$$0, 1, \ldots, ab-1.$$

If $$a\beta+b\alpha = \rho,$$

then (5.6.2) implies $$U+V = m.$$

There is one pair α, β satisfying the first equation, and $m+1$ pairs U, V satisfying the second, and each set U, V, α, β gives a solution of (5.6.2). On the other hand, if

$$a\beta+b\alpha = ab+\rho',$$

then (5.6.2) gives $$U+V = m-1,$$

and in this case there are m solutions only.

arranged in order of magnitude, and with each λ_ν counted as often as it occurs, and
$$f(s) = \Sigma e^{-\lambda_\nu s},$$
then[1]
$$N^*(\eta) = \sum_{\lambda_\nu \leqq \eta}{}' 1 = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} f(s)\frac{e^{\eta s}}{s}\,ds.$$
Here $c > 0$ and the dash has the meaning explained in § 5.3. But
$$f(s) = \sum_{u,v} e^{-(au+bv)s} = \frac{1}{(1-e^{-as})(1-e^{-bs})},$$
so that

(5.7.1) $$N^*(\eta) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{e^{\eta s}}{(1-e^{-as})(1-e^{-bs})}\frac{ds}{s}.$$

We calculate the integral as a sum of residues. The integrand has

(i) a treble pole at the origin, with residue

(5.7.2) $$P(\eta) = \frac{\eta^2}{2ab} + \frac{\eta}{2a} + \frac{\eta}{2b} + \frac{a^2+3ab+b^2}{12ab};$$

(ii) double poles at the points
$$s = 2k\pi i,$$
where k is an integer, with residues
$$\frac{e^{2k\eta\pi i}}{2kab\pi i}\left(\eta - \frac{1}{2k\pi i} + \tfrac{1}{2}a + \tfrac{1}{2}b\right);$$

(iii) simple poles at the points
$$s = \frac{2k\pi i}{a}\,(a \nmid k), \quad s = \frac{2k\pi i}{b}\,(b \nmid k),\text{[2]}$$
with residues
$$-\frac{1}{4k\pi}\frac{e^{2k\pi i(\eta+\frac{1}{2}b)/a}}{\sin(kb\pi/a)}, \quad -\frac{1}{4k\pi}\frac{e^{2k\pi i(\eta+\frac{1}{2}a)/b}}{\sin(ka\pi/b)}$$
respectively: and it is not difficult to show that the integral is equal to the sum of all these residues.[3] A straightforward calculation then leads to the formula

(5.7.3) $$N^*(\eta) = P(\eta) + Q(\eta) + T_1(\eta) + T_2(\eta),$$

where $P(\eta)$ is defined by (5.7.2),

(5.7.4) $$Q(\eta) = -\frac{\eta + \frac{1}{2}a + \frac{1}{2}b}{ab}(\eta - [\eta] - \tfrac{1}{2}) + \frac{1}{2ab}\{(\eta - [\eta] - \tfrac{1}{2})^2 - \tfrac{1}{12}\},$$

[1] By Perron's formula already quoted on p. 69.

[2] $x \nmid y$ means 'x is not a divisor of y' (the contrary of $x \mid y$).

[3] We apply Cauchy's theorem to a rectangle
$$c - iT,\ c + iT,\ -\xi + iT,\ -\xi - iT,$$
where T is so chosen that the horizontal sides of the rectangle do not pass within a fixed distance δ of any pole, and then make T and ξ tend to infinity.

$$(5.7.5)\quad T_1(\eta) = -\frac{1}{4\pi}\Sigma'\frac{\cos\frac{2k\pi}{a}(\eta+\frac{1}{2}b)}{k\sin\frac{bk\pi}{a}}, \quad T_2(\eta) = -\frac{1}{4\pi}\Sigma'\frac{\cos\frac{2k\pi}{b}(\eta+\frac{1}{2}a)}{k\sin\frac{ak\pi}{b}},$$

the k in the last two series running through all integral values which are not multiples of a and b respectively.

It is plain that

$$Q(\eta) = -\frac{\eta}{ab}(\eta-[\eta]-\tfrac{1}{2})+O(1),$$

and $T_1(\eta)$ and $T_2(\eta)$ are periodic (with periods a and b respectively), and so bounded. Hence

$$(5.7.6)\qquad N^*(\eta) = P(\eta)-\frac{\eta}{ab}(\eta-[\eta]-\tfrac{1}{2})+O(1)$$

for large η. The second term has a discontinuity

$$\frac{n}{ab}+O(1)$$

when η passes through an integral value n, in accordance with our conclusions in § 5.6.

Irrational θ

5.8. The problem is naturally more difficult when θ is irrational. It has been proved, in the first instance, that

$$(5.8.1)\qquad R(\eta) = o(\eta)$$

for all irrational θ, so that

$$(5.8.2)\qquad \Omega(\eta) = \frac{\eta^2}{2\omega\omega'}+\frac{\eta}{2\omega}+\frac{\eta}{2\omega'}$$

is a genuine approximation to $N(\eta)$; and there are sharper results for special classes of θ. These depend upon the nature of the rational approximations to θ, or (what is the same thing) on the behaviour of the quotients a_n in the expression

$$\theta = a_0+\frac{1}{a_1+}\frac{1}{a_2+}\cdots$$

of θ as a continued fraction. If the a_n do not increase very rapidly, then

$$(5.8.3)\qquad R(\eta) = O(\eta^\alpha),$$

for an α between 0 and 1; and if the a_n are bounded, then

$$(5.8.4)\qquad R(\eta) = O(\log\eta).$$

In particular (5.8.3) is true for all algebraic and (5.8.4) for all quadratic θ.

The θ of Ramanujan's problem is transcendental. Its irrationality is trivial, since

$$\frac{\log 2}{\log 3} = \frac{a}{b}$$

would involve $3^a = 2^b$. That it is transcendental follows from the theorem of Gelfond and Schneider, that

$$\alpha^\beta$$

is transcendental whenever α and β are algebraic and β irrational. For 3 and

$$3^\theta = 2$$

are rational, and therefore θ cannot be algebraic.

The most that we can prove about $R(\eta)$, in Ramanujan's case, is that

$$R(\eta) = o\left(\frac{\eta}{\log \eta}\right). \tag{5.8.5}$$

This I prove in § 5.15. The slightly less precise equation

$$R(\eta) = O\left(\frac{\eta}{\log \eta}\right), \tag{5.8.6}$$

which I prove in § 5.12, is substantially one of Ostrowski's theorems.

5.9. There is an "identity" for $N^*(\eta)$, when θ is irrational, which is similar in form to (5.7.3) but involves series which are not convergent in the ordinary sense. I shall not use this, but I shall give a proof of (5.8.1) and (5.8.6) which is due substantially to Heilbronn. Heilbronn's argument is the same in principle as Ostrowski's, but simpler. It is effective when we have to prove $R(\eta)$ "only a little less than η", as it is in (5.8.1) and (5.8.6), but does not lead, in this simple form, to the more precise results which, like (5.8.4), are true only for very "simple" θ.

We define $\{x\}$ by $\qquad \{x\} = x - [x] - \frac{1}{2}.$

This function has the analytic representation

$$\{x\} = -\frac{1}{\pi}\left(\sin 2\pi x + \frac{\sin 4\pi x}{2} + \dots\right),$$

but we shall not use this here.

The ordinate $u = n$ cuts the hypotenuse of the triangle in the point

$$n, \frac{\eta - n\omega}{\omega'},$$

and the number of lattice-points on it is

$$1 + \left[\frac{\eta - n\omega}{\omega'}\right].$$

Hence $N(\eta) = \sum_{n \leq \eta/\omega} \left(1 + \left[\frac{\eta - n\omega}{\omega'}\right]\right) = \sum_{n \leq \eta/\omega} \left(\frac{\eta - n\omega}{\omega'} + \frac{1}{2}\right) - \sum_{n \leq \eta/\omega} \left\{\frac{\eta - n\omega}{\omega'}\right\}.$

If
$$\frac{\eta}{\omega} = \left[\frac{\eta}{\omega}\right] + f,$$
then the first sum is
$$\left(\left[\frac{\eta}{\omega}\right] + 1\right)\left(\frac{\eta}{\omega'} + \frac{1}{2}\right) - \frac{1}{2}\left[\frac{\eta}{\omega}\right]\left(\left[\frac{\eta}{\omega}\right] + 1\right)\frac{\omega}{\omega'}$$
$$= \left(\frac{\eta}{\omega} + 1 - f\right)\left(\frac{\eta}{\omega'} + \frac{1}{2}\right) - \frac{1}{2}\left(\frac{\eta}{\omega} - f\right)\left(\frac{\eta}{\omega} + 1 - f\right)\frac{\omega}{\omega'} = \frac{\eta^2}{2\omega\omega'} + \frac{\eta}{2\omega} + \frac{\eta}{2\omega'} + O(1).$$
Hence
$$(5.9.1) \quad N(\eta) = \frac{\eta^2}{2\omega\omega'} + \frac{\eta}{2\omega} + \frac{\eta}{2\omega'} - S(\eta) + O(1) = \Omega(\eta) - S(\eta) + O(1),$$
where
$$(5.9.2) \qquad S(\eta) = \sum_{n \leq \eta/\omega} \left\{\frac{\eta}{\omega'} - n\theta\right\};$$
and the proof of (5.8.1) is reduced to a proof that
$$(5.9.3) \qquad S(\eta) = o(\eta).$$

Ostrowski, in his paper referred to on p. 70, is concerned with the sum
$$\sum_{n \leq x} \{n\theta\},$$
a series of much the same type as (5.9.2), and his arguments may be applied with equal effect to (5.9.2).

5.10. If
$$\frac{p}{q} = \frac{p_m}{q_m}$$
is a convergent to the continued fraction for θ, then q_m tends to infinity with m and
$$\theta - \frac{p}{q} = \theta - \frac{p_m}{q_m} = \frac{(-1)^m}{q_m q'_{m+1}},$$
where
$$q'_{m+1} = a'_{m+1} q_m + q_{m-1}$$
and a'_{m+1} is the complete quotient corresponding to a_{m+1}. We suppose that $q < \eta/\omega$ and
$$\left[\frac{\eta}{\omega}\right] = rq + s \qquad (r \geq 1, 0 \leq s < q).$$
Then
$$(5.10.1) \qquad S(\eta) = \sum_{n=0}^{rq-1} \left\{\frac{\eta}{\omega'} - n\theta\right\} + O(s) = S^*(\eta) + O(s).$$

If we write
$$n = \mu q + \nu \qquad (\mu = 0, 1, \ldots, r-1;\ \nu = 0, 1, \ldots, q-1),$$

then

$$S^*(\eta) = \sum_{\mu=0}^{r-1}\sum_{\nu=0}^{q-1}\left\{\frac{\eta}{\omega'} - (\mu q+\nu)\,\theta\right\} = \sum_{\mu=0}^{r-1} S_\mu(\eta), \tag{5.10.2}$$

where

$$S_\mu(\eta) = \sum_{0}^{q-1}\{\alpha - \nu\theta\}, \quad \alpha = \alpha_\mu = \frac{\eta}{\omega'} - \mu q\theta. \tag{5.10.3}$$

We prove in the next section that

$$S_\mu(\eta) = O(1), \tag{5.10.4}$$

uniformly in α. If we assume this for a moment, then (5.10.1), (5.10.2), and (5.10.4) will give

$$S(\eta) = O(r) + O(s) = O\left(\frac{\eta}{q_m}\right) + O(q_m). \tag{5.10.5}$$

Proof that $S_\mu(\eta)$ is bounded

5.11. Let us suppose, to fix our ideas, that m is even. Then

$$0 < \theta - \frac{p}{q} < \frac{1}{q^2}.$$

Also

$$\alpha = \frac{a}{q} + \delta,$$

where a is an integer and

$$0 \leqq \delta < \frac{1}{q}.$$

Thus

$$\alpha - \nu\theta - \frac{a-\nu p}{q} = \delta - \nu\left(\theta - \frac{p}{q}\right)$$

is less than $1/q$ and greater than $-1/q$, so that

$$\left|\alpha - \nu\theta - \frac{a-\nu p}{q}\right| < \frac{1}{q}. \tag{5.11.1}$$

If m is odd, so that θ is less than p/q, then we define a by

$$\alpha = \frac{a}{q} - \delta \qquad \left(0 \leqq \delta < \frac{1}{q}\right),$$

and (5.11.1) is then still true. Hence in any case

$$\alpha - \nu\theta, \quad \frac{a-\nu p}{q} \tag{5.11.2}$$

differ by less than $1/q$. But $(p, q) = 1$, and therefore

$$\frac{a-\nu p}{q} = i_\nu + r_\nu,$$

where i_ν is an integer and r_ν runs, in some order, through the values

$$0, \frac{1}{q}, \frac{2}{q}, \ldots, \frac{q-1}{q}.$$

The integral parts of the numbers (5.11.2) can differ only if there is an integer between the numbers, and this can happen only if $r_\nu = 0$, and therefore once only. In this case the difference

$$[\alpha - \nu\theta] - \left[\frac{a-\nu p}{q}\right]$$

is 0 or -1. We call this difference ϵ.

It follows that

$$\sum_0^{q-1} [\alpha - \nu\theta] = \sum_0^{q-1} \left[\frac{a-\nu p}{q}\right] + \epsilon$$
$$= \sum_0^{q-1} \frac{a-\nu p}{q} - \frac{1}{q} - \frac{2}{q} - \ldots - \frac{q-1}{q} + \epsilon$$
$$= a - \tfrac{1}{2}(p+1)(q-1) + \epsilon,$$
$$S_\mu(q) = \sum_0^{q-1} \{\alpha - \nu\theta\} = \sum_0^{q-1} (\alpha - \nu\theta - \tfrac{1}{2}) - \sum_0^{q-1} [\alpha - \nu\theta]$$
$$= q(\alpha - \tfrac{1}{2}) - \tfrac{1}{2}q(q-1)\theta - a + \tfrac{1}{2}(p+1)(q-1) - \epsilon$$
$$= q\alpha - a - \tfrac{1}{2}q(q-1)\left(\theta - \frac{p}{q}\right) - \tfrac{1}{2} - \epsilon,$$
$$|S_\mu(q)| \leqq |q\delta| + \tfrac{1}{2}q^2 \cdot \frac{1}{q^2} + \tfrac{1}{2} + 1 < 3.$$

This completes the proof of (5.10.4) and so of (5.10.5).

5.12. We have thus

$$S(\eta) = O\left(\frac{\eta}{q_m}\right) + O(q_m).$$

Given any positive δ, and any sufficiently large η, we can choose m so that

$$\frac{1}{\delta} < q_m < \delta\eta,$$

$$\frac{\eta}{q_m} + q_m < 2\delta\eta.$$

Hence $S(\eta) = o(\eta)$ for all irrational θ; and this, as we have seen, is equivalent to (5.8.1).

Suppose next that θ has Ramanujan's value. Then

(5.12.1) $$|2^q - 3^p| \geqq 1$$

for all p, q. This is $$|1 - e^{p \log 3 - q \log 2}| \geqq 2^{-q},$$

from which it follows that

(5.12.2) $$|q\theta-p|>A2^{-q}$$

for a constant A and all q.

If p and q are p_m and q_m, then

$$|q_m\theta-p_m|<\frac{1}{q_{m+1}},$$

and so $$q_{m+1}<A2^{q_m}<e^{Aq_m}.$$[1]

We can choose m so that

$$q_m \leqq \frac{\eta}{\log\eta}<q_{m+1}.$$

Then $$e^{Aq_m}>q_{m+1}>\frac{\eta}{\log\eta},$$

$$q_m>A\log\eta,$$

$$\frac{\eta}{q_m}+q_m<A\frac{\eta}{\log\eta},$$

and this proves (5.8.6).

Proof of (5.8.5)

5.13. It remains to prove (5.8.5). This is a very small improvement on (5.8.6), and no doubt says very much less than the final truth, so that it may seem odd to insist on it. But the proof depends on a very interesting theorem of Pillai, and it is for the sake of this theorem that I include it.

We write down the sequence

$$2,\quad 3,\quad 4,\quad 8,\quad 9,\quad 16,\quad 27,\quad 32,\quad 64,\quad 81,\ldots,$$

formed by the powers of 2 and 3 arranged in order of magnitude, and call the n-th term of the sequence u_n. Then Pillai's theorem asserts that $u_{n+1}-u_n$, the n-th gap in the series, tends to infinity nearly as rapidly as u_n. More precisely, given any positive δ,

$$|2^x-3^y|>2^{(1-\delta)x}$$

for all integral x and y with $x>x_0(\delta)$.

It follows in particular that the equation

(5.13.1) $$2^x-3^y=k$$

can have only a finite number of integral solutions for any given k. This is easier to prove. A very well-known theorem of Thue says that

(5.13.2) $$AX^3-BY^3=l$$

[1] The various A's are naturally not the same.

(with any integral A, B, l) has only a finite number of solutions. If we now put

$$x = 3\xi + \rho, \quad y = 3\eta + \sigma \qquad (\rho, \sigma = 0, 1, 2),$$

we obtain nine equations

$$2^\rho(2^\xi)^3 - 3^\sigma(3^\eta)^3 = k$$

of the form (5.13.2). Each of these has only a finite number of solutions, and so therefore has (5.13.1).

5.14. Pillai proves, more generally, that *if m, n, a, b are given positive integers,*

$$am^x - bn^y \neq 0$$

for any integral x and y, and δ is positive, then

$$| am^x - bn^y | > m^{(1-\delta)x}$$

for all $x > x_0(\delta)$. Here $x_0(\delta)$, of course, depends on m, n, a and b as well as on δ. Similarly, a K which appears in the argument may depend on m, n, a and b, as well as on any parameter indicated specially.

We use one deep theorem of Siegel: *if ξ is an algebraic number of degree r, then there is an $A(\xi)$, depending only on ξ, such that*

$$\left|\xi - \frac{p}{q}\right| > \frac{A(\xi)}{q^{2r^{\frac{1}{2}}}}$$

for all integral p and q. There is a familiar theorem of Liouville in which r stands in the place of $2r^{\frac{1}{2}}$. Thue showed that r could be replaced by any number greater than $\frac{1}{2}r + 1$, but neither Liouville's nor Thue's theorems would be strong enough for Pillai's application. What is essential is to have an exponent of lower order of magnitude than r.

Suppose now that u and v are positive integers, that a/b is not a perfect r-th power, and that

$$\tfrac{1}{2}au^r < bv^r < au^r.$$

If

$$w = \left(\frac{a}{b}\right)^{1/r} u = \alpha u,$$

then α is an algebraic number of degree r at most, and

$$au^r - bv^r = b(w^r - v^r) > brv^{r-1}(w - v)$$

$$> K(r)\, u^{r-1}(w - v) = K(r)\, u^r\left(\alpha - \frac{v}{u}\right).$$

Hence

(5.14.1) $$au^r - bv^r > K(r)u^{r-2r^{\frac{1}{2}}},$$

by Siegel's theorem. It is obvious that this is also true if $0 < bv^r \leqq \frac{1}{2}au^r$, so that it holds whenever $au^r - bv^r$ is positive. Similarly

(5.14.2) $$bv^r - au^r > K(r)v^{r-2r^{\frac{1}{2}}}$$

whenever it is positive, and a moment's consideration shows that we can unite (5.14.1) and (5.14.2) in the form

$$(5.14.3) \qquad |au^r - bv^r| > K(r)\, z^{r-2r^{\frac{1}{2}}},$$

u and v being any positive integers, and z being u or v at our discretion.[1] Here $K(r)$ of course depends on a and b as well as on r.

We can now prove Pillai's theorem. We take δ small and positive and

$$r = \frac{16}{\delta^2},$$

and write $\qquad x = sr+h \;\; (0 \leqq h < r), \quad y = tr+l \;\; (0 \leqq l < r),$

$$u = m^s, \quad v = n^t.$$

Then $\qquad |am^x - bn^y| = |am^h . u^r - bn^l . v^r| > K(r,h,l)\, u^{r-2r^{\frac{1}{2}}},$

by (5.14.3); and so $\qquad |am^x - bn^y| > K(\delta)\, u^{r-2r^{\frac{1}{2}}},$

since the number of values of h and l concerned depends only on r, and r only on δ.

Also $\qquad u^r = m^{x-h} > K(\delta)\, m^x,$

$$u^{r-2r^{\frac{1}{2}}} = u^{(1-\frac{1}{2}\delta)r} > K(\delta)\, m^{(1-\frac{1}{2}\delta)x};$$

and so $\qquad |am^x - bn^y| > K(\delta)\, m^{(1-\frac{1}{2}\delta)x}.$

From this it follows that

$$|am^x - bn^y| > m^{(1-\delta)x}$$

for $x > x_0(\delta)$, and this is Pillai's theorem.

5.15. Pillai's theorem is just sufficient to improve (5.8.6) into (5.8.5). Returning to the argument of § 5.12, we can replace (5.12.1) by

$$|2^q - 3^p| \geqq 2^{q(1-\delta)}$$

(for any positive δ and sufficiently large p, q); and (5.12.2) by

$$|q\theta - p| > 2^{-\delta q} = e^{-\epsilon q},$$

where $\epsilon = \delta \log 2$. It follows that

$$q_{m+1} < e^{\epsilon q_m}$$

for sufficiently large m. We choose m so that

$$q_m \leqq \frac{2\epsilon\eta}{\log \eta} < q_{m+1}.$$

Then $\qquad q_m > \frac{1}{\epsilon} \log q_{m+1} > \frac{1}{\epsilon}(\log \eta - \log\log \eta + \log 2\epsilon) > \frac{1}{2\epsilon} \log \eta$

[1] Since a/b is not an r-th power, $au^r - bv^r$ cannot be 0. If it is of smaller order than u^r or v^r, then u and v are of the same order.

for large η and m, and

$$q_m + \frac{\eta}{q_m} < \frac{4\epsilon\eta}{\log\eta},$$

so that

$$S(\eta) = o\left(\frac{\eta}{\log\eta}\right)$$

We have thus replaced the O of (5.8.6) by o. It may seem rather disappointing that the use of such powerful weapons should lead to so small an improvement in the final result, but such disappointments are common in this kind of analysis.

NOTES ON LECTURE V

§ 5.1. There are two proofs of (5.1.1) in Hardy and Wright, 268–269.

The essential difficulty of the circle problem is that of determining Θ, the smallest value of ξ such that

$$N(x) = \pi x + O(x^{\xi+\epsilon})$$

for every positive ϵ. It follows from (5.1.1) that $\Theta \leqq \frac{1}{2}$. Sierpinski proved in 1906 that $\Theta \leqq \frac{1}{3}$, van der Corput in 1923 that $\Theta < \frac{1}{3}$, Littlewood and Walfisz in 1924 that $\Theta \leqq \frac{37}{112}$. In the other direction, Hardy and Landau proved independently in 1915 that $\Theta \geqq \frac{1}{4}$. There is a profound study of the problem, up to this stage, in Landau, *Vorlesungen*, ii, 183–308.

The results have been improved since by Nieland, Titchmarsh, and Vinogradov. The best result is Vinogradov's $\Theta \leqq \frac{17}{53}$.

Fuller references will be found in Bohr and Cramer, *Enzykl. d. Math. Wiss.* II C 8 (1922), 823–824, and in two papers by Titchmarsh, *Quarterly Journal of Math.* (Oxford), 2 (1931), 161–173 and *Proc. London Math. Soc.* (2), 38 (1935), 96–115 and 555. The paper by Littlewood and Walfisz is in *Proc. Royal Soc.* (A), 106 (1924), 478–488.

§ 5.2. For Dirichlet's proof of (5.2.1) see, for example, Hardy and Wright, 262–263.

The history of this problem is very similar to that of the circle problem, though recent writers have tended to concentrate on the latter. In this problem Θ is the smallest ξ for which

$$N(x) = x\log x + (2\gamma - 1)\,x + O(x^{\xi+\epsilon}).$$

It follows from (5.2.1) that $\Theta \leqq \frac{1}{2}$. Voronoi proved in 1903 that $\Theta \leqq \frac{1}{3}$, Hardy and Landau in 1916 that $\Theta \geqq \frac{1}{4}$, and van der Corput in 1922 that $\Theta < \frac{33}{100}$. For fuller references see Bohr and Cramér's article quoted above, 815–822.

The best known result here seems to be $\Theta \leqq \frac{27}{82}$, proved by van der Corput in a later paper in *Math. Annalen*, 98 (1928), 697–717.

§ 5.3. One example is the formula

$$\text{(1)} \qquad \sum_0^\infty \frac{r(n)}{\sqrt{(n+a)}}\, e^{-2\pi\sqrt{\{(n+a)b\}}} = \sum_0^\infty \frac{r(n)}{\sqrt{(n+b)}}\, e^{-2\pi\sqrt{\{(n+b)a\}}}.$$

Here a and b are positive and $r(n)$ has the meaning of § 5.1.

It is easy to prove (1) by writing the series on the left as

$$\frac{1}{\sqrt{\pi}}\int_0^\infty e^{-ax-\pi^2 b/x}\{\Sigma r(n)\, e^{-nx}\}\frac{dx}{\sqrt{x}} = \frac{1}{\sqrt{\pi}}\int_0^\infty e^{-ax-\pi^2 b/x}\,\vartheta^2(x)\frac{dx}{\sqrt{x}},$$

where

$$\vartheta(x) = 1 + 2e^{-x} + 2e^{-4\pi x} + \dots,$$

and using the functional equation

$$\vartheta(x) = \sqrt{\left(\frac{\pi}{x}\right)}\vartheta\left(\frac{\pi^2}{x}\right).$$

There is a generalisation for the ellipsoid, which I quote in the *Quarterly Journal of Math.* 46 (1915), 283.

The formula remains valid so long as $\sqrt{a}$ and $\sqrt{b}$ have positive real parts. If we put $a = xe^{i\theta}$, where $0<\theta<\pi$, and x is positive and non-integral, make $\theta\to\pi$, take the imaginary parts, and then put $b = 0$, we obtain

$$(2) \qquad \sum_{0\leq n<x} \frac{r(n)}{\sqrt{(x-n)}} = 2\pi\sqrt{x} + \sum_{1}^{\infty} \frac{r(n)}{\sqrt{n}} \sin\{2\pi\sqrt{(nx)}\}.$$

This is also the result of putting $\alpha = -\frac{1}{2}$ in a formula for $\Sigma(x-n)^{\alpha} r(n)$ which I proved (for $\alpha>0$) in a paper in *Proc. London Math. Soc.* (2), 15 (1916), 192–213 (205). Neither of these deductions, of course, is a *proof* of (2), and I do not know that there is any proof standing in the literature. The formula is of the same type as the identity

$$\sum_{0\leq n\leq x}' r(n) = \pi x + \sqrt{x}\sum_{1}^{\infty}\frac{r(n)}{\sqrt{n}} J_1\{2\pi\sqrt{(nx)}\}$$

(first proved in my paper in the *Quarterly Journal* quoted above).

I imagine that the series on the right of (2) is summable by Cesàro or Rieszian means of any positive order, so long as x is not an integer.

§ 5.4. *Papers*, xxiv (3).

§ 5.5. The principal papers by Hardy and Littlewood are in *Proc. London Math. Soc.* (2), 20 (1922), 15–36 and *Abh. math. Semin. Hamburg*, 1 (1922), 212–249; and that by Ostrowski in *Abh. math. Semin. Hamburg*, 1 (1922), 77–98 and 250–251. There is an account of these problems, with an exhaustive bibliography, in Koksma, "Diophantische Approximationen", *Ergebnisse der Math.* IV 4 (1936), Kap. IX.

§ 5.8. The problem of proving the transcendentality of α^{β}, under the conditions stated, was the seventh of the problems proposed by Hilbert in his address "Mathematische Probleme" to the seventh international congress of mathematicians in Paris in 1900. This address was published in German in *Göttinger Nachrichten* (1900), 253–297, and in French, under the title "Sur les problèmes futurs des mathématiques", in the official report of the congress (Paris, 1902). See Koksma, *l.c. supra*, Kap. IV, especially pp. 64–65, where there are references to the papers of Gelfond and Schneider.

§ 5.9. For the "identity" see Theorem 4 of the second of the papers by Hardy and Littlewood referred to under § 5.5.

Dr Heilbronn communicated his proof to me personally.

§ 5.10. The notation for continued fractions is that of Hardy and Wright, ch. X.

§§ 5.13–14. Pillai's theorem was proved in *Journal Indian Math. Soc.* 19 (1931), 1–11.

Pólya, *Math. Zeitschrift*, 1 (1918), 143–148, proved a general theorem from which it follows, as a very particular case, that (5.13.1) has only a finite number of solutions. Herschfeld, *Bull. Amer. Math. Soc.* 42 (1936), 231–234, has proved that there is at most *one* solution when k is sufficiently large. There are further results of this kind in another paper by Pillai, *Journal Indian Math. Soc.* (2), 2 (1936), 119–122 and 215.

For Thue's and Siegel's theorems see Landau, *Vorlesungen*, iii, 37–56. Siegel's theorem is Satz 691. It is stated there in a slightly different form, and for $r\geqq 3$, but it is trivial when $r = 2$.

VI

RAMANUJAN'S WORK ON PARTITIONS

6.1. A *partition* of n is a division of n into any number of positive integral parts. Thus

$$4 = 3+1 = 2+2 = 2+1+1 = 1+1+1+1$$

has 5 partitions. The order in which the parts are arranged is irrelevant, so that we may think of them, if we please, as arranged in descending order. We denote the number of partitions of n by $p(n)$; thus $p(1) = 1$ and $p(4) = 5$. It is convenient to define $p(0)$ as 1.

A partition of n may be represented graphically by an array of dots or "nodes". Thus

represents the partition $6+3+3+2+1$ of 15. We could also read the graph vertically, when it would represent the partition $5+4+3+1+1+1$. Two partitions so related are called *conjugate*.

The largest part in either of these partitions is equal to the number of parts in the other. Generally, a graph with m rows represents, when read horizontally, a partition into m parts, while read vertically it represents a partition into parts the largest of which is m. It follows that the number of partitions of n into m parts is equal to the number of partitions into parts of which the largest is m; and that the number of partitions into at most m parts is equal to the number of partitions into parts which do not exceed m.

6.2. There are many less obvious theorems about partitions which can be proved by a direct study of their graphs. I choose as an example F. Franklin's beautiful proof of a famous identity of Euler.

Euler's identity is

$$(1-x)(1-x^2)(1-x^3)\ldots = 1-x-x^2+x^5+x^7-\ldots. \tag{6.2.1}$$

The right-hand side is

$$1+\sum_{k=1}^{\infty}(-1)^k\{x^{\frac{1}{2}k(3k-1)}+x^{\frac{1}{2}k(3k+1)}\} = \sum_{k=-\infty}^{\infty}(-1)^k x^{\frac{1}{2}k(3k+1)}.$$

It may also be written as $$1+\sum_{1}^{\infty} c_n x^n,$$

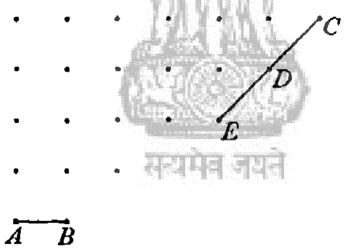
where c_n is 0 unless $n = \frac{1}{2}k(3k \pm 1)$, and then $(-1)^k$.

Let us write $$(1-x)(1-x^2)(1-x^3)\ldots = \sum \gamma(n)\, x^n,$$

and multiply out the product so as to obtain an arithmetical interpretation of the coefficient $\gamma(n)$. There is a term in x^n corresponding to every partition of n into unequal parts: thus the partition $6 = 3+2+1$ gives rise to a term $(-1)^3 x^6$. Generally, a partition of n into μ unequal parts contributes $(-1)^\mu$ to $\gamma(n)$, so that
$$\gamma(n) = p_e(n) - p_o(n),$$

where $p_e(n)$ and $p_o(n)$ are the numbers of partitions of n into an even and an odd number of unequal parts. We try to establish, as far as we can, a one-to-one correspondence between partitions of these two types. The correspondence cannot be complete, since a complete correspondence would show that $p_e(n) = p_o(n)$ and $\gamma(n) = 0$ for every n.

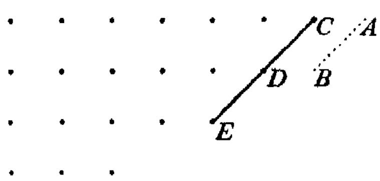
We take the graph G_1 which represents (read horizontally) any partition of n into any number of unequal parts, in descending order. We call



(G_1)

the lowest line AB the *base* β of the graph. From the extreme north-east node, we draw the longest south-westerly line possible in the graph; it might of course contain one node only. This line CDE we call the slope σ of the graph. We write $\beta < \sigma$ when (as in graph G_1) there are more nodes in σ than in β, and use a similar notation in other cases. Then there are three possibilities.

(i) $\beta < \sigma$. We move β into a position parallel to and outside σ, as shown in graph G_2. This gives a new partition into decreasing unequal parts, and



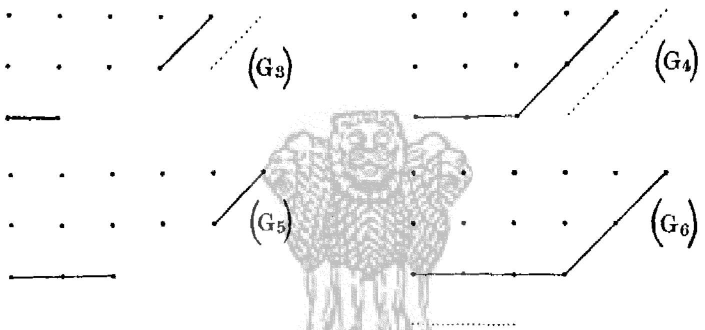
(G_2)

into a number of such parts whose parity is opposite to that of the number in G_1.

We call this operation O, and the converse operation (removing σ and placing it below β) Ω. It is plain that Ω is not possible, when $\beta < \sigma$, without violating the conditions of the graph.

(ii) $\beta = \sigma$. In this case O is possible (as in graph G_3) unless β meets σ (as in graph G_4), when it is impossible. Ω is not possible in either case.

(iii) $\beta > \sigma$. In this case O is always impossible. Ω is possible (as in graph G_5) unless β meets σ and $\beta = \sigma + 1$ (as in graph G_6). Ω is impossible in the last case because it would lead to a partition with two equal parts.



To sum up: there is a (1, 1) correspondence between the two types of partitions except in the cases exemplified by (G_4) and (G_6). In the first of these exceptional cases n is of the form

$$k + (k+1) + (k+2) + \dots + (2k-1) = \tfrac{1}{2}(3k^2 - k),$$

and in this case there is an excess of one even or one odd partition according as k is even or odd. In the second case n is of the form

$$(k+1) + (k+2) + (k+3) + \dots + 2k = \tfrac{1}{2}(3k^2 + k),$$

and the excess is the same. Hence $\gamma(n)$ is 0 unless $n = \tfrac{1}{2}(3k^2 \pm k)$, when $\gamma(n) = (-1)^k$. That is to say, $\gamma(n) = c_n$, and this is Euler's theorem.

6.3. Franklin's proof is a very striking example of what may be done by elementary "combinatorial" arguments; but most of the theory of partitions requires a more analytical setting.

The analytical theory was founded by Euler, and rests, like the analytical theory of primes, on the idea of a *generating function*. But the generating functions of the theory of partitions are *power-series*

$$F(x) = \Sigma f(n)\, x^n.$$

The function $F(n)$ is called the generating function of $f(n)$, and is also said to *enumerate* $f(n)$.

It is easy to find the generating function of $p(n)$. This is the function

$$F(x) = \frac{1}{(1-x)(1-x^2)(1-x^3)\dots}$$

(a function fundamental in the theory of elliptic functions). In fact, expanding each factor of $F(x)$ by the binomial theorem, we have

$$F(x) = (1+x+x^2+x^3+\dots)(1+x^2+x^4+x^6+\dots)(1+x^3+x^6+x^9+\dots)\dots,$$

and a moment's consideration shows that every partition of n contributes just 1 to the coefficient of x^n. Hence

$$F(x) = \Sigma p(n)\,x^n.$$

It is equally easy to find the generating functions which enumerate partitions of n into parts restricted in various ways. Thus

$$\frac{1}{(1-x)(1-x^3)(1-x^5)\dots}$$

enumerates partitions into *odd* parts;

$$(1+x)(1+x^2)(1+x^3)\dots$$

partitions into *unequal* parts; and

$$(1+x)(1+x^3)(1+x^5)\dots$$

partitions into *odd and unequal* parts. Euler's product

$$(1-x)(1-x^2)(1-x^3)\dots$$

enumerates the function $\gamma(n)$ of § 6.2, the excess of the number of partitions of n into an even number of unequal parts over that of its partitions into an odd number.

Similarly (I take examples which I shall want to refer to later)

$$\frac{1}{(1-x)(1-x^2)\dots(1-x^m)}$$

enumerates partitions into parts not exceeding m or (what we have seen to be equivalent) into at most m parts;

$$\frac{x^N}{(1-x)(1-x^2)\dots(1-x^m)}$$

enumerates the number of partitions of $n-N$ into at most m parts; and

$$\frac{x^N}{(1-x^2)(1-x^4)\dots(1-x^{2m})}$$

the number of partitions of $n-N$ into at most m even parts, or of $\frac{1}{2}(n-N)$ into at most m parts of any kind. Finally

$$\frac{1}{(1-x)(1-x^4)(1-x^6)(1-x^9)\ldots},$$

where the indices of x are the numbers $5m+1$ and $5m+4$, enumerates the partitions of n into parts of these two forms.

Ramanujan's congruences

6.4. Very little is known about the arithmetical properties of $p(n)$; we do not know, for example, when $p(n)$ is odd or even. Ramanujan was the first, and up to now the only, mathematician to discover any such properties; and his theorems were discovered, in the first instance, by observation. MacMahon had calculated, for other purposes to which I shall refer later, a table of $p(n)$ for the first 200 values of n, and Ramanujan observed that the table indicated certain simple congruence properties of $p(n)$. In particular, the numbers of the partitions of numbers $5m+4$, $7m+5$, and $11m+6$ are divisible by 5, 7 and 11 respectively: i.e.

(6.4.1) $$p(5m+4) \equiv 0 \pmod 5,$$

(6.4.2) $$p(7m+5) \equiv 0 \pmod 7,$$

(6.4.3) $$p(11m+6) \equiv 0 \pmod{11}.$$

Thus $p(4) = 5$ and $p(5) = 7$.

6.5. Ramanujan found comparatively simple proofs of (6.4.1) and (6.4.2). These depend on two formulae which belong properly to the theory of elliptic functions, Euler's formula (6.2.1) and Jacobi's formula

(6.5.1) $$\{(1-x)(1-x^2)(1-x^3)\ldots\}^3 = 1-3x+5x^3-7x^6+\ldots,$$

where the indices on the right are the triangular numbers $\frac{1}{2}k(k+1)$. We have proved (6.2.1), but there is no equally simple proof of (6.5.1). We may also write (6.5.1) as

$$\{(1-x)(1-x^2)(1-x^3)\ldots\}^3 = \tfrac{1}{2}\sum_{-\infty}^{\infty}(-1)^k(2k+1)x^{\frac{1}{2}k(k+1)}.$$

Ramanujan now argues as follows. We have

$$\begin{aligned} x\{(1-x)(1-x^2)\ldots\}^4 &= x\{(1-x)(1-x^2)\ldots\}.\{(1-x)(1-x^2)\ldots\}^3 \\ &= x(1-x-x^2+x^5+\ldots)(1-3x+5x^3-7x^6+\ldots), \end{aligned}$$

by (6.2.1) and (6.5.1). We write this as

(6.5.2) $$x\{(1-x)(1-x^2)\ldots\}^4 = \tfrac{1}{2}\Sigma\Sigma(-1)^{\mu+\nu}(2\nu+1)x^{1+\frac{1}{2}\mu(3\mu+1)+\frac{1}{2}\nu(\nu+1)},$$

both μ and ν running from $-\infty$ to ∞, and we consider in what circumstances the index of x is divisible by 5. This demands that

$$2(\mu+1)^2+(2\nu+1)^2 = 8\{1+\tfrac{1}{2}\mu(3\mu+1)+\tfrac{1}{2}\nu(\nu+1)\}-10\mu^2-5$$

shall also be a multiple of 5. Now

$$2(\mu+1)^2 \equiv 0, 2, \text{ or } 3 \pmod 5$$

and

$$(2\nu+1)^2 \equiv 0, 1, \text{ or } 4 \pmod 5,$$

as we can see by enumerating the possible cases; and the sum of one residue from each set can be 0 or 5 only if each is 0. Hence, if the index in (6.5.2) is a multiple of 5, the coefficient $2\nu+1$ is also a multiple of 5, and therefore *the coefficient of* x^{5m+5} *in*

$$x\{(1-x)(1-x^2)\ldots\}^4$$

is a multiple of 5.

Next, in the binomial expansion of

$$\frac{1}{(1-x)^5},$$

all the coefficients are divisible by 5 except those of $1, x^5, x^{10}, \ldots$, which have residue 1 (mod 5). That is to say

$$\frac{1}{(1-x)^5} \equiv \frac{1}{1-x^5} \pmod 5$$[1]

or

$$\frac{1-x^5}{(1-x)^5} \equiv 1 \pmod 5.$$

Hence the coefficient of x^{5m+5} in

$$x\frac{(1-x^5)(1-x^{10})\ldots}{(1-x)(1-x^2)\ldots} = x\{(1-x)(1-x^2)\ldots\}^4\frac{(1-x^5)(1-x^{10})\ldots}{\{(1-x)(1-x^2)\ldots\}^5}$$

is a multiple of 5, and so therefore is that in

$$\frac{x}{(1-x)(1-x^2)(1-x^3)\ldots};$$

and this coefficient is $p(5m+4)$.

We can prove (6.4.2) similarly. In this case we write

$$x^2\{(1-x)(1-x^2)\ldots\}^6 = x^2(1-3x+5x^3-7x^6+\ldots)^2$$
$$= \tfrac{1}{4}\Sigma\Sigma(-1)^{\mu+\nu}(2\mu+1)(2\nu+1)x^{2+\frac{1}{2}\mu(\mu+1)+\frac{1}{2}\nu(\nu+1)},$$

and observe that

$$(2\mu+1)^2+(2\nu+1)^2 = 8\{2+\tfrac{1}{2}\mu(\mu+1)+\tfrac{1}{2}\nu(\nu+1)\}-14$$

can be divisible by 7 only if $2\mu+1$ and $2\nu+1$ are both divisible by 7. The proof may then be completed on the same lines as before. There does not seem to be any equally simple proof of (6.4.3).

[1] Corresponding coefficients are congruent (mod 5).

6.6. Ramanujan went a good deal further. He proved congruences with moduli 5^2, 7^2 and 11^2, that for 5^2 being

$$p(25m+24) \equiv 0 \pmod{5^2},$$

and put forward a general conjecture: *if*

$$\delta = 5^a 7^b 11^c$$

and $$24\lambda \equiv 1 \pmod{\delta},$$

then $$p(m\delta+\lambda) \equiv 0 \pmod{\delta}$$

for every m. It would be sufficient to prove the congruence for the special moduli 5^a, 7^b and 11^c, the general congruence being a corollary.

This conjecture has led to a good deal of work, and it has been found that Ramanujan generalised too far. Gupta, extending MacMahon's calculations of $p(n)$ up to $n = 300$, found that

$$p(243) = 133978259344888,$$

a number not divisible by 7^3; and S. Chowla observed that, since

$$24.243 \equiv 1 \pmod{7^3},$$

this contradicts Ramanujan's conjecture. On the other hand Krečmar has proved the congruence

$$p(125m+99) \equiv 0 \pmod{5^3},$$

and Watson the congruence for general 5^a; and D. H. Lehmer has verified the conjecture, in certain special cases, for 11^3 and 11^4. Lehmer's work involves the calculation of some particular very large values of $p(n)$ by a method which I shall explain in Lecture VIII: the largest is

(6.6.1) $$p(14031) = 92\ 85303\ 04759\ 09931\ 69434\ 85156\ 67127\ 75089\ 29160\ 56358\ 46500\ 54568\ 28164\ 58081\ 50403\ 46756\ 75123\ 95895\ 59113\ 47418\ 88383\ 22063\ 43272\ 91599\ 91345\ 00745,$$

a number divisible by 11^4.

6.7. There is another proof of (6.4.1) which is much more difficult than the one I quoted, but which says much more and led Ramanujan much deeper into the theory of elliptic modular functions. In the same paper in which he proved (6.4.1) and (6.4.2), Ramanujan stated without proof the two remarkable identities

(6.7.1) $$p(4)+p(9)x+p(14)x^2+\ldots = 5\frac{\{(1-x^5)(1-x^{10})(1-x^{15})\ldots\}^5}{\{(1-x)(1-x^2)(1-x^3)\ldots\}^6}$$

and

$$(6.7.2)\quad p(5)+p(12)x+p(17)x^2+\ldots = 7\frac{\{(1-x^7)(1-x^{14})(1-x^{21})\ldots\}^3}{\{(1-x)(1-x^2)(1-x^3)\ldots\}^4} + 49x\frac{\{(1-x^7)(1-x^{14})(1-x^{21})\ldots\}^7}{\{(1-x)(1-x^2)(1-x^3)\ldots\}^8}.$$

These make (6.4.1) and (6.4.2) intuitive, and also provide proofs of the congruences to moduli 5^2 and 7^2. Thus, if we assume (6.7.1), we have

$$\frac{p(4)x+p(9)x^2+\ldots}{5\{(1-x^5)(1-x^{10})\ldots\}^4} = \frac{x}{(1-x)(1-x^2)\ldots}\cdot\frac{(1-x^5)(1-x^{10})\ldots}{\{(1-x)(1-x^2)\ldots\}^5}$$

$$\equiv \frac{x}{(1-x)(1-x^2)\ldots} \pmod 5.$$

Hence (after what we have proved already) the coefficient of x^{5m+5} on the left-hand side is a multiple of 5; and from this it follows that

$$p(25m+24)\equiv 0 \pmod{5^2}.$$

Similarly (6.7.2) leads to

$$p(49m+47)\equiv 0 \pmod{7^2}.$$

Ramanujan never published a complete proof of (6.7.1) or (6.7.2); but proofs have been found by Darling and Mordell.

The Rogers-Ramanujan identities

6.8. I come next to two formulae, the "Rogers-Ramanujan identities", in which Ramanujan had been anticipated by a much less famous mathematician, but which are certainly as remarkable as any which even he ever wrote down.

The Rogers-Ramanujan identities are

$$(6.8.1)\quad 1+\frac{x}{1-x}+\frac{x^4}{(1-x)(1-x^2)}+\ldots+\frac{x^{m^2}}{(1-x)(1-x^2)\ldots(1-x^m)}+\ldots = \frac{1}{(1-x)(1-x^6)\ldots(1-x^4)(1-x^9)\ldots}$$

and

$$(6.8.2)\quad 1+\frac{x^2}{1-x}+\frac{x^6}{(1-x)(1-x^2)}+\ldots+\frac{x^{m(m+1)}}{(1-x)(1-x^2)\ldots(1-x^m)}+\ldots = \frac{1}{(1-x^2)(1-x^7)\ldots(1-x^3)(1-x^8)\ldots}$$

The exponents in the denominators on the right form in each case two arithmetical progressions with the difference 5. This is the surprise of the formulae; the "basic series" on the left are of a comparatively familiar type.

The formulae have a very curious history. They were found first in 1894 by Rogers, a mathematician of great talent but comparatively little reputation, now remembered mainly from Ramanujan's rediscovery of his work. Rogers was a fine analyst, whose gifts were, on a smaller scale, not unlike Ramanujan's; but no one paid much attention to anything he did, and the particular paper in which he proved the formulae was quite neglected.

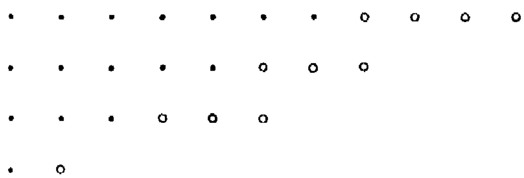
Ramanujan rediscovered the formulae sometime before 1913. He had then no proof (and knew that he had none), and none of the mathematicians to whom I communicated the formulae could find one. They are therefore stated without proof in the second volume of MacMahon's *Combinatory analysis*.

The mystery was solved, trebly, in 1917. In that year Ramanujan, looking through old volumes of the *Proceedings of the London Mathematical Society*, came accidentally across Rogers's paper. I can remember very well his surprise, and the admiration which he expressed for Rogers's work. A correspondence followed in the course of which Rogers was led to a considerable simplification of his original proof. About the same time I. Schur, who was then cut off from England by the war, rediscovered the identities again. Schur published two proofs, one of which is "combinatorial" and quite unlike any other proof known. There are now seven published proofs, the four referred to already, the two much simpler proofs found later by Rogers and Ramanujan and published in the *Papers*, and a much later proof by Watson based on quite different ideas. None of these proofs can be called both "simple" and "straightforward", since the simplest are essentially verifications; and no doubt it would be unreasonable to expect a really easy proof.

6.9. MacMahon and Schur showed that the theorems have a simple combinatorial interpretation. I take the first. We can exhibit a square m^2 as

$$1+3+5+\ldots+(2m-1),$$

or in the manner shown by the black dots of (G_7). If we now take any partition of $n-m^2$ into m parts at most, with the parts in descending order,



(G_7)

and add it to the graph, as shown by the circles of (G_7), where $m=4$ and $n=4^2+11=27$, we obtain a partition of n (here

$$27=11+8+6+2)$$

into parts without repetitions or sequences,[1] or *parts whose minimal difference is* 2. The partitions of n of this type, associated with a particular m, are enumerated by

$$\frac{x^{m^2}}{(1-x)(1-x^2)\dots(1-x^m)},$$

the general term of the series on the left in (6.8.1); and the whole series enumerates all such partitions of n.

On the other hand the right-hand side enumerates partitions into numbers $5m+1$ and $5m+4$. Hence (6.8.1) may be restated as a "combinatorial" theorem: *the number of partitions of n with minimal difference 2 is equal to the number of partitions into parts $5m+1$ and $5m+4$.* Thus when $n = 9$ there are 5 partitions of each type;

$$9,\ 8+1,\ 7+2,\ 6+3,\ 5+3+1$$

of the first kind, and

$$9,\ 6+1+1+1,\ 4+4+1,\ 4+1+1+1+1+1,$$
$$1+1+1+1+1+1+1+1+1$$

of the second. There is a similar combinatorial interpretation of (6.8.2).

These forms of the theorems are MacMahon's (or Schur's); neither Rogers nor Ramanujan ever considered their combinatorial aspect. It is natural to ask for a proof in which we set up, by "combinatorial" arguments, a direct correspondence between the two sets of partitions, but no such proof is known. Schur's "combinatorial" proof is based, not on (6.8.1) itself, but on a transformation of the formula which I will mention in a moment.[2] It is not unlike Franklin's proof of (6.2.1), but a good deal more complicated.

6.10. The proofs given ultimately by Rogers and Ramanujan are much the same, but Rogers's form is a little easier to follow.

We can write the right-hand side of (6.8.1) as

$$\frac{1}{\Pi\{(1-x^{5m+1})(1-x^{5m+4})\}} = \frac{\Pi\{(1-x^{5m})(1-x^{5m+2})(1-x^{5m+3})\}}{(1-x)(1-x^2)(1-x^3)\dots};$$

and the numerator on the right can be transformed, by a standard formula from the theory of the theta-functions, into

$$1-x^2-x^3+x^9+x^{11}-\dots,$$

where the indices are the numbers

$$\tfrac{1}{2}(5n^2 \pm n) \qquad (n = 0, 1, 2, \dots).$$

[1] Parts differing by 1.

[2] (6.10.1), with each side multiplied by $(1-x)(1-x^2)\dots$.

We have therefore to prove that

$$(6.10.1)\quad 1+\frac{x}{1-x}+\frac{x^4}{(1-x)(1-x^2)}+\ldots=\frac{1-x^2-x^3+x^9+x^{11}-\ldots}{(1-x)(1-x^2)(1-x^3)\ldots}.$$

Similarly (6.8.2) is equivalent to

$$(6.10.2)\quad 1+\frac{x^2}{1-x}+\frac{x^6}{(1-x)(1-x^2)}+\ldots=\frac{1-x-x^4+x^7+x^{13}-\ldots}{(1-x)(1-x^2)(1-x^3)\ldots},$$

the indices in the numerator on the right being the numbers $\frac{1}{2}(5n^2\pm 3n)$.

6.11. We use the auxiliary function

$$(6.11.1)\quad G_k=G_k(a,x)=\sum_{n=0}^{\infty}(-1)^n a^{2n}x^{\frac{1}{2}n(5n+1)-kn}(1-a^kx^{2kn})\,C_n,$$

where k is 0, 1, or 2 and

$$C_0=1,\quad C_n=\frac{(1-a)(1-ax)\ldots(1-ax^{n-1})}{(1-x)(1-x^2)\ldots(1-x^n)}.$$

Thus

$$(6.11.2)\quad G_k=(1-a^k)\,C_0-a^2x^{3-k}(1-a^kx^{2k})\,C_1+a^4x^{11-2k}(1-a^kx^{4k})\,C_2-\ldots.$$

If $a\neq 0$ then $G_0=0$ for all x. Also

$$(6.11.3)\qquad G_1(x,x)=1-x-x^4+x^7+x^{13}-\ldots$$

and

$$(6.11.4)\qquad G_2(x,x)=1-x^2-x^3+x^9+x^{11}-\ldots$$

are the series which occur in (6.10.2) and (6.10.1).

If the operator η is defined by

$$\eta f(a,x)=f(ax,x),$$

then

$$(6.11.5)\qquad \eta C_n=\frac{(1-ax)\ldots(1-ax^n)}{(1-x)\ldots(1-x^n)}=\frac{1-ax^n}{1-a}C_n$$

and

$$(6.11.6)\qquad \eta C_{n-1}=\frac{(1-ax)\ldots(1-ax^{n-1})}{(1-x)\ldots(1-x^{n-1})}=\frac{1-x^n}{1-a}C_n.$$

Hence

$$(6.11.7)\quad (1-x^n)\,C_n=(1-a)\,\eta C_{n-1},\quad (1-ax^n)\,C_n=(1-a)\,\eta C_n,$$

and in particular $\qquad \eta C_0=C_0=1.$

If k is 1 or 2, then

$$G_k-G_{k-1}=\sum_{n=0}^{\infty}(-1)^n a^{2n}x^{\frac{1}{2}n(5n+1)-kn}\{1-a^kx^{2kn}-x^k(1-a^{k-1}x^{2(k-1)n})\}\,C_n$$

$$=a^{k-1}(1-a)+\sum_{n=1}^{\infty}(-1)^n a^{2n}x^{\frac{1}{2}n(5n+1)-kn}\{(1-x^k)+a^{k-1}x^{(2k-1)n}(1-ax^n)\}\,C_n$$

$$=a^{k-1}(1-a)\,\eta C_0+(1-a)\sum_{n=1}^{\infty}(-1)^n a^{2n}x^{\frac{1}{2}n(5n+1)-kn}\{\eta C_{n-1}+a^{k-1}x^{(2k-1)n}\eta C_n\},$$

by (6.11.7). When we rearrange this series in terms of ηC_0, ηC_1, ..., the coefficient of ηC_n is

$$(-1)^n(1-a)\{a^{2n+k-1}x^{\frac{1}{2}n(5n+1)+(k-1)n} - a^{2n+2}x^{\frac{1}{2}(n+1)(5n+6)-k(n+1)}\}$$
$$= (-1)^n(1-a)a^{2n+k-1}x^{\frac{1}{2}n(5n+1)+(k-1)n}\{1-a^{3-k}x^{(3-k)(2n+1)}\}.$$

Hence

$$G_k - G_{k-1} = (1-a)a^{k-1}\sum_{n=0}^{\infty}(-1)^n a^{2n}x^{\frac{1}{2}n(5n+1)+(k-1)n}\{1-a^{3-k}x^{(3-k)(2n+1)}\}\eta C_n.$$

But $$G_{3-k} = \sum_{n=0}^{\infty}(-1)^n a^{2n}x^{\frac{1}{2}n(5n+1)-(3-k)n}\{1-a^{3-k}x^{(3-k)2n}\}C_n,$$

and so $$\eta G_{3-k} = \sum_{n=0}^{\infty}(-1)^n a^{2n}x^{\frac{1}{2}n(5n+1)+(k-1)n}\{1-a^{3-k}x^{(3-k)(2n+1)}\}C_n;$$

and therefore

(6.11.8) $$G_k - G_{k-1} = (1-a)a^{k-1}\eta G_{3-k} \qquad (k=1,2).$$

6.12. If now

$$H_k = H_k(a,x) = \frac{G_k(a,x)}{(1-a)(1-ax)(1-ax^2)\dots}$$

(so that $H_0 = 0$), then (6.11.8) becomes

$$H_k - H_{k-1} = a^{k-1}\eta H_{3-k}.$$

In particular

(6.12.1) $$H_1 = \eta H_2, \quad H_2 - H_1 = a\eta H_1$$

and so

(6.12.2) $$H_2 = \eta H_2 + a\eta^2 H_2.$$

Suppose now that $H_2 = 1 + c_1 a + c_2 a^2 + \dots$,

where the coefficients depend on x only. Substituting into (6.12.2), we obtain

$$1 + c_1 a + c_2 a^2 + \dots = 1 + c_1 ax + c_2 a^2 x^2 + \dots + a(1 + c_1 ax^2 + c_2 a^2 x^4 + \dots).$$

Hence, equating coefficients,

$$c_1 = \frac{1}{1-x}, \quad c_2 = \frac{x^2}{1-x^2}c_1, \quad c_3 = \frac{x^4}{1-x^3}c_2, \dots$$

and $$c_n = \frac{x^{n(n-1)}}{(1-x)(1-x^2)\dots(1-x^n)};$$

and hence

$$\frac{G_2(a,x)}{(1-a)(1-ax)\dots} = H_2(a,x) = 1 + \frac{a}{1-x} + \frac{a^2x^2}{(1-x)(1-x^2)} + \frac{a^3x^6}{(1-x)(1-x^2)(1-x^3)} + \dots.$$

Also

$$\frac{G_1(a,x)}{(1-a)(1-ax)\ldots} = H_1(a,x) = \eta H_2(a,x) = 1+\frac{ax}{1-x} + \frac{a^2x^4}{(1-x)(1-x^2)} + \frac{a^3x^9}{(1-x)(1-x^2)(1-x^3)} + \ldots.$$

Finally, putting $a = x$ in these two formulae, and using (6.11.4) and (6.11.3), we obtain (6.10.1) and (6.10.2).

This proof is elementary, and reasonably simple; but it is undeniably rather artificial. It is a "verification"; we verify that the series (6.11.1) satisfies a functional equation, and the argument gives no explanation of our choice of this particular series.

6.13. There is another proof by Rogers which seems to assume a little more but is really more illuminating.[1]

We shorten our formulae by writing

$$x_n = 1-x^n, \quad x_n! = x_1 x_2 \ldots x_n,$$

and begin by expanding the function

$$f(a) = \prod_1^\infty (1+ax^n)$$

in powers of a. The function satisfies

$$f(a) = (1+ax)f(ax).$$

Substituting a power series in a for $f(a)$, and equating coefficients, we find without difficulty that

$$f(a) = 1+\frac{x}{x_1!}a+\frac{x^3}{x_2!}a^2+\ldots+\frac{x^{\frac{1}{2}n(n+1)}}{x_n!}a^n+\ldots.$$

Replacing a by $ae^{i\theta}$ and $ae^{-i\theta}$, and multiplying the resulting series, we find that

$$(6.13.1) \quad \Phi(x,\theta,a) = \prod_1^\infty (1+2ax^n\cos\theta+a^2x^{2n})$$

$$= \left(1+\frac{x}{x_1!}ae^{i\theta}+\frac{x^3}{x_2!}a^2e^{2i\theta}+\ldots\right)\left(1+\frac{x}{x_1!}ae^{-i\theta}+\frac{x^3}{x_2!}a^2e^{-2i\theta}+\ldots\right)$$

$$= 1+\sum_1^\infty \frac{B_n(\theta)}{x_n!}a^n,$$

[1] The argument of §§ 10–12, if regarded as a proof of (6.10.1), assumes *nothing*, though some knowledge of theta-functions is required to identify (6.8.1) and (6.10.1). In the proof here we use formulae from the theory of theta-functions in the proof.

where

$$(6.13.2)\quad \frac{B_{2n}(\theta)}{x_{2n}!}=\frac{x^{n(n+1)}}{x_n!\,x_n!}\left(1+\frac{x_n}{x_{n+1}}2x\cos 2\theta+\frac{x_n x_{n-1}}{x_{n+1}x_{n+2}}2x^4\cos 4\theta+\dots\right.$$
$$\left.+\frac{x_n x_{n-1}\dots x_1}{x_{n+1}x_{n+2}\dots x_{2n}}2x^{n^2}\cos 2n\theta\right),$$

$$(6.13.3)\quad \frac{B_{2n+1}(\theta)}{x_{2n+1}!}=\frac{x^{(n+1)^2}}{x_n!\,x_{n+1}!}\left(2\cos\theta+\frac{x_n}{x_{n+2}}2x^2\cos 3\theta\right.$$
$$\left.+\frac{x_n x_{n-1}}{x_{n+2}x_{n+3}}2x^6\cos 5\theta+\dots+\frac{x_n x_{n-1}\dots x_1}{x_{n+2}x_{n+3}\dots x_{2n+1}}2x^{n(n+1)}\cos(2n+1)\theta\right).$$

Finally, we replace a in (6.13.1) by $x^{-\frac{1}{2}}$, and use a standard formula from the theory of the theta-functions, viz.

$$\Phi(x,\theta,x^{-\frac{1}{2}})=\Pi(1+2x^{n-\frac{1}{2}}\cos\theta+x^{2n-1})$$
$$=\frac{1+2x^{\frac{1}{2}}\cos\theta+2x^{\frac{4}{2}}\cos 2\theta+2x^{\frac{9}{2}}\cos 3\theta+\dots}{(1-x)(1-x^2)(1-x^3)\dots}.$$

We thus obtain

$$(6.13.4)\quad \frac{1+2x^{\frac{1}{2}}\cos\theta+2x^{\frac{4}{2}}\cos 2\theta+2x^{\frac{9}{2}}\cos 3\theta+\dots}{(1-x)(1-x^2)(1-x^3)\dots}=1+\sum_1^\infty\frac{B_n(\theta)}{x_n!}x^{-\frac{1}{2}n}.$$

6.14. If we replace $B_n(\theta)$, in (6.13.4), by its explicit expression (6.13.2) or (6.13.3), and rearrange the right-hand side as a trigonometrical series, we obtain an equality between two convergent trigonometrical series. Such an equality must be an identity, the coefficients of $\cos n\theta$ in the two series being the same. It follows that *we may replace*

$$1,\ 2\cos\theta,\ 2\cos 2\theta,\ 2\cos 3\theta,\ \dots$$

by any numbers for which the series remain convergent.

If we replace

$$1,\ 2\cos 2\theta,\ 2\cos 4\theta,\ \dots,\ 2\cos 2n\theta,\ \dots$$

by $$1,\ -(1+x),\ x(1+x^2),\ \dots,\ (-1)^n x^{\frac{1}{2}n(n-1)}(1+x^n),\ \dots$$

and all the odd cosines by 0, then $B_{2n}(\theta)$ becomes

$$(6.14.1)\quad \beta_{2n}=\frac{x_{2n}!}{x_n!\,x_n!}x^{n(n+1)}\left\{1-\frac{x_n}{x_{n+1}}x(1+x)+\frac{x_n x_{n-1}}{x_{n+1}x_{n+2}}x^5(1+x^2)-\dots\right.$$
$$\left.+(-1)^n\frac{x_n x_{n-1}\dots x_1}{x_{n+1}x_{n+2}\dots x_{2n}}x^{\frac{1}{2}n(3n-1)}(1+x^n)\right\},$$

and we obtain

$$(6.14.2)\quad 1+\frac{\beta_2}{x_2!}x^{-1}+\frac{\beta_4}{x_4!}x^{-2}+\dots=\frac{1-x^2-x^3+x^9+x^{11}-\dots}{(1-x)(1-x^2)(1-x^3)\dots},$$

where the right-hand side is the same as in (6.10.1). On the other hand, if we replace

$$2\cos\theta,\ 2\cos 3\theta,\ \ldots,\ 2\cos(2n+1)\theta,\ \ldots$$

by $$(1-x),\ -(1-x^3),\ \ldots,\ (-1)^n x^{\frac{1}{2}n(n-1)}(1-x^{2n+1}),$$

and the even cosines by 0, then $B_{2n+1}(\theta)$ becomes

(6.14.3)

$$\beta_{2n+1} = \frac{x_{2n+1}!}{x_n!\,x_{n+1}!}\,x^{(n+1)^2}\left\{1-\frac{x_n}{x_{n+2}}\,x^2(1-x^3)+\frac{x_n x_{n-1}}{x_{n+2}x_{n+3}}\,x^7(1-x^5)-\ldots\right.$$
$$\left.+(-1)^n\frac{x_n x_{n-1}\ldots x_1}{x_{n+2}x_{n+3}\ldots x_{2n+1}}\,x^{\frac{1}{2}n(3n+1)}(1-x^{2n+1})\right\}.$$

Multiplying by $x^{-\frac{1}{4}}$, we obtain

(6.14.4) $$\frac{\beta_1}{x_1!}x^{-1}+\frac{\beta_3}{x_3!}x^{-2}+\ldots = \frac{1-x-x^4+x^7+x^{13}-\ldots}{(1-x)(1-x^2)(1-x^3)\ldots},$$

where the right-hand side is the same as in (6.10.2). It remains to prove that the series on the left in (6.14.2) and (6.14.4) are the same as in (6.10.1) and (6.10.2).

6.15. We can do this by evaluating β_n in an elementary manner. But before doing this I observe that the substitutions of § 6.14 correspond to linear analytical transformations. Thus if $x = e^{-\delta}$ then

$$\int_{-\infty}^{\infty} e^{-2\theta^2/\delta}\cos\theta\,.\,2\cos 2n\theta\,.\,d\theta = \int_{-\infty}^{\infty} e^{-2\theta^2/\delta}\{\cos(2n-1)\theta+\cos(2n+1)\theta\}\,d\theta$$
$$= \sqrt{(\tfrac{1}{2}\pi\delta)}\,\{e^{-\frac{1}{8}(2n-1)^2\delta}+e^{-\frac{1}{8}(2n+1)^2\delta}\} = \sqrt{(\tfrac{1}{2}\pi\delta)}\,x^{\frac{1}{8}}\,.\,x^{\frac{1}{2}n(n-1)}(1+x^n).$$

Hence the first substitution of § 6.14 corresponds to the result of (i) replacing θ by $\theta+\frac{1}{2}\pi$ and (ii) operating with

$$\sqrt{\left(\frac{2}{\pi\delta}\right)}\,x^{-\frac{1}{8}}\int_{-\infty}^{\infty} e^{-2\theta^2/\delta}\cos\theta\ldots d\theta.$$

Similarly it may be verified that the second is the result of (i) replacing θ by $\theta+\frac{1}{2}\pi$ and (ii) operating with

$$\sqrt{\left(\frac{2}{\pi\delta}\right)}\,x^{-\frac{1}{2}}\int_{-\infty}^{\infty} e^{-2\theta^2/\delta}\sin 2\theta\ldots d\theta.$$

6.16. We now write

(6.16.1) $$\beta_{2n} = \frac{x_{2n}!}{x_n!\,x_n!}\,x^{n(n+1)}\gamma_{2n},\quad \beta_{2n+1} = \frac{x_{2n+1}!}{x_n!\,x_{n+1}!}\,x^{(n+1)^2}\gamma_{2n+1},$$

and prove that

(6.16.2) $$\gamma_{2n} = x_n!,\quad \gamma_{2n+1} = x_{n+1}!.$$

We shall then have

$$(6.16.3)\quad \beta_{2n} = x^{n(n+1)}x_{n+1}x_{n+2}\dots x_{2n}, \quad \beta_{2n+1} = x^{(n+1)^2}x_{n+1}x_{n+2}\dots x_{2n+1},$$

and it may be verified at once that the series on the left of (6.14.2) and (6.14.4) reduce to the Rogers-Ramanujan series.

To prove (6.16.2) it is enough to prove that

$$(6.16.4)\qquad \gamma_{2n+1} = x_{n+1}\gamma_{2n}, \quad \gamma_{2n+2} = \gamma_{2n+1}.$$

Now

$$\gamma_{2n} = 1 - \frac{x_n}{x_{n+1}}x(1+x) + \frac{x_n x_{n-1}}{x_{n+1}x_{n+2}}x^5(1+x^2) - \frac{x_n x_{n-1} x_{n-2}}{x_{n+1}x_{n+2}x_{n+3}}x^{12}(1+x^3) + \dots$$

$$= \left(1 - x\frac{x_n}{x_{n+1}}\right) - x^2\frac{x_n}{x_{n+1}}\left(1 - x^3\frac{x_{n-1}}{x_{n+2}}\right) + x^7\frac{x_n x_{n-1}}{x_{n+1}x_{n+2}}\left(1 - x^5\frac{x_{n-2}}{x_{n+3}}\right) - \dots$$

$$= \frac{1}{x_{n+1}}(1-x) - x^2\frac{x_n}{x_{n+1}x_{n+2}}(1-x^3) + x^7\frac{x_n x_{n-1}}{x_{n+1}x_{n+2}x_{n+3}}(1-x^5) - \dots$$

$$= \frac{\gamma_{2n+1}}{x_{n+1}},$$

and

$$\gamma_{2n+1} = (1-x) - \frac{x_n}{x_{n+2}}x^2(1-x^3) + \frac{x_n x_{n-1}}{x_{n+2}x_{n+3}}x^7(1-x^5) - \frac{x_n x_{n-1} x_{n-2}}{x_{n+2}x_{n+3}x_{n+4}}x^{15}(1-x^7) + \dots$$

$$= 1 - x\left(1 + x\frac{x_n}{x_{n+2}}\right) + x^5\frac{x_n}{x_{n+2}}\left(1 + x^2\frac{x_{n-1}}{x_{n+3}}\right) - x^{12}\frac{x_n x_{n-1}}{x_{n+2}x_{n+3}}\left(1 + x^3\frac{x_{n-2}}{x_{n+4}}\right) + \dots$$

$$= 1 - \frac{x_{n+1}}{x_{n+2}}x(1+x) + \frac{x_{n+1}x_n}{x_{n+2}x_{n+3}}x^5(1+x^2) - \frac{x_{n+1}x_n x_{n-1}}{x_{n+2}x_{n+3}x_{n+4}}x^{12}(1+x^3) + \dots$$

$$= \gamma_{2n+2}.$$

These are the relations required.

The equations (6.16.2) are

$$(6.16.5)\quad 1 - \frac{1-x^n}{1-x^{n+1}}x(1+x) + \frac{(1-x^n)(1-x^{n-1})}{(1-x^{n+1})(1-x^{n+2})}x^5(1+x^2) - \dots = (1-x)(1-x^2)\dots(1-x^n)$$

and

$$(6.16.6)\quad (1-x) - \frac{1-x^n}{1-x^{n+2}}x^2(1-x^3) + \frac{(1-x^n)(1-x^{n-1})}{(1-x^{n+2})(1-x^{n+3})}x^7(1-x^5) - \dots = (1-x)(1-x^2)\dots(1-x^{n+1}).$$

Each of them reduces, when $n \to \infty$, to

$$1 - x - x^2 + x^5 + x^7 - \dots = (1-x)(1-x^2)(1-x^3)\dots,$$

Euler's identity; and the argument of this section gives a particularly simple proof of the identity. We shall be led to the formulae (6.16.5) and (6.16.6) in a different manner later.[1]

6.17. It follows from (6.12.2), or may be verified directly, that

$$F(a) = H_1(a,x) = 1+\frac{ax}{1-x}+\frac{a^2x^4}{(1-x)(1-x^2)}+\dots$$

satisfies the functional equation

$$F(a) = F(ax)+axF(ax^2).$$

From this it follows that

$$\frac{F(a)}{F(ax)} = 1+ax\frac{F(ax^2)}{F(ax)} = 1+\cfrac{ax}{1+ax^2\dfrac{F(ax^3)}{F(ax^2)}}$$

$$= 1+\frac{ax}{1+}\,\frac{ax^2}{1+}\,\frac{ax^3}{1+\dots}.$$

In particular

$$1+\frac{x}{1+}\,\frac{x^2}{1+}\,\frac{x^3}{1+\dots} = \frac{F(1)}{F(x)}$$

$$= \frac{(1-x^2)(1-x^7)\dots(1-x^3)(1-x^8)\dots}{(1-x)(1-x^6)\dots(1-x^4)(1-x^9)\dots}$$

$$= \frac{1-x^2-x^3+x^9+x^{11}-\dots}{1-x-x^4+x^7+x^{13}-\dots}$$

is a quotient of elliptic theta-functions, which may be evaluated for certain special values of x. This formula is the key to Ramanujan's evaluations of the continued fraction for special values of x, which I quoted in my first lecture.

NOTES ON LECTURE VI

This lecture contains a good deal of the substance of Hardy and Wright, ch. 19, and there is inevitably a certain amount of repetition; but the account here is naturally less systematic. There is nothing in Hardy and Wright corresponding to §§ 6.13–16.

§ 6.2. See Hardy and Wright, § 19.11, or MacMahon, *Combinatory analysis*, ii, 21–23. Franklin's proof was first published in *Comptes rendus*, 92 (1881), 448–450.

Both Hardy and Wright and MacMahon give other examples of 'graphical' proofs.

§ 6.3. For a more rigorous proof that $F(x)$ enumerates $p(n)$, see Hardy and Wright, § 19.3.

§§ 6.4–5. Compare Hardy and Wright, § 19.12, where however there is no proof of (6.4.2). There are alternative proofs of (6.4.1) and (6.4.2), and a proof of (6.4.3), in no. 30 of the *Papers*. Darling (3) gave further proofs of (6.4.1) and (6.4.2).

[1] See Lecture VII, § 7.8.

There are interesting remarks on the parity of $p(n)$ in MacMahon's paper **2**. MacMahon does not prove any general theorem, but gives recurrence congruences (mod 2) by which it is possible to calculate the parity of $p(n)$, for quite large n, very quickly. Thus he proves, in 'about five minutes work', that $p(1000)$ is odd.

One of the standard proofs of (6.5.1) is reproduced in Hardy and Wright, § 19.9.

§§ 6.6–7. Ramanujan's proofs of the congruences for moduli 5^2, 7^2, and 11^2 are contained in an unpublished manuscript now in the possession of Prof. Watson. Darling (**2**) gave a proof of (6.7.1), and Mordell (**1**) much shorter proofs of both (6.7.1) and (6.7.2).

The references relevant to the work of Chowla, Gupta, Krečmar, Lehmer, and Watson are S. Chowla (**1**); Gupta (**1, 2, 3**); Krečmar (**1**); D. H. Lehmer (**1, 3**); and Watson (**24**).

§ 6.8. Rogers (**1**): the identities are formulae (1) and (2) of § 5. Rogers also anticipated 'Hölder's inequality' (and is quoted by Hölder), but without writing it in the standard form or recognising its fundamental importance. See Hardy, Littlewood, and Pólya, *Inequalities*, 25 and 311.

Roger's two later proofs are in his papers **2** and **4**: the latter contains the proof given in §§ 6.10–6.12, the former that given in §§ 6.13–6.16.

For Ramanujan's own proof see no. 26 of the *Papers*. Schur's two proofs appeared in *Berliner Sitzungsberichte* (1917), 301–321, and Watson's in Watson (**3**).

Ramanujan does not seem to have stated the formulae explicitly in his letters to me, but formulae IX, (4)–(7), of his first letter depend upon them. See Lecture I, formulae (1.10)–(1.12); *Papers*, xxvii; and Watson (**4**). Ramanujan proposed the formulae as a problem in *Journal Indian Math. Soc.* 6 (1914), 199; see *Papers*, 330.

§ 6.9. See Hardy and Wright, § 19.13, and MacMahon, *Combinatory analysis*, ii, 33–36.

§§ 6.10–6.12. See Hardy and Wright, § 19.14. The theta-function formulae required at the beginning of the proof are proved in §§ 19.8–19.9 (Theorems 355 and 356).

§ 6.13. The expansion of $f(a)$ in powers of a goes back to Euler. The theta-function formula is proved in Hardy and Wright, § 19.8, or in any of the standard treatises on elliptic functions.

§ 6.17. See Lecture I, formulae (1.10)–(1.12). Proofs were given by Watson (**4**).

VII

HYPERGEOMETRIC SERIES

7.1. Ramanujan's work on hypergeometric series is contained in two chapters of his notebook which I analysed and edited after his death. This analysis has since occasioned a small flood of papers, by Bailey, Watson, Whipple and others, in the publications of the London Mathematical Society. There is an excellent account of the results of all these researches in Bailey's tract.

The problem which dominates Ch. 10 of the notebook is that of the summation of the series

(7.1.1) $$F\binom{\alpha_1, \alpha_2, \alpha_3}{\beta_1, \beta_2} = 1+\frac{\alpha_1\alpha_2\alpha_3}{1.\beta_1\beta_2}+\frac{\alpha_1(\alpha_1+1)\,\alpha_2(\alpha_2+1)\,\alpha_3(\alpha_3+1)}{1.2.\beta_1(\beta_1+1)\beta_2(\beta_2+1)}+\ldots,$$

and the chapter contains practically everything known about this series before 1922. When $\beta_2 = \alpha_3$ the series reduces to an ordinary hypergeometric series, and its sum is given by Gauss's formula

(7.1.2)

$$F\binom{\alpha_1, \alpha_2}{\beta_1} = 1+\frac{\alpha_1\alpha_2}{1.\beta_1}+\frac{\alpha_1(\alpha_1+1)\,\alpha_2(\alpha_2+1)}{1.2.\beta_1(\beta_1+1)}+\ldots = \frac{\Gamma(\beta_1)\,\Gamma(\beta_1-\alpha_1-\alpha_2)}{\Gamma(\beta_1-\alpha_1)\,\Gamma(\beta_1-\alpha_2)}.$$ [1]

It will be necessary to use hypergeometric series of higher order, and I must begin by explaining the standard notations. We write

$$a^{(n)} = a(a+1)\ldots(a+n-1), \quad a_{(n)} = a(a-1)\ldots(a-n+1),$$

and $${}_pF_q\left(\begin{matrix}\alpha_1, \alpha_2, \ldots, \alpha_p\\ \beta_1, \beta_2, \ldots, \beta_q\end{matrix}; x\right) = \sum_{n=0}^{\infty} \frac{\alpha_1^{(n)}\alpha_2^{(n)}\ldots\alpha_p^{(n)}}{n!\,\beta_1^{(n)}\beta_2^{(n)}\ldots\beta_q^{(n)}}x^n.$$

The series is generally convergent for all x if $p \leqq q$, and for $|x|<1$ when $p = q+1$. When $p>q+1$ it is divergent for all x unless one of the α is 0 or a negative integer. We shall usually omit the argument 1 when $x = 1$. Thus

$$F\binom{\alpha_1, \alpha_2, \alpha_3}{\beta_1, \beta_2} = {}_3F_2\binom{\alpha_1, \alpha_2, \alpha_3}{\beta_1, \beta_2} = {}_3F_2\left(\begin{matrix}\alpha_1, \alpha_2, \alpha_3\\ \beta_1, \beta_2\end{matrix}; 1\right).$$

[1] Here and elsewhere I ignore questions of convergence. The reader should be able to supply the conditions for the validity of any formulae which I quote or prove.

7.2. The key formula of the chapter is

$$(7.2.1)\quad \sum_{n=0}^{\infty}(-1)^n(s+2n)\frac{s^{(n)}(x+y+z+u+2s+1)^{(n)}}{n!\,(x+y+z+u+s)_{(n)}}\prod_{x,y,z,u}\frac{x_{(n)}}{(x+s+1)^{(n)}}$$

$$=\frac{s}{\Gamma(s+1)\,\Gamma(x+y+z+u+s+1)}\prod_{x,y,z,u}\frac{\Gamma(x+s+1)\,\Gamma(y+z+u+s+1)}{\Gamma(z+u+s+1)},$$

where one of the five arguments

$$(7.2.2)\qquad x,\ y,\ z,\ u,\ -x-y-z-u-2s-1$$

is a positive integer. If this condition (which is essential) is satisfied, then the series terminates, and the formula is an algebraical identity. The whole chapter, indeed, is essentially a chapter in elementary formal algebra; we are concerned, at bottom, with identities between polynomials. From these we deduce identities between infinite series, but the passages to the limit which are necessary, though some of them have points of interest, do not present any serious difficulty to a competent analyst.

Ramanujan seems to have found the formula about 1910 or 1911, but he had been anticipated by Dougall. The formula looks formidable, but Dougall's proof is very simple. I imagine that Ramanujan argued similarly,[1] but there is nothing in the notebooks to show.

If we observe that $\qquad a_{(n)} = (-1)^n(-a)^{(n)}$

and
$$\frac{(1+\tfrac{1}{2}s)^{(n)}}{(\tfrac{1}{2}s)^{(n)}}=\frac{s+2n}{s},$$

we can write (7.2.1) in the form

$$(7.2.3)\quad {}_7F_6\left(\begin{matrix} s,\,1+\tfrac{1}{2}s,\,-x,\,-y,\,-z,\,-u,\,x+y+z+u+2s+1 \\ \tfrac{1}{2}s,\,x+s+1,\,y+s+1,\,z+s+1,\,u+s+1,\,-x-y-z-u-s\end{matrix};1\right)$$

$$=\frac{1}{\Gamma(s+1)\,\Gamma(x+y+z+u+s+1)}\prod_{x,y,z,u}\frac{\Gamma(x+s+1)\,\Gamma(y+z+u+s+1)}{\Gamma(z+u+s+1)}.$$

If we write $-v$ for $x+y+z+u+2s+1$, so that

$$(7.2.4)\qquad x+y+z+u+v=-2s-1,$$

then the formula becomes

$$(7.2.5)\quad {}_7F_6\left(\begin{matrix} s,\,1+\tfrac{1}{2}s,\,-x,\,-y,\,-z,\,-u,\,-v \\ \tfrac{1}{2}s,\,x+s+1,\,y+s+1,\,z+s+1,\,u+s+1,\,v+s+1\end{matrix};1\right)$$

$$=\frac{1}{\Gamma(s+1)}\,\frac{1}{\Gamma(x+y+z+u+s+1)}\prod_{x,y,z,u}\frac{\Gamma(x+s+1)\,\Gamma(y+z+u+s+1)}{\Gamma(z+u+s+1)}.$$

[1] By such verifications in particular cases as, when associated appropriately, yield a rigorous proof.

The left-hand side here is obviously symmetrical in x, y, z, u, v. The right-hand side is symmetrical in the four arguments x, y, z, u, and, since the five arguments are connected by the symmetrical relation (7.2.4), it is symmetrical in the five.

One of the arguments is to be a positive integer. If, say,

$$x = m,$$

then the series terminates, and the right-hand side becomes

$$\frac{(s+1)^{(m)}(z+u+s+1)^{(m)}(u+y+s+1)^{(m)}(y+z+s+1)^{(m)}}{(y+s+1)^{(m)}(z+s+1)^{(m)}(u+s+1)^{(m)}(y+z+u+s+1)^{(m)}}. \tag{7.2.6}$$

And if we multiply both sides by

$$(y+s+1)^{(m)}(y+z+u+s+1)^{(m)},$$

then the formula asserts the identity of two polynomials in y, each of degree $2m$.

We assume the truth of the formula for

$$x = 0, 1, 2, \ldots, m-1,$$

and prove it for $x = m$. It is sufficient, after our last remark, to prove that it is true for $2m+1$ different values of y.

Now the formula is true for

$$y = 0, 1, 2, \ldots, m-1$$

from the inductive hypothesis and the symmetry in x and y, and for

$$v = 0, 1, 2, \ldots, m-1,$$

i.e. for

$$y = -2s-z-u-m-1,\ -2s-z-u-m-2,\ \ldots,\ -2s-z-u-2m,$$

from the inductive hypothesis and the symmetry in x and v. Hence it is true for $2m$ values of y, in general different, and it is sufficient to verify its truth for one more value. We choose the value

$$y = -s-m.$$

This value of y is a pole of the last term only of the series (7.2.5); and it is sufficient to prove that the residue of this term is equal to the residue of the product (7.2.6). It is easily verified that each residue is

$$(-1)^{m-1}\frac{(s+1)^{(m)}}{(m-1)!}\frac{(s+z+u+1)^{(m)}(z-m+1)^{(m)}(u-m+1)^{(m)}}{(s+z+1)^{(m)}(s+u+1)^{(m)}(z+u-m+1)^{(m)}},$$

and this completes the proof.[1]

[1] I give a little more detail in § 7.7, where I prove a more general formula which includes (7.2.3) as a particular case.

7.3. There is a whole crowd of striking particular cases. If we suppose that x, y or z is a positive integer, divide by s, write $t-u$ for s, and then make $u\to\infty$, we obtain

$$(7.3.1)\quad F\begin{pmatrix} -x,\ -y,\ -z \\ t+1,\ -x-y-z-t \end{pmatrix} = \frac{\Gamma(t+1)\,\Gamma(y+z+t+1)\,\Gamma(z+x+t+1)\,\Gamma(x+y+t+1)}{\Gamma(x+t+1)\,\Gamma(y+t+1)\,\Gamma(z+t+1)\,\Gamma(x+y+z+t+1)},$$

which gives the value of (7.1.1) when

$$\alpha_1+\alpha_2+\alpha_3 = \beta_1+\beta_2-1$$

and one α is a negative integer. This result was found by Saalschütz in 1890.

If on the other hand we suppose that u is a positive integer, divide by s, put $z=-\frac{1}{2}s$, and then make $u\to\infty$, we obtain

$$(7.3.2)\quad F\begin{pmatrix} -x,\ -y,\ s \\ x+s+1,\ y+s+1 \end{pmatrix} = \frac{\Gamma(\frac{1}{2}s+1)\,\Gamma(x+s+1)\,\Gamma(y+s+1)\,\Gamma(x+y+\frac{1}{2}s+1)}{\Gamma(s+1)\,\Gamma(x+\frac{1}{2}s+1)\,\Gamma(y+\frac{1}{2}s+1)\,\Gamma(x+y+s+1)}.$$

This formula, due to A. C. Dixon, gives the value of (7.1.1) when

$$\alpha_1+\beta_1 = \alpha_2+\beta_2 = \alpha_3+1.$$

It includes F. Morley's formula

$$1+\left(\frac{m}{1}\right)^3+\left(\frac{m(m+1)}{1.2}\right)^3+\ldots = \frac{\Gamma(1-\frac{3}{2}m)}{\{\Gamma(1-\frac{1}{2}m)\}^3}\cos\tfrac{1}{2}m\pi$$

for the sum of the cubes of the coefficients in the binomial series for $(1-x)^{-m}$.

The formulae (7.3.1) and (7.3.2) are two of the three most important in the theory of the series (7.1.1). The third is

$$(7.3.3)\quad \frac{\Gamma(x+y+s+1)}{\Gamma(x+s+1)\,\Gamma(y+s+1)}\,F\begin{pmatrix} -a,\ -b,\ x+y+s+1 \\ x+s+1,\ y+s+1 \end{pmatrix} = \frac{\Gamma(a+b+s+1)}{\Gamma(a+s+1)\,\Gamma(b+s+1)}\,F\begin{pmatrix} -x,\ -y,\ a+b+s+1 \\ a+s+1,\ b+s+1 \end{pmatrix}.$$

This formula of Thomae's was also rediscovered by Ramanujan. It is not a consequence of the Dougall-Ramanujan identity, which is my main subject here, but I give Ramanujan's proof.[1] By Gauss's formula

$$\frac{\Gamma(x+y+s+n+1)}{\Gamma(x+s+n+1)\,\Gamma(y+s+n+1)} = \frac{1}{\Gamma(s+n+1)}\,F\begin{pmatrix} -x,\ -y \\ s+n+1 \end{pmatrix} = \frac{1}{\Gamma(-x)\,\Gamma(-y)}\sum_{m=0}^{\infty}\frac{\Gamma(-x+m)\,\Gamma(-y+m)}{m!\,\Gamma(s+m+n+1)}.$$

[1] This is one of the few cases in which Ramanujan gives an explicit proof in the notebook. The proof is essentially the same as Thomae's.

Hence

$$\frac{\Gamma(x+y+s+1)}{\Gamma(x+s+1)\,\Gamma(y+s+1)}F\left(\begin{matrix}-a,\,-b,\,x+y+s+1\\ x+s+1,\,y+s+1\end{matrix}\right)$$

$$=\frac{1}{\Gamma(-a)\,\Gamma(-b)}\sum_{n=0}^{\infty}\frac{\Gamma(-a+n)\,\Gamma(-b+n)\,\Gamma(x+y+s+n+1)}{n!\,\Gamma(x+s+n+1)\,\Gamma(y+s+n+1)}$$

$$=\frac{1}{\Gamma(-a)\,\Gamma(-b)\,\Gamma(-x)\,\Gamma(-y)}$$

$$\times\sum_{m,n=0}^{\infty}\frac{\Gamma(-x+m)\,\Gamma(-y+m)\,\Gamma(-a+n)\,\Gamma(-b+n)}{m!\,n!\,\Gamma(s+m+n+1)},$$

and the conclusion follows by symmetry.

7.4. I return to the Dougall-Ramanujan identity. The notebook contains a mass of elegant summations, most of which, though not all, can be derived from (7.2.1).[1] I cannot quote many here, but I will say something about the formulae (1.2), (1.3) and (1.4) of my first lecture.

If we suppose u, in (7.2.1), a positive integer, and make $u\to\infty$, we obtain

$$(7.4.1)\qquad \sum_{n=0}^{\infty}(-1)^n(s+2n)\frac{s^{(n)}}{n!}\prod_{x,y,z}\frac{x_{(n)}}{(x+s+1)^{(n)}}$$

$$=\frac{s\Gamma(x+y+z+s+1)}{\Gamma(s+1)}\prod_{x,y,z}\frac{\Gamma(x+s+1)}{\Gamma(y+z+s+1)}.$$

The special cases $\qquad x=y=-s,\quad z=\infty,$

and $\qquad x=y=z=-s$

give

$$(7.4.2)\qquad s-(s+2)\left(\frac{s}{1}\right)^3+(s+4)\left(\frac{s(s+1)}{1.2}\right)^3-\ldots=\frac{\sin s\pi}{\pi},$$

which reduces to (1.2) for $s=\frac{1}{2}$, and

$$(7.4.3)\qquad s+(s+2)\left(\frac{s}{1}\right)^4+(s+4)\left(\frac{s(s+1)}{1.2}\right)^4+\ldots=\frac{\sin^2 s\pi}{2\pi^2\cos s\pi}\frac{\{\Gamma(s)\}^2}{\Gamma(2s)},$$

which reduces to (1.3) for $s=\frac{1}{4}$.

The formula (1.4) is more difficult, and it is not possible to deduce it from (7.2.1). Proofs have been given by myself and by Whipple, the first depending on the theory of the Legendre polynomials, and the second on

[1] There are a large number of examples in my paper 8, and on p. 96 of Bailey's tract.

a generalisation of (7.2.1) due to Whipple himself. Each proof shows that Ramanujan's series is equal to

$$\frac{2}{\pi}\left\{1+\left(\frac{1}{2}\right)^3+\left(\frac{1.3}{2.4}\right)^3+\ldots\right\}$$

and then sums this series by means of (7.3.2). It would be very interesting to know Ramanujan's proof.

Another interesting summation is

$$(7.4.4)\qquad 1-\left(\frac{1}{2}\right)^3+\left(\frac{1.3}{2.4}\right)^3-\ldots=\left\{\frac{\Gamma(\frac{9}{8})}{\Gamma(\frac{5}{4})\,\Gamma(\frac{7}{8})}\right\}^2.$$

This also is not a consequence of (7.2.1), but may be effected by means of the two formulae

$$(7.4.5)\qquad \left\{{}_2F_1\left(\begin{matrix}\alpha,\beta\\ \alpha+\beta+\frac{1}{2}\end{matrix};x\right)\right\}^2={}_3F_2\left(\begin{matrix}2\alpha,\alpha+\beta,2\beta\\ \alpha+\beta+\frac{1}{2},2\alpha+2\beta\end{matrix};x\right)$$

and

$$(7.4.6)\qquad {}_2F_1\left(\begin{matrix}\alpha,\beta\\ 1+\alpha-\beta\end{matrix};-1\right)=\frac{\Gamma(1+\alpha-\beta)\,\Gamma(1+\frac{1}{2}\alpha)}{\Gamma(1+\alpha)\,\Gamma(1+\frac{1}{2}\alpha-\beta)},$$

due to Clausen and Kummer respectively. Both of these formulae are in the notebook. If we put

$$\alpha=\beta=\tfrac{1}{4}$$

in (7.4.5) and (7.4.6), and $x=-1$ in (7.4.5), we obtain

$$1-\left(\frac{1}{2}\right)^3+\left(\frac{1.3}{2.4}\right)^3-\ldots={}_3F_2\left(\begin{matrix}\frac{1}{2},\frac{1}{2},\frac{1}{2}\\ 1,1\end{matrix};-1\right)$$

$$=\left\{{}_2F_1\left(\begin{matrix}\frac{1}{4},\frac{1}{4}\\ 1\end{matrix};-1\right)\right\}^2=\left\{\frac{\Gamma(\frac{9}{8})}{\Gamma(\frac{5}{4})\,\Gamma(\frac{7}{8})}\right\}^2.$$

7.5. Another curious formula which has attracted a good deal of attention is

$$(7.5.1)\qquad \frac{1}{n}+\left(\frac{1}{2}\right)^2\frac{1}{n+1}+\left(\frac{1.3}{2.4}\right)^2\frac{1}{n+2}+\ldots$$

$$=\left\{\frac{\Gamma(n)}{\Gamma(n+\frac{1}{2})}\right\}^2\left\{1+\left(\frac{1}{2}\right)^2+\left(\frac{1.3}{2.4}\right)^2+\ldots\text{ to } n \text{ terms}\right\}.$$

I prove this as a special case of a more general formula found by Bailey, viz.

$$(7.5.2)\qquad \left\{\frac{\Gamma(m+\frac{1}{2})}{\Gamma(m)}\right\}^2\left\{\frac{1}{m}+\left(\frac{1}{2}\right)^2\frac{1}{m+1}+\left(\frac{1.3}{2.4}\right)^2\frac{1}{m+2}+\ldots\text{ to } n \text{ terms}\right\}$$

$$=\left\{\frac{\Gamma(n+\frac{1}{2})}{\Gamma(n)}\right\}^2\left\{\frac{1}{n}+\left(\frac{1}{2}\right)^2\frac{1}{n+1}+\left(\frac{1.3}{2.4}\right)^2\frac{1}{n+2}+\ldots\text{ to } m \text{ terms}\right\}.$$

We can write this as an equation between two terminating series of the type ${}_4F_3$. For

$$\frac{1}{m}+\left(\frac{1}{2}\right)^2\frac{1}{m+1}+\left(\frac{1.3}{2.4}\right)^2\frac{1}{m+2}+\dots \text{ to } n \text{ terms}$$

$$=\left\{\frac{1.3\dots(2n-3)}{2.4\dots(2n-2)}\right\}^2\frac{1}{m+n-1}\left\{1+\frac{(2n-2)^2(m+n-1)}{(2n-3)^2(m+n-2)}\right.$$

$$\left.+\frac{(2n-2)^2(2n-4)^2(m+n-1)(m+n-2)}{(2n-3)^2(2n-5)^2(m+n-2)(m+n-3)}+\dots\right\}$$

$$=\frac{1}{(m+n-1)\pi}\left\{\frac{\Gamma(n-\frac{1}{2})}{\Gamma(n)}\right\}^2 F\begin{pmatrix}1, -n+1, -n+1, -m-n+1\\ -n+\frac{3}{2}, -n+\frac{3}{2}, -m-n+2\end{pmatrix}.[1]$$

Hence (7.5.2) reduces to

$$(7.5.3)\qquad (m-\tfrac{1}{2})^2 F\begin{pmatrix}1, -n+1, -n+1, -m-n+1\\ -n+\frac{3}{2}, -n+\frac{3}{2}, -m-n+2\end{pmatrix}$$

$$=(n-\tfrac{1}{2})^2 F\begin{pmatrix}1, -m+1, -m+1, -m-n+1\\ -m+\frac{3}{2}, -m+\frac{3}{2}, -m-n+2\end{pmatrix}.$$

Now $${}_2F_1\begin{pmatrix}a, b\\ c\end{pmatrix}; x = (1-x)^{c-a-b}\,{}_2F_1\begin{pmatrix}c-a, c-b\\ c\end{pmatrix}; x$$

and so

$${}_2F_1\left(\begin{matrix}-n+1, -n+1\\ -m-n+2\end{matrix}; x\right){}_2F_1\left(\begin{matrix}-m+\frac{1}{2}, -m+\frac{1}{2}\\ -m-n+1\end{matrix}; x\right)$$

$$={}_2F_1\left(\begin{matrix}-m+1, -m+1\\ -m-n+2\end{matrix}; x\right){}_2F_1\left(\begin{matrix}-n+\frac{1}{2}, -n+\frac{1}{2}\\ -m-n+1\end{matrix}; x\right).$$

If we equate the coefficients of x^{m+n-1} on the two sides of this equation, it will be found that we obtain (7.5.3).

7.6. Most of the preceding formulae, and in particular (7.2.1), can be generalised for "basic" series. I prove the generalisation of (7.2.1), since it is interesting in itself and has a curious connection with the Rogers-Ramanujan identities.

The "basic" generalisation of the hypergeometric series was first studied systematically by Heine. Suppose that, in the hypergeometric series

$$(7.6.1)\qquad 1+\frac{\alpha\beta}{1.\gamma}+\frac{\alpha(\alpha+1)\beta(\beta+1)}{1.2.\gamma(\gamma+1)}+\dots$$

we write $$\frac{1-q^{\lambda+n}}{1-q^{\mu+n}},$$

where $0<q<1$, for $$\frac{\lambda+n}{\mu+n}$$

[1] Using the notation of § 7.1.

(its limit when $q \to 1$), and then write l, m for q^λ and q^μ; and that we also introduce a factor q^n in the n-th term of the series. We thus obtain the series

$$(7.6.2) \qquad 1+\frac{(1-a)(1-b)}{(1-q)(1-c)}q+\frac{(1-a)(1-aq)(1-b)(1-bq)}{(1-q)(1-q^2)(1-c)(1-cq)}q^2+\dots,$$

which reduces to (7.6.1) when $q \to 1$. This is the "basic series" corresponding to (7.6.1).

More generally, we may write

$$(a)_q^n = (1-a)(1-aq)\dots(1-aq^{n-1})$$

and

$$(7.6.3) \qquad \Phi\begin{pmatrix} a_1, a_2, \dots, a_r \\ b_1, b_2, \dots, b_s \end{pmatrix} = \sum_0^\infty \frac{(a_1)_q^n (a_2)_q^n \dots (a_r)_q^n}{(q)_q^n (b_1)_q^n \dots (b_s)_q^n} q^n.$$

If $a_1 = q^{\alpha_1}, \dots, b_1 = q^{\beta_1}, \dots$, and $q \to 1$, then (7.6.3) reduces to

$$F\begin{pmatrix} \alpha_1, \alpha_2, \dots, \alpha_r \\ \beta_1, \beta_2, \dots, \beta_s \end{pmatrix}.$$

An additive relation between the α and β, such as $\alpha_1+\beta_1 = 1$, will correspond to a multiplicative relation between the a and b, such as $a_1 b_1 = q$.

The analogue of the series (7.2.5) is

$$(7.6.4) \qquad \Phi\begin{pmatrix} a, q\sqrt{a}, -q\sqrt{a}, 1/b, 1/c, 1/d, 1/e, 1/f \\ \sqrt{a}, -\sqrt{a}, abq, acq, adq, aeq, afq \end{pmatrix}.$$

Here we have replaced $\qquad s, x, y, z, u, v$

by $\qquad q^s = a, q^x = b, q^y = c, q^z = d, q^u = e, q^v = f.$

The effect of the four parameters

$$q\sqrt{a}, -q\sqrt{a}, \sqrt{a}, -\sqrt{a}$$

is to contribute a factor $\qquad \dfrac{1-aq^{2n}}{1-a}$

in the general term of the series, just as the effect of the two parameters $1+\frac{1}{2}s$ and $\frac{1}{2}s$ in (7.2.5) was to contribute a factor $(s+2n)/s$. The relation (7.2.4) is replaced by

$$(7.6.5) \qquad a^2bcdefq = 1,$$

and one of the parameters, say b, must be of the form q^m.

We can transform the product of gamma-functions on the right of (7.2.1), and the finite product (7.2.6), similarly. We have

$$\frac{\Gamma(\mu_1+1)\Gamma(\mu_2+1)}{\Gamma(\lambda_1+1)\Gamma(\lambda_2+1)} = \prod_1^\infty \frac{(\lambda_1+n)(\lambda_2+n)}{(\mu_1+n)(\mu_2+n)}$$

when $\lambda_1+\lambda_2 = \mu_1+\mu_2$, and this is replaced by

$$\frac{f(l_1)f(l_2)}{f(m_1)f(m_2)},$$

where
$$f(l) = \prod_1^\infty (1-lq^n).$$

Hence the product of gamma-functions becomes

$$(7.6.6) \qquad f(a)f(abcde) \prod_{b,c,d,e} \frac{f(abc)}{f(ab)f(abcd)};$$

and the finite product becomes

$$(7.6.7) \qquad \frac{(aq)_q^m\,(aqde)_q^m\,(aqec)_q^m\,(aqcd)_q^m}{(aqc)_q^m\,(aqd)_q^m\,(aqe)_q^m\,(aqcde)_q^m};$$

and we are led to conjecture that, if (7.6.5) is satisfied, and one of b, c, d, e, f, say b, is q^m, then (7.6.4) is equal to either of (7.6.6) or (7.6.7). It is plain that (7.6.4) is symmetrical in b, c, d, e, f (since the restriction can be imposed upon any one of these parameters), while (7.6.6) is symmetrical in four of five symmetrically connected parameters, and so symmetrical in all of them.

7.7. We can prove the identity, which was found first by F. H. Jackson, by an argument which runs strictly parallel to that of § 7.2. We assume it true for
$$b = 1, q, q^2, \ldots, q^{m-1}$$
and prove it for $b = q^m$. If $b = q^m$, and we multiply both sides by
$$(aqc)_q^m\,(aqcde)_q^m,$$
then it becomes an identity between two polynomials in c of degree $2m$, and it is enough to verify it for $2m+1$ different values of c. Now it is true for
$$c = 1, q, q^2, \ldots, q^{m-1},$$
by the inductive hypothesis and the symmetry in b and c, and also for the m values of c corresponding to the same values of f. It is therefore enough to verify it for one more value of c.

We choose the value
$$c = a^{-1}q^{-m},$$
which is a pole of the last term only of the series (7.6.4). It is sufficient to prove that the residue at this pole is equal to the residue of (7.6.7). The residues are, apart from a factor $-a^{-1}q^{-m}$, the values of the coefficients of

$$\frac{1}{1-acq^m} = -\frac{1}{aq^m}\frac{1}{c-a^{-1}q^{-m}}$$

when $c = a^{-1}q^{-m}$; and we have to prove that these coefficients are equal for this particular value of c.

To calculate the first coefficient we observe that

$$b = q^m, \quad c = \frac{1}{aq^m}, \quad \frac{1}{f} = adeq,$$

and we obtain

$$(7.7.1) \quad \frac{1-aq^{2m}}{1-a}\frac{(a)_q^m}{(q)_q^m}\frac{(q^{-m})_q^m}{(aq^{m+1})_q^m}\frac{(aq^m)_q^m}{(q^{-m+1})_q^{m-1}}\frac{(1/d)_q^m\,(1/e)_q^m\,(adeq)_q^m}{(adq)_q^m\,(aeq)_q^m\,(1/de)_q^m}q^m.$$

Since

$$\left(\frac{1}{d}\right)_q^m = \left(1-\frac{1}{d}\right)\left(1-\frac{q}{d}\right)\dots\left(1-\frac{q^{m-1}}{d}\right) = (-1)^m q^{\frac{1}{2}m(m-1)}d^{-m}(dq^{-m+1})_q^m,$$

the terms in $1/e$ and $1/de$ may be transformed similarly, and

$$(q)_q^m = (1-q)(1-q^2)\dots(1-q^m) = (-1)^m q^{\frac{1}{2}m(m+1)}(q^{-m})_q^m,$$

$$(7.7.1) \text{ is } \quad \frac{1-aq^{2m}}{1-a}\frac{(a)_q^m\,(aq^m)_q^m}{(aq^{m+1})_q^m}\frac{(adeq)_q^m\,(dq^{-m+1})_q^m\,(eq^{-m+1})_q^m}{(q^{-m+1})_q^{m-1}\,(adq)_q^m\,(aeq)_q^m\,(deq^{-m+1})_q^m}.$$

Also

$$\frac{1-aq^{2m}}{1-a}\frac{(a)_q^m\,(aq^m)_q^m}{(aq^{m+1})_q^m} = \frac{1-aq^{2m}}{1-a}(1-a)(1-aq)\dots(1-aq^{m-1})\frac{1-aq^m}{1-aq^{2m}} = (aq)_q^m.$$

Hence finally (7.7.1) reduces to

$$\frac{(aq)_q^m\,(adeq)_q^m\,(dq^{-m+1})_q^m\,(eq^{-m+1})_q^m}{(q^{-m+1})_q^{m-1}\,(adq)_q^m\,(aeq)_q^m\,(deq^{-m+1})_q^m},$$

which is also the coefficient occurring in (7.6.7).

This completes the proof of Jackson's identity, which includes the Dougall-Ramanujan identity as a limiting case.

7.8. Let us suppose, in particular, that

$$a = 1, \quad b = q^m, \quad c = d = e = \epsilon, \quad f = a^{-2}q^{-m-1}\epsilon^{-3},$$

and make ϵ tend to zero. Then

$$\frac{(1-c^{-1})(1-c^{-1}q)\dots(1-c^{-1}q^{n-1})}{(1-acq)(1-acq^2)\dots(1-acq^n)} \sim (-1)^n q^{\frac{1}{2}n(n-1)}\epsilon^{-n},$$

and similarly for the factors in d and e; and

$$\frac{(1-f^{-1})(1-f^{-1}q)\dots(1-f^{-1}q^{n-1})}{(1-afq)(1-afq^2)\dots(1-afq^n)} \sim (-1)^n q^{-\frac{1}{2}n(n-1)+mn}\epsilon^{3n}.$$

After a little reduction, we obtain

$$\sum_{n=0}^{m}(-1)^n(1+q^n)\frac{(1-q^m)(1-q^{m-1})\dots(1-q^{m-n+1})}{(1-q^{m+1})(1-q^{m+2})\dots(1-q^{m+n})}q^{\frac{1}{2}n(3n-1)}$$
$$= (1-q)(1-q^2)\dots(1-q^m),$$

which is the first of the two formulae of Rogers proved directly in § 6.16. If we take $a = q$ instead of $a = 1$, we obtain the second formula. These formulae lead, as we saw in § 6.16, to a proof of the Rogers-Ramanujan identities. The proof thus obtained is a compromise between Rogers's proof given in §§ 6.13–16 and the proof found later by Watson. Watson uses an identity which is more complex in form than that of §§ 7.6–7, but which leads to the Rogers-Ramanujan identities by a direct limiting process. We use the simpler identity, the direct generalisation of the Dougall-Ramanujan identity, to avoid the rather tricky algebra of § 6.16, but retain the more straightforward part of Rogers's argument.

NOTES ON LECTURE VII

§ 7.1. My analysis of the two chapters is in Hardy (**8**). There I call the chapter with which I am primarily concerned ch. XII, but it is ch. x in Watson's copy of the 'second edition'. See Watson, **10**, 139.

Almost all of the formulae discussed in the lecture are proved in Bailey's tract *Generalised hypergeometric series* (Cambridge 1935). This tract contains a full bibliography, and I do not repeat references to papers published before 1914.

The notation for generalised hypergeometric series which has become standardized was introduced by Barnes, *Proc. London Math. Soc.* (2), 5 (1907), 59–116.

§ 7.2. The Dougall-Ramanujan identity is the first formula in ch. x of the note-book. Formulae (1.2) and (1.3) of Lecture I are special cases, but Ramanujan does not seem ever to have printed the general formula, which I found in the note-book only after his death.

§ 7.3. The special case of Morley's formula in which m is a negative integer had been found by Dixon as early as 1891 (and proved more shortly by Richmond in 1892). Morley published his formula in 1902, and it was his result which led Dixon, in the same year, to (7.3.2). Dixon's original proof was very complicated.

The corresponding sum for the expansion of $(1+x)^m$ is a ${}_3F_2$ with argument -1, not summable as a finite product of gamma-functions.

Formula (7.3.3) is equivalent to (1), § 3.2, of Bailey's tract. It is an expression of the theorem that

$$\frac{1}{\Gamma(\beta_1)\Gamma(\beta_2)\Gamma(\beta_1+\beta_2-\alpha_1-\alpha_2-\alpha_3)} F\begin{pmatrix}\alpha_1, \alpha_2, \alpha_3 \\ \beta_1, \beta_2\end{pmatrix}$$

is a symmetric function of the five arguments

$$\beta_1, \quad \beta_2, \quad \beta_1+\beta_2-\alpha_2-\alpha_3, \quad \beta_1+\beta_2-\alpha_3-\alpha_1, \quad \beta_1+\beta_2-\alpha_1-\alpha_2.$$

§ 7.4. The limit process used in the deduction of (7.4.1) requires a little attention. See p. 27 of Bailey's tract, and the paper of Dougall's to which he refers.

For (1.4) see Hardy (**4**) and Whipple (**1**).

Whipple gives two generalisations of (7.4.4) in **3**. These generalisations give the sums of the series

$${}_3F_2\begin{pmatrix}\frac{1}{2}, \frac{1}{2}+x, \frac{1}{2}-x \\ 1+x, 1-x\end{pmatrix}; -1\Bigg)$$

and
$${}_3F_2\begin{pmatrix}\frac{1}{2}, \frac{1}{2}+x, \frac{1}{2}+2x \\ 1+x, 1+2x\end{pmatrix}; -1\Bigg),$$

which reduce to Ramanujan's series when $x = 0$. The second can be proved by the method stated in the lecture.

§ 7.5. The first proofs of (7.5.1) were given by Darling (**1**) and Watson (**5**); and generalisations by Bailey (**2**, **4**), Hodgkinson (**1**) and Whipple (**2**). The formula has an interesting application to the 'Lebesgue' and 'Landau' constants: see Watson, *Quarterly Journal of Math.* (Oxford), 1 (1930), 310–318.

Bailey (**4**) proved (7.5.2), and the still more general formula

$$\frac{\Gamma(x+m)\,\Gamma(y+m)}{\Gamma(m)\,\Gamma(x+y+m)}\left[{}_3F_2\begin{pmatrix}x, y, v+m-1\\ v, x+y+n\end{pmatrix}\right]_n = \frac{\Gamma(x+n)\,\Gamma(y+n)}{\Gamma(n)\,\Gamma(x+y+n)}\left[{}_3F_2\begin{pmatrix}x, y, v+n-1\\ v, x+y+n\end{pmatrix}\right]_m.$$

Here $[\ldots]_n$ and $[\ldots]_m$ mean that only the first n or m terms of the series are retained. This reduces to (7.5.2) when $x = y = \frac{1}{2}$, $v = 1$.

Formula (7.5.3) is a special case of formula (1), § 7.2, of Bailey's tract. Replace x, y, z, u, v, w, n by

$$-n+1, \quad -n+1, \quad 1, \quad -m-n-2, \quad -n+\tfrac{3}{2}, \quad -n+\tfrac{3}{2}, \quad m+n-1.$$

Further information will be found in § 10.4 of the tract.

§ 7.6. See Heine, *Theorie der Kugelfunktionen*, i (1878), 97–125.

§ 7.7. Jackson, *Messenger of Math.* 50 (1921), 101–112. See § 8.3 of Bailey's tract. Watson's proof of the Rogers-Ramanujan identities is given in §§ 8.5–6.

VIII

ASYMPTOTIC THEORY OF PARTITIONS

8.1. In this lecture I shall be concerned with the question: how large is $p(n)$ for large n? It is very remarkable, if we consider how much has been written about approximate or asymptotic values of arithmetical functions, that this question should never have been asked before 1917, when Ramanujan and I published the memoir which is now no. 36 in his *Papers*. We were very lucky in finding a problem which proved to have so much individuality and to admit so complete and so surprising a solution.

8.2. There is one obvious method of attack on any such problem (if no strictly elementary method suggests itself). If a_n is positive, and the power series

$$F(x) = \Sigma a_n x^n$$

has radius of convergence 1, then there is a general correspondence between the order of magnitude of a_n, for large n, and that of $F(x)$ for x near 1. The first step, therefore, is to determine the order of magnitude of Euler's function

$$F(x) = \frac{1}{(1-x)(1-x^2)(1-x^3)\dots} \tag{8.2.1}$$

when $0<x<1$ and $x\to 1$.[1] This is quite simple, if we are content with a rough approximation. For

$$\log F(x) = \sum_n \log \frac{1}{1-x^n} = \sum_{m,n} \frac{x^{mn}}{m} = \Sigma \frac{x^m}{m(1-x^m)}$$

and

$$mx^{m-1}(1-x) < 1-x^m < m(1-x),$$

so that

$$\frac{1}{1-x}\Sigma\frac{x^m}{m^2} < \log F(x) < \frac{1}{1-x}\Sigma\frac{x}{m^2}. \tag{8.2.2}$$

Each of the series in (8.2.2) has the limit $\frac{1}{6}\pi^2$ when $x\to 1$, and so

$$\log F(x) \sim \frac{\pi^2}{6(1-x)} \tag{8.2.3}$$

or

$$F(x) = \exp\left\{\frac{\frac{1}{6}\pi^2 + o(1)}{1-x}\right\}. \tag{8.2.4}$$

[1] I write x now instead of q, q being wanted for other purposes.

It follows that the order of magnitude of $F(x)$ is, to a first approximation, that of

(8.2.5) $$\exp\frac{\pi^2}{6(1-x)}.$$

We want to know the order of a_n corresponding to this order for

$$F(x) = \Sigma a_n x^n.$$

If $a_n = n^\alpha$, where $\alpha > -1$, and $x = e^{-y}$, so that $y \to 0$, then

$$F(x) = \Sigma n^\alpha x^n = \Sigma n^\alpha e^{-ny} \sim \int_0^\infty t^\alpha e^{-ty} dt = \frac{\Gamma(\alpha+1)}{y^{\alpha+1}} \sim \frac{\Gamma(\alpha+1)}{(1-x)^{\alpha+1}}.$$

On the other hand, if a_n were as large as $e^{\delta n}$, for some positive δ, then the series would diverge before x reaches 1. It is plain then that a_n must be smaller than this, but larger than any power of n. It is natural to conjecture that the right order is about

$$e^{Bn^b}$$

for some b between 0 and 1 and some B.

The order of $$G(x) = \Sigma e^{Bn^b} x^n = \Sigma e^{Bn^b - ny}$$

may be calculated roughly from that of its maximum term. This occurs when $Bbn^{b-1} = y$, approximately; and the maximum term is then about

$$\exp\{C(1-x)^{-\frac{b}{1-b}}\},$$

where $$C = B^{\frac{1}{1-b}} b^{\frac{b}{1-b}} (1-b).$$

This agrees with (8.2.5) if $b = \frac{1}{2}$ and $B^2 = \frac{2}{3}\pi^2$; and we conclude that the order of $p(n)$ should be about

$$e^{Kn^{\frac{1}{2}}},$$

where

(8.2.6) $$K = \pi\sqrt{(\tfrac{2}{3})}.$$

8.3. It is actually true that

(8.3.1) $$\log p(n) \sim Kn^{\frac{1}{2}}$$

or

(8.3.2) $$p(n) = e^{(K+o(1))n^{\frac{1}{2}}},$$

but we cannot prove this very simply. It is however quite easy to prove that

(8.3.3) $$e^{An^{\frac{1}{2}}} < p(n) < e^{Bn^{\frac{1}{2}}}$$

for $n > 0$ and *some* positive A and B.

If $x = e^{-y}$, then $y \to 0$ when $x \to 1$, and

(8.3.4) $$1 - x \sim y.$$

We write

$$F(x) = \Sigma p(n)e^{-ny} = G(y); \tag{8.3.5}$$

and we suppose that

$$0 < E < C = \tfrac{1}{6}\pi^2 < D. \tag{8.3.6}$$

(i) It follows from (8.2.3), (8.3.4) and (8.3.6) that

$$p(n)e^{-ny} < G(y) < \exp\frac{D}{y}$$

for all n and small y. Hence, taking $y = n^{-\frac{1}{2}}$, we obtain

$$p(n) < \exp\left(ny + \frac{D}{y}\right) = e^{Bn^{\frac{1}{2}}},$$

with $B = D+1$, so that B may be any number greater than $C+1$.

(ii) On the other hand

$$G(y) > \exp\frac{E}{y}$$

for small y; and so

$$G_1(y) + G_2(y) = \sum_0^m p(n)e^{-ny} + \sum_{m+1}^{\infty} p(n)e^{-ny} > \exp\frac{E}{y}$$

for every m. But $p(n)$ increases with n, and therefore

$$(m+1)\,p(m) > G_1(y) > \exp\frac{E}{y} - G_2(y).$$

Also
$$G_2(y) = \sum_{m+1}^{\infty} p(n)e^{-ny} < \sum_{m+1}^{\infty} \exp\,(Bn^{\frac{1}{2}} - ny),$$

after what we proved under (i); and so

$$p(m) > \frac{1}{m+1}\left\{\exp\frac{E}{y} - \sum_{m+1}^{\infty} \exp\,(Bn^{\frac{1}{2}} - ny)\right\}.$$

We choose H so that $H > 2B$, and take $y = Hm^{-\frac{1}{2}}$. Then

$$\sum_{m+1}^{\infty} \exp\,(Bn^{\frac{1}{2}} - ny) \leqq \sum_{m+1}^{\infty} \exp\,(Bn^{\frac{1}{2}} - 2Bnm^{-\frac{1}{2}}) < \sum_0^{\infty} e^{-Bn^{\frac{1}{2}}} < L,$$

say. It follows that

$$p(m) > \frac{1}{m+1}\left(\exp\frac{E}{y} - L\right) = \frac{1}{m+1}\left(\exp\frac{E}{H}m^{\frac{1}{2}} - L\right) > \exp Am^{\frac{1}{2}}$$

for large m, A being any number less than E/H. Since E may be any number less than C, and H any number greater than $2B = 2(1+D)$, i.e. any number greater than $2(1+C)$, A may be any number less than $\frac{1}{2}C/(1+C)$.

8.4. To prove (8.3.1) or (8.3.2) we require a general theorem of the so-called "Tauberian" kind. It is in fact true, whenever $a_n \geqq 0$ for all n, that

$$\log \Sigma a_n x^n \sim \frac{C}{1-x}$$

implies $$\log A_n = \log(a_0 + a_1 + \ldots + a_n) \sim 2(Cn)^{\frac{1}{2}}.$$

When, as here, a_n increases with n, we can replace the last equation by

$$\log a_n \sim 2(Cn)^{\frac{1}{2}},$$

and this gives (8.3.1).

We have thus an asymptotic formula for the logarithm of $p(n)$. But an asymptotic formula for the logarithm of an arithmetical function is a very crude result; it does not distinguish, for example, between the orders of magnitude

$$n^{\pm 1000} e^{Kn^{\frac{1}{2}}}.$$

We should like, at the least, an asymptotic formula for $p(n)$ itself. Actually

$$p(n) \sim \frac{1}{4n\sqrt{3}} e^{K\sqrt{n}}; \tag{8.4.1}$$

but we cannot hope to prove so much as this by arguments of so elementary and general a kind. It is essential to use more powerful methods which pay proper regard to the more intimate peculiarities of $F(x)$.

8.5. Our natural resource is Cauchy's theorem. This tells us that

$$p(n) = \frac{1}{2\pi i}\int_C \frac{F(x)}{x^{n+1}}\,dx, \tag{8.5.1}$$

where C is a contour round the origin. We must move C into the most advantageous position and study the integral directly. There is of course nothing in the least novel in this idea, which is that which dominates the whole analytic theory of numbers, and in particular the theory of primes; but the setting here is quite different, and it is instructive to compare the two problems.

In the theory of primes our generating functions were Dirichlet's series $\Sigma a_n n^{-s}$, and the proof of the Prime Number Theorem depended upon the integral

$$\psi^*(x) = -\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{\zeta'(s)}{\zeta(s)}\frac{x^s}{s}\,ds,$$

where $c > 1$. We moved the contour of integration to the left, across the pole at $s = 1$, and inferred that $\psi^*(x)$ and $\psi(x)$ differed from the residue x at the pole by an error of smaller order than x.[1]

[1] See Lecture II.

The conclusion was correct, but the argument was difficult to justify because $\zeta(s)$ behaves in a very complicated way at infinity. In particular the location of its zeros is still highly mysterious. On the other hand there is no difficulty at all about the singularity which yields the dominant term, a pole at $s = 1$ of the simplest possible character.

8.6. The singularities of the $F(x)$ of our present problem are very much more complicated. They cover the unit circle $|x| = 1$. The circle is a "barrier" for the function, which does not exist outside it, and there can be no question of "moving C across the singularities". All that we can hope to do is to move C *close to* the singularities and study each part of it in detail.

For all this, however, there are strong consolations. The function $F(x)$ is one of a well-known class, the elliptic modular functions, whose properties have been studied intensively and are very exactly known. These functions all have the same peculiarities as $F(x)$, and exist only inside the circle; but they satisfy remarkable functional equations which enable us to determine their behaviour, near any point of the circle, very precisely. In particular, $F(x)$ satisfies the equation

$$F(x) = \frac{x^{\frac{1}{24}}}{(2\pi)^{\frac{1}{2}}}\left(\log\frac{1}{x}\right)^{\frac{1}{2}}\exp\left\{\frac{\pi^2}{6\log(1/x)}\right\}F(x'), \tag{8.6.1}$$

where

$$\log\frac{1}{x}\log\frac{1}{x'} = 4\pi^2, \quad x' = \exp\left\{-\frac{4\pi^2}{\log(1/x)}\right\}. \tag{8.6.2}$$

If, for example, x is positive and near to 1, then x' is extravagantly small and $F(x')$ is practically 1; so that (8.6.1) expresses $F(x)$, effectively, in terms of elementary functions. There are similar formulae associated with other points of the circle, such as

$$-1, e^{\frac{2}{3}\pi i}, e^{-\frac{2}{3}\pi i}, i, -i, e^{\frac{2}{5}\pi i}, \ldots$$

(generally, with all primitive roots of unity); but (8.6.1) alone is enough to enable us to make great progress.

In particular, if we take C to be a circle with just the right radius, a little less than 1, we can substitute from (8.6.1) into (8.5.1), and replace $F(x')$ by 1, with an error which turns out to be of order

$$e^{Hn^{\frac{1}{2}}},$$

where $$H < K = \pi\sqrt{(\tfrac{2}{3})}.$$

There are then only elementary functions in the integral, and we can calculate it very precisely.

The result is the formula

(8.6.3) $$p(n) = \frac{1}{2\pi\sqrt{2}}\frac{d}{dn}\left(\frac{e^{K\lambda_n}}{\lambda_n}\right) + O(e^{Hn^{\frac{1}{2}}}),$$

where

(8.6.4) $$\lambda_n = \sqrt{(n - \tfrac{1}{24})}, \quad H < K.$$

This includes (8.4.1) and is very much more precise. The form of the dominant term is at first sight rather mysterious, but it arises naturally from the analysis. In particular, the " $-\frac{1}{24}$" in λ_n arises naturally from the index $\frac{1}{24}$ in (8.6.1).

8.7. This however is by no means the end of the matter. The formula (8.6.1) is that appropriate for the study of $F(x)$ near $x = 1$. There are, as I remarked, similar formulae associated with other "rational points"

(8.7.1) $$x_{p,q} = e^{2p\pi i/q}$$

on the unit circle. One may say (naturally very roughly) that these "rational singularities" are the *heaviest* singularities of $F(x)$, that $F(x)$ is bigger near them than near other points of the circle, and that their contributions to the integral (8.5.1) may be expected to outweigh those of other points.

Further, these rational singularities diminish in weight as q increases. When $x \to 1$ along a radius, $F(x)$ behaves roughly like

$$\exp\left\{\frac{\pi^2}{6(1-x)}\right\},$$

while when $x \to x_{p,q}$ it behaves roughly like

$$\exp\left\{\frac{\pi^2}{6q^2(1-|x|)}\right\}.$$

It is reasonable to expect that

(8.7.2) $$p(n) = P_1(n) + P_2(n) + \ldots + P_Q(n) + R(n),$$

where $P_1(n)$ is the dominant term in (8.6.3), $P_2(n), P_3(n), \ldots, P_Q(n)$ are similar in form, but with smaller numbers $K_2, K_3, \ldots, K_Q$ in the place of K, and $R(n)$ is an error of lower order than $e^{K_Q n^{\frac{1}{2}}}$.

The proof of all this can be put through without much additional difficulty. The form of $P_q(n)$ is

(8.7.3) $$P_q(n) = L_q(n)\,\phi_q(n),$$

where

(8.7.4) $$\phi_q(n) = \frac{q^{\frac{1}{2}}}{2\pi\sqrt{2}}\frac{d}{dn}\left(\frac{e^{K\lambda_n/q}}{\lambda_n}\right),$$

(8.7.5) $$L_q(n) = \sum_p \omega_{p,q} e^{-2np\pi i/q},$$

p runs through the integers less than and prime to q,[1] and $\omega_{p,q}$ is a certain 24th root of unity. Thus

$$L_1(n) = 1, \quad \phi_1(n) = P_1(n), \quad K_q = \frac{K}{q}.$$

And

(8.7.6) $$R(n) = O(e^{H_Q n^{\frac{1}{2}}}),$$

where $$H_Q < K_Q$$

(so that $H_Q \to 0$ when $Q \to \infty$). We can thus find $p(n)$ with error $O(e^{\delta n^{\frac{1}{2}}})$ and an arbitrarily small positive δ.

8.8. At this point we might have stopped had it not been for Major MacMahon's love of calculation. MacMahon was a practised and enthusiastic computer, and made us a table of $p(n)$ up to $n = 200$. In particular he found that

(8.8.1) $$p(200) = 3972999029388,$$

and we naturally took this value as a test for our asymptotic formula. We expected a good result, with an error of perhaps one or two figures, but we had never dared to hope for such a result as we found. Actually 8 terms of our formulae gave $p(200)$ with an error of 0·004. We were inevitably led to ask whether the formula could not be used to calculate $p(n)$ *exactly* for any large n.

It is plain that, if this is possible, it will be necessary to use a "large" number of terms of the series, that is to say to make Q a function of n. Our final result was as follows. There are constants α, M such that

(8.8.2) $$p(n) = \sum_{q < \alpha n^{\frac{1}{2}}} P_q(n) + R(n),$$

where

(8.8.3) $$|R(n)| < Mn^{-\frac{1}{4}},$$

and, since $p(n)$ is an integer, (8.8.2) will give its value exactly for sufficiently large n. The formula is one of the rare formulae which are both asymptotic and exact; it tells us all we want to know about the order and approximate form of $p(n)$, and it appears also to be adapted for exact calculation. It was in fact from this formula that D. H. Lehmer first calculated the value of $p(721)$.

8.9. It was however necessary, until very recently, to make a curious reservation at this point. The values of $p(200)$ and $p(243)$ were *known*, because they had been calculated directly by MacMahon and Gupta, but calculations based upon (8.8.2) were not decisive. We cannot use the formula

[1] When $q = 1$, $p = 0$.

to *prove* that $p(721)$ has a particular value until we have found numerical values for α and M; Ramanujan and I had merely proved their existence. It was necessary to go over all our analysis, give numerical values to all our "constants", and replace all our "O" terms by terms with numerical bounds. In the meantime Lehmer's calculations, which used 21 terms of the series, and led to the value

$$161061755750279477635534762{\cdot}0041,$$

gave a very strong presumption about the value of $p(721)$ but were not conclusive.

The gap has now been filled by Rademacher, who (trying at first merely to simplify our analysis) was led to make a very fortunate formal change. Ramanujan and I worked, not exactly with the function

$$\phi(n) = \frac{1}{2\pi\sqrt{2}}\frac{d}{dn}\left(\frac{e^{K\lambda_n}}{\lambda_n}\right),$$

but with the "nearly equivalent" function

$$\frac{1}{\pi\sqrt{2}}\frac{d}{dn}\left(\frac{\cosh K\lambda_n - 1}{\lambda_n}\right)$$

(afterwards discarding the less important parts of the function). Rademacher works with

$$\psi(n) = \frac{1}{\pi\sqrt{2}}\frac{d}{dn}\left(\frac{\sinh K\lambda_n}{\lambda_n}\right),$$

which is also "nearly equivalent"; and this apparently slight change has a very important effect, since it leads to an *identity* for $p(n)$.

We have

$$\frac{d}{dn}\left(\frac{1}{\lambda_n}\sinh\frac{K\lambda_n}{q}\right) = \frac{d}{dn}\left(\frac{K}{q} + \frac{K^3(n-\frac{1}{24})}{6q^3} + \ldots\right) = O\left(\frac{1}{q^3}\right)$$

for fixed n and large q. Hence the function

$$\text{(8.9.1)} \qquad \psi_q(n) = \frac{q^{\frac{1}{2}}}{\pi\sqrt{2}}\frac{d}{dn}\left\{\frac{\sinh(K\lambda_n/q)}{\lambda_n}\right\}$$

behaves for large q like a multiple of $q^{\frac{1}{2}}.q^{-3} = q^{-\frac{5}{2}}$; and

$$\text{(8.9.2)} \qquad |L_q(n)| = \left|\sum_p \omega_{p,q} e^{-2np\pi i/q}\right| \leq q,$$

so that

$$\text{(8.9.3)} \qquad \sum_q L_q(n)\,\psi_q(n)$$

is convergent. The series $\Sigma L_q(n)\,\phi_q(n)$ is not convergent; this question Ramanujan and I left doubtful, but it has been settled since by Lehmer.

Rademacher proved that

$$p(n) = \sum_{q=1}^{\infty} L_q(n)\,\psi_q(n), \tag{8.9.4}$$

and that the remainder after Q terms is less than

$$CQ^{-\frac{1}{2}} + D\left(\frac{Q}{n}\right)^{\frac{1}{2}} \sinh \frac{Kn^{\frac{1}{2}}}{Q},$$

where C and D are constants for which he found definite values. The remainder is of order $n^{-\frac{1}{4}}$ when Q is of order $n^{\frac{1}{2}}$, as in our older work.

Proof of Rademacher's identity

8.10. The rest of this lecture will be devoted to the proof of (8.9.4).

The Farey series $\mathfrak{F}_N$ of order N is the set of irreducible fractions p/q, between 0 and 1, whose denominators do not exceed N. We include 0 and 1 in the forms $\frac{0}{1}$ and $\frac{1}{1}$: thus $\mathfrak{F}_5$ is

$$\tfrac{0}{1},\ \tfrac{1}{5},\ \tfrac{1}{4},\ \tfrac{1}{3},\ \tfrac{2}{5},\ \tfrac{1}{2},\ \tfrac{3}{5},\ \tfrac{2}{3},\ \tfrac{3}{4},\ \tfrac{4}{5},\ \tfrac{1}{1}.$$

If

$$x = re^{2\pi i\theta}$$

and θ runs through the values of $\mathfrak{F}_N$ (so that the first and last values of x are each r), we define a set of "Farey points" $re^{2p\pi i/q}$ on the circle $|x| = r$.

Suppose that $q > 1$ and

$$\frac{p''}{q''},\quad \frac{p}{q},\quad \frac{p'}{q'}$$

are three consecutive fractions of $\mathfrak{F}_N$. We associate with p/q the interval $\xi_{p,q}$ or

$$\frac{p}{q} - \chi''_{p,q},\quad \frac{p}{q} + \chi'_{p,q},$$

where

$$\chi''_{p,q} = \frac{1}{q(q+q'')},\quad \chi'_{p,q} = \frac{1}{q(q+q')}.$$

These intervals just fill up the interval (0, 1), and the length of each of the parts into which $\xi_{p,q}$ is divided by p/q lies between

$$\frac{1}{2Nq},\quad \frac{1}{Nq}.$$

The definitions naturally require modification when $q = 1$. Then p/q is $\frac{0}{1}$ or $\frac{1}{1}$ and the interval has one part only.

We shall also use $\xi_{p,q}$ for the arc of $|x| = r$ defined by the same values of θ; the two extreme arcs, which now abut at $x = r$, are to be amalgamated into one. We call this dissection of the circle the ***Farey dissection*** of order N.

We apply (8.5.1) to the circle C defined by

$$|x| = r = e^{-2\pi/N^2}.$$

We write

$$(8.10.1)\qquad p(n)=\frac{1}{2\pi i}\int\frac{F(x)}{x^{n+1}}dx=\sum_{p,q}\frac{1}{2\pi i}\int_{\xi_{p,q}}\frac{F(x)}{x^{n+1}}dx=\sum_{p,q}j_{p,q},$$

and study each integral separately. The range of summation is defined by

$$(8.10.2)\qquad 0<p<q,\quad (p,q)=1,\quad 1\leqq q\leqq N$$

(except that, when $q=1$, p assumes the single value 0).

8.11. On $\xi_{p,q}$ we write

$$(8.11.1)\qquad x=re^{2\pi i\theta},\quad r=e^{-2\pi/N^2},\quad \theta=\frac{p}{q}+\phi$$

so that

$$(8.11.2)\qquad x=\exp\left\{\frac{2p\pi i}{q}-2\pi\left(\frac{1}{N^2}-i\phi\right)\right\}=\exp\left(\frac{2p\pi i}{q}-\frac{2\pi z}{q}\right),$$

where

$$(8.11.3)\qquad z=q\left(\frac{1}{N^2}-i\phi\right).$$

Thus $\phi=0$ gives the Farey point of $\xi_{p,q}$,

$$(8.11.4)\qquad -\chi''=-\chi''_{p,q}\leqq\phi\leqq\chi'_{p,q}=\chi'$$

and

$$(8.11.5)\qquad \frac{1}{2qN}<\frac{\chi'}{\chi''}<\frac{1}{qN},\quad |\phi|<\frac{1}{qN}.$$

We also collect here some simple inequalities which will be useful later. Since $|q\phi|<1/N$ and $q\leqq N$, we have

$$(8.11.6)\qquad |z|=q(N^{-4}+\phi^2)^{\frac12}\leqq(q^2N^{-4}+N^{-2})^{\frac12}\leqq 2^{\frac12}N^{-1}$$

(so that $|z|$ is small, uniformly, for large N); also

$$(8.11.7)\qquad \left|\exp\left(-\frac{\pi z}{12q}\right)\right|=\exp\left(-\frac{\pi}{12N^2}\right)<1,$$

and

$$(8.11.8)\qquad \frac{1}{q}\Re\frac{1}{z}=\frac{N^{-2}}{q^2(N^{-4}+\phi^2)}>\frac{N^{-2}}{q^2N^{-4}+N^{-2}}\geqq\tfrac12.$$

8.12. We now use the formula of the type (8.6.1) but associated with $e^{2p\pi i/q}$ instead of with 1. This formula is

$$(8.12.1)\qquad F(x)=\omega_{p,q}z^{\frac12}\exp\left(\frac{\pi}{12qz}-\frac{\pi z}{12q}\right)F(x'),$$

where

$$(8.12.2)\qquad x=\exp\left(\frac{2p\pi i}{q}-\frac{2\pi z}{q}\right),\quad x'=\exp\left(\frac{2p_1\pi i}{q}-\frac{2\pi}{qz}\right),$$

$\omega_{p,q}$ is the 24th root of unity referred to in § 8.7, $z^{\frac{1}{2}}$ has its principal value, and $pp_1 \equiv -1 \pmod q$. The formula reduces to (8.6.1) when $q = 1$.

If we substitute from (8.12.1) into the integral $j_{p,q}$ of (8.10.1), write

(8.12.3) $$\Psi(z) = \Psi_q(z) = z^{\frac{1}{2}} \exp\left(\frac{\pi}{12qz} - \frac{\pi z}{12q}\right),$$

and observe that

$$\frac{dx}{x} = 2\pi i\, d\phi, \quad x^{-n} = \exp\left(\frac{2n\pi}{N^2} - \frac{2np\pi i}{q} - 2n\pi\phi i\right),$$

we obtain

(8.12.4) $$e^{-2n\pi/N^2} j_{p,q} = \omega_{p,q} e^{-2np\pi i/q} \int_{-\chi''}^{\chi'} \Psi(z)\, F(x')\, e^{-2n\pi\phi i}\, d\phi.$$

It is convenient to have also a formula for $j_{p,q}$ in which the variable of integration is z. This formula, which is a trivial transformation of (8.12.4), is

(8.12.5) $$j_{p,q} = \frac{i}{q} \omega_{p,q} e^{-2np\pi i/q} \int \Psi(z)\, F(x')\, e^{2n\pi z/q}\, dz,$$

the path of integration being the straight line joining the points

$$q\left(\frac{1}{N^2} + i\chi''\right), \quad q\left(\frac{1}{N^2} - i\chi'\right)$$

in the plane of z.

8.13. When z is small and positive (i.e. when x is near to $e^{2p\pi i/q}$, on the radius to that point),

$$|x'| = e^{-2\pi/qz}$$

is extremely small, and $F(x')$ is practically 1. We may hope to replace $F(x')$ by 1, without serious error, on the whole of $\xi_{p,q}$. We therefore write

(8.13.1) $$j_{p,q} = J_{p,q} + J'_{p,q}$$

and

(8.13.2) $$p(n) = \Sigma j_{p,q} = \Sigma J_{p,q} + \Sigma J'_{p,q} = P(n) + P'(n),$$

say, $J_{p,q}$ and $J'_{p,q}$ being the integrals obtained from $j_{p,q}$ when we replace $F(x')$ by

$$1, \quad F(x') - 1,$$

respectively. Of course $P(n)$ and $P'(n)$ depend upon N as well as n, and we have to study their limits when $N \to \infty$.

8.14. We prove first that

(8.14.1) $$P'(n) \to 0.$$

We have $$\Psi(z)\{F(x') - 1\} = z^{\frac{1}{2}} \exp\left\{-\frac{\pi}{12q}\left(z - \frac{1}{z}\right)\right\} \sum_1^\infty p(\nu)\, x'^{\nu}.$$

Now
$$|e^{-\pi z/12q}|<1,$$
by (8.11.7). Also
$$|e^{\pi/12qz}x'^{\nu}| = \left|\exp\left\{\frac{\pi}{12qz}+\nu\left(\frac{2p_1\pi i}{q}-\frac{2\pi}{qz}\right)\right\}\right|$$
$$= \exp\left\{-\frac{2\pi}{q}(\nu-\tfrac{1}{24})\,\Re\frac{1}{z}\right\} \leqq e^{-(\nu-\frac{1}{24})\pi},$$
by (8.11.8), so that
$$\left|e^{\pi/12qz}\sum_1^{\infty} p(\nu)\,x'^{\nu}\right| \leqq \sum_1^{\infty} p(\nu)\,e^{-(\nu-\frac{1}{24})\pi} = B,$$
where B is a constant. Finally,
$$|z|^{\frac12} \leqq 2^{\frac14}N^{-\frac12},$$
by (8.11.6), and the interval of integration in (8.12.4) is less than $2/qN$, by (8.11.5). Hence
$$|J'_{p,q}| < e^{2n\pi/N^2}\cdot\frac{2}{qN}\cdot 2^{\frac14}N^{-\frac12}B = e^{2n\pi/N^2}\frac{2^{\frac54}B}{qN^{\frac32}},$$
and
$$|P'(n)| = O\left(N^{-\frac32}\sum_{p,q}\frac{1}{q}\right) = O\left(N^{-\frac32}\sum_{q\leqq N}1\right) = O(N^{-\frac12}).$$

Combining (8.14.1) and (8.13.2), and remembering that $p(n)$ is independent of N, we see that

(8.14.2)
$$p(n) = \lim_{N\to\infty} P(n).$$

8.15. In discussing $P(n)$ we use the formula (8.12.5), with $F(x')$ replaced by 1. Putting
$$z = qZ$$
we obtain

(8.15.1)
$$J_{p,q} = \omega_{p,q}\,e^{-2np\pi i/q}R_{p,q},$$
where

(8.15.2)
$$R_{p,q} = \frac{q^{\frac12}}{i}\int Z^{\frac12}\exp\left\{\frac{\pi}{12q^2Z}+2\pi(n-\tfrac{1}{24})\,Z\right\}dZ.$$

The path of integration is now the line L of Fig. 2. Thus

(8.15.3)
$$p(n) = \lim_{N\to\infty}P(n) = \lim_{N\to\infty}\sum_{q=1}^{N}\sum_{p}\omega_{p,q}R_{p,q}e^{-2np\pi i/q} = \lim_{N\to\infty}\sum_{q=1}^{N}T_q,$$
where

(8.15.4)
$$T_q = \sum_{p}\omega_{p,q}R_{p,q}e^{-2np\pi i/q}.$$

We have now to transform $R_{p,q}$. We apply Cauchy's theorem to the contour indicated in the figure. Here $Z^{\frac12}$ is positive at N^{-2}, is $-i(-Z)^{\frac12}$, where $(-Z)^{\frac12}$ is positive, on the line (1), and $i(-Z)^{\frac12}$ on (6). Also
$$\epsilon < N^{-2}$$
and ϵ will be made to tend to 0 before N tends to ∞.

Cauchy's theorem gives

$$(8.15.5)\qquad R_{p,q} = \frac{q^{\frac{1}{2}}}{i}\left(\int_{-\infty}^{(0+)} - \int_{(1)} - \int_{(2)} - \int_{(3)} - \int_{(4)} - \int_{(5)} - \int_{(6)}\right)$$

$$= 2q^{\frac{1}{2}}U_q + iq^{\frac{1}{2}}(I_1+I_2+I_3+I_4+I_5+I_6),$$

say, where U_q is an integral from $-\infty$ to $-\infty$ enclosing the origin in the positive direction, and is independent of p.[1]

It is plain that

$$(8.15.6)\qquad I_1+I_6 \to -2iV_q,$$

where

$$(8.15.7)\qquad V_q = \int_0^\infty t^{\frac{1}{2}} \exp\left\{-\frac{\pi}{12q^2t} - 2\pi(n-\tfrac{1}{24})t\right\} dt$$

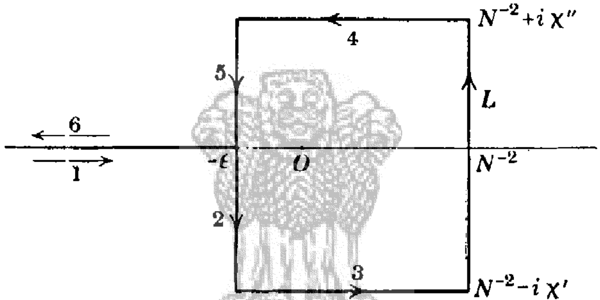


Fig. 2.

(so that V_q also is independent of p), when $\epsilon \to 0$. If we assume provisionally that the other integrals I_2, I_3, I_4, and I_5 can be neglected, then we shall have

$$p(n) = \lim_{N\to\infty} \sum_{q=1}^{N} T_q = \lim_{N\to\infty} \lim_{\epsilon\to 0} \sum_{q=1}^{N} T_q$$

$$= \lim_{N\to\infty} \lim_{\epsilon\to 0} \sum_{q=1}^{N} \{2q^{\frac{1}{2}}U_q + iq^{\frac{1}{2}}(I_1+I_6)\} \sum_p \omega_{p,q} e^{-2np\pi i/q}$$

$$= \lim_{N\to\infty} \sum_{q=1}^{N} 2q^{\frac{1}{2}}(U_q+V_q) \sum_p \omega_{p,q} e^{-2np\pi i/q}$$

$$= \lim_{N\to\infty} 2\sum_{q=1}^{N} q^{\frac{1}{2}} L_q(U_q+V_q) = 2\sum_{q=1}^{\infty} q^{\frac{1}{2}} L_q(U_q+V_q).$$

Here $L_q = L_q(n)$ is defined by (8.7.5). And if we can then prove that

$$(8.15.8)\qquad U_q+V_q = \frac{1}{2\pi\sqrt{2}}\frac{d}{dn}\left(\frac{1}{\lambda_n}\sinh\frac{K\lambda_n}{q}\right),$$

the proof of Rademacher's identity will be completed.

[1] $R_{p,q}$ depends upon p as well as on q, because χ' and χ'' do so; and so do I_2, I_3, I_4, I_5. But U_q, V_q, and the upper bounds found for $I_2, \ldots, I_5$ in § 8.16, are independent of p.

8.16. We must first verify the provisional assumption made in § 8.15. It is sufficient to prove that

$$\lim_{N\to\infty}\sum_{q=1}^{N} q^{\frac{1}{2}}\sum_{p}\overline{\lim_{\epsilon\to 0}}\,|I_2+I_3+I_4+I_5| = 0,$$

or that

(8.16.1) $$\lim_{N\to\infty}\sum_{q=1}^{N} q^{\frac{3}{2}}\,\overline{\lim_{\epsilon\to 0}}\,\operatorname*{Max}_{p}\,|I_2+I_3+I_4+I_5| = 0.$$

In I_2, $$Z = -\epsilon - iY \qquad (0 < Y < \chi'),$$

$$\Re(Z) < 0, \quad \Re\left(\frac{1}{Z}\right) = -\frac{\epsilon}{\epsilon^2+Y^2} < 0,$$

and $$|I_2| < (\epsilon^2+\chi'^2)^{\frac{1}{4}}\chi' < \left(\epsilon^2+\frac{1}{q^2N^2}\right)^{\frac{1}{4}}\frac{1}{qN}.$$

Plainly I_5 satisfies the same inequality, and

(8.16.2) $$\overline{\lim_{\epsilon\to 0}}\,|I_2+I_5| \leqq 2(qN)^{-\frac{3}{2}}$$

(for each p).

In I_3, $$Z = X - i\chi' \qquad (-\epsilon < X < N^{-2}),$$

$$|Z|^{\frac{1}{2}} < 2^{\frac{1}{4}}q^{-\frac{1}{2}}N^{-\frac{1}{2}},$$

by (8.11.6); $$|e^{2\pi(n-\frac{1}{24})Z}| < e^{2n\pi/N^2};$$

$$\frac{1}{q^2}\Re\frac{1}{Z} = \frac{X}{q^2(X^2+\chi'^2)} \leqq \frac{N^{-2}}{q^2\chi'^2} \leqq 4,$$

by (8.11.5); and, for each p,

(8.16.3) $$|I_3| < 2N^{-2}.2^{\frac{1}{4}}q^{-\frac{1}{2}}N^{-\frac{1}{2}}.e^{2n\pi/N^2}.e^{\frac{1}{3}\pi} < BN^{-\frac{5}{2}}q^{-\frac{1}{2}}e^{2n\pi/N^2},$$

where B is again a constant (in particular, independent of p). Plainly I_4 also satisfies the same inequality, and so

(8.16.4) $$\overline{\lim_{\epsilon\to 0}}\,|I_3+I_4| < 2BN^{-\frac{5}{2}}q^{-\frac{1}{2}}e^{2n\pi/N^2}.$$

Thus it is sufficient to prove that

$$\sum_{q=1}^{N} q^{\frac{3}{2}}(qN)^{-\frac{3}{2}} \to 0$$

and $$\sum_{q=1}^{N} q^{\frac{3}{2}}.N^{-\frac{5}{2}}q^{-\frac{1}{2}} \to 0;$$

and each of these is $O(N^{-\frac{1}{2}})$.

8.17. It remains only to evaluate the integrals U_q and V_q, and so prove (8.15.8). The integral U_q is one of a standard type, and may be calculated by formulae to be found in Watson's *Bessel functions*. We have

$$U_q = \frac{1}{2i}\int_{-\infty}^{(0+)} Z^{\frac{1}{2}} \exp\left\{\frac{\pi}{12q^2Z} + 2\pi(n-\tfrac{1}{24})\,Z\right\}dZ$$

$$= \frac{1}{4\pi i}\frac{d}{dn}\int_{-\infty}^{(0+)} Z^{-\frac{1}{2}} \exp\left\{\frac{\pi}{12q^2Z} + 2\pi(n-\tfrac{1}{24})\,Z\right\}dZ.$$

If we put
$$Z = \frac{t}{2\pi(n-\frac{1}{24})} = \frac{t}{2\pi\lambda_n^2},$$
we obtain
$$\frac{1}{4\pi i}\frac{d}{dn}\left\{\frac{1}{\lambda_n\sqrt{(2\pi)}}\int_{-\infty}^{(0+)} t^{-\frac{1}{2}}\exp\left(t-\frac{\mu^2}{4t}\right)dt\right\},$$
with
$$\mu^2 = -\frac{2\pi^2\lambda_n^2}{3q^2},\quad \mu = i\pi\sqrt{(\tfrac{2}{3})}\,\frac{\lambda_n}{q} = i\,\frac{K\lambda_n}{q};$$
and this is
$$\frac{1}{2i}\frac{d}{dn}\left\{\frac{1}{(2\pi)^{\frac{1}{2}}\lambda_n}\,.\,2\pi i(\tfrac{1}{2}\mu)^{\frac{1}{2}}J_{-\frac{1}{2}}(\mu)\right\} = \frac{1}{2\pi\sqrt{2}}\frac{d}{dn}\left(\frac{1}{\lambda_n}\cosh\frac{K\lambda_n}{q}\right),$$
so that
$$U_q = \frac{1}{2\pi\sqrt{2}}\frac{d}{dn}\left(\frac{1}{\lambda_n}\cosh\frac{K\lambda_n}{q}\right).$$

As for V_q, we have

$$V_q = \int_0^{\infty} t^{\frac{1}{2}}\exp\left\{-\frac{\pi}{12q^2t} - 2\pi(n-\tfrac{1}{24})\,t\right\}dt$$

$$= -\frac{1}{2\pi}\frac{d}{dn}\int_0^{\infty} t^{-\frac{1}{2}}\exp\left\{-\frac{\pi}{12q^2t} - 2\pi(n-\tfrac{1}{24})\,t\right\}dt.$$

Since
$$\int_0^{\infty} e^{-a^2t^2-b^2/t^2}\,dt = \frac{\sqrt{\pi}}{2a}\,e^{-2ab}$$
when a and b are positive, this gives
$$V_q = -\frac{1}{2\pi\sqrt{2}}\frac{d}{dn}\left(\frac{e^{-K\lambda_n/q}}{\lambda_n}\right);$$
and
$$U_q + V_q = \frac{1}{2\pi\sqrt{2}}\frac{d}{dn}\left(\frac{1}{\lambda_n}\sinh\frac{K\lambda_n}{q}\right),$$
which is (8.15.8).

8.18. It is easy to estimate the error involved in taking only N terms of the series. The value of N which we choose will depend upon n, and we must now use approximations valid uniformly in n. In the first place

$$|\,L_q(n)\,| \leq q$$

by (8.9.2). Also

$$\frac{d}{dn}\left(\frac{1}{\lambda_n}\sinh\frac{K\lambda_n}{q}\right) = \frac{d}{dn}\left(\frac{K}{q}+\frac{K^3(n-\frac{1}{24})}{6q^3}+\ldots\right) = O\left(\frac{1}{q^3}\right)$$

if

$$n < Aq^2,$$

for a fixed A, and q is large (and uniformly in any such range of n). The error will involve all values of q greater than N, so that N must be at least of order $n^{\frac{1}{2}}$.

This being so, we have $\psi_q(n) = O(q^{-\frac{3}{2}})$

for $q > N$, and $\sum_{N+1}^{\infty} L_q(n)\,\psi_q(n) = O\left(\sum_{N+1}^{\infty} q^{-\frac{3}{2}}\right) = O(N^{-\frac{1}{2}})$.

In particular, if we take N of order $n^{\frac{1}{2}}$, we obtain $p(n)$, for large n, with an error $O(n^{-\frac{1}{4}})$.

A rather closer analysis is required for numerical computation, but there is no difficulty of principle. Thus Rademacher, using only the crude estimate (8.9.2), found that the error after N terms is less than

$$B_1 N^{-\frac{1}{2}} + B_2\left(\frac{N}{n}\right)^{\frac{1}{2}} \sinh\frac{Kn^{\frac{1}{2}}}{N}$$

(with definite values for B_1 and B_2). If we wish to use $\phi_q(n)$ instead of $\psi_q(n)$, as in the Hardy-Ramanujan formula (and this is more convenient), then there is a slight additional error which is easily estimated. Thus Rademacher was able to show that if $n = 721$ and $N = 21$, the error is less than 0·38. Since $p(n)$ is an integer, any bound less than $\frac{1}{2}$ is good enough to determine $p(n)$.

A cruder estimate will suffice for special purposes. These calculations, for example, were undertaken with a view to deciding whether

$$p(721) \equiv 0 \pmod{11^3}$$

in accordance with one of Ramanujan's conjectures. Since Ramanujan proved that $p(721)$ is a multiple of 11^2, any bound for the error less than 60·5 is sufficient for this purpose.

I have set out in tabular form the values of the first 21 terms of the Hardy-Ramanujan formula.[1] The rapidity of the convergence is very impressive, and the actual error is much less than that given by Rademacher's analysis. The largest value of $p(n)$ which has been found in this way is

$p(14031) =$ 92 85303 04759 09931 69434 85156 67127
75089 29160 56358 46500 54568 28164
58081 50403 46756 75123 95895 59113
47418 88383 22063 43272 91599 91345
00745,

[1] The corresponding table for $n = 14031$ is too long to print conveniently.

recently computed by Lehmer. This requires 62 terms of the series, and a more refined estimate of $L_q(n)$; the crude inequality (8.9.2) is insufficient. Lehmer has, however, shown that

$$|L_q(n)| < 2q^{\frac{5}{6}}, \tag{8.18.1}$$

and this was enough for the purpose. It turns out that

$$p(14031) \equiv 0 \pmod{11^4},$$

as predicted by Ramanujan.

q	$P_q(721)$
1	161061755750279601828302117·84821
2	− 124192062781·96844
3	− 706763·61926
4	2169·16829
5	0·00000[1]
6	14·20724
7	6·07837
8	0·18926
9	0·04914
10	0·00000[1]
11	0·08814
12	− 0·03525
13	0·03247
14	− 0·00687
15	0·00000[1]
16	− 0·01133
17	0·00000[1]
18	− 0·00553
19	0·00859
20	0·00000[1]
21	− 0·00524
	161061755750279477635534762·0041

[1] These terms vanish identically.

NOTES ON LECTURE VIII

§ 8.1. Most of the material of this lecture is to be found in Hardy and Ramanujan, *Proc. London Math. Soc.* (2), 17 (1918), 75–115 (no. 36 of the *Papers*), and Rademacher (2).

§ 8.3. Hardy and Ramanujan (*l.c.* 285–287) prove the more precise inequalities

$$Hn^{-1}e^{2n^{\frac{1}{2}}} < p(n) < Kn^{-1}e^{2(2n)^{\frac{1}{2}}}$$

by elementary methods.

§ 8.4. The Tauberian theorem is proved, in a more general form, by Hardy and Ramanujan, *Proc. London Math. Soc.* (2), 16 (1917), 112–132 (no. 34 of the *Papers*). There are references at the end of this paper to similar theorems of Valiron and earlier writers.

The 'Tauberian' methods are naturally effective, so far as they go, over a wide range of problems. Thus we can prove that the number of partitions of n into primes is

$$\exp\left\{\frac{2\pi}{\sqrt{3}}\left(\frac{n}{\log n}\right)^{\frac{1}{2}}\right\},$$

with the same degree of accuracy as in (8.3.1) or (8.3.2).

§ 8.6. The functional equations for $F(x)$, required here and later (in particular in § 8.12), are the formulae for the linear transformation of the function $h(\tau)$ of Tannery and Molk. See Tannery and Molk, *Fonctions elliptiques*, ii, 264–267 (tables XLV–XLVI).

The formula (8.6.3) was found independently by Uspensky, *Bull. de l'acad. des sciences de l'URSS* (6), 14 (1920), 199–218. Uspensky's paper was published a little after ours, and we developed the solution much further, so that his proof of (8.6.3), which is simpler than ours, has been noticed less than it deserves.

§ 8.7. The abstract of no. 36 which appeared in 1917 in the *Comptes rendus* (no. 31 of the *Papers*) does not advance beyond this stage.

The idea of approximating to an arithmetical function by a 'singular series', in which each term corresponds to a 'rational point' on the circle of convergence of the generating series, is one of these which dominate the work of Hardy and Littlewood on Waring's problem.

Various expressions for $\omega_{p,q}$ have been given by Hermite, Tannery and Molk, Hardy and Ramanujan, and Rademacher (1). Rademacher's is

$$\omega_{p,q} = e^{\pi i s_{p,q}},$$

where
$$s_{p,q} = \frac{1}{q}\sum_{\mu=1}^{q-1}\mu\left(\frac{\mu p}{q} - \left[\frac{\mu p}{q}\right] - \frac{1}{2}\right).$$

D. H. Lehmer (4) has proved that $L_q(n)$ has a 'multiplicative' property which enables us to reduce its calculation to cases in which q is a prime or a power of a prime, and that in these cases it is the product of $q^{\frac{1}{2}}$, a power of 2, a symbol of quadratic residuality, and the cosine of a rational multiple of π.

§§ 8.8–9. MacMahon worked with the recurrence formula

$$p(n) - p(n-2) - p(n-3) + p(n-5) + \dots = 0$$

obtained by equating coefficients in the identity

$$(1 - x^2 - x^3 + x^5 + \dots)\,\Sigma p_n x^n = 1.$$

Gupta (1, 2) used additional devices.

Lehmer calculated $p(721)$, from the Hardy-Ramanujan formula, in his paper 1, but the result was first made decisive by Rademacher (2). Both Lehmer and Rademacher also calculated $p(599)$, as a test of Ramanujan's conjecture (§ 6.6) for modulus 5^4 (the result being again affirmative). In the meantime Gupta (2) had verified the value of $p(599)$ by direct computation.

In his paper 3 Lehmer finds the values of $p(n)$ for

$$n = 1224, 2052, 2474, 14031$$

and verifies their divisibility by

$$5^4, 11^3, 5^5, 11^4$$

respectively.

Rademacher and Zuckermann (**1**) and Zuckermann (**3**) have found identities for the coefficients in other modular functions. Some of these functions are of the same type as $F(x)$, while others are of a different type also considered by Hardy and Ramanujan [*Proc. Royal Soc.* (A), 95 (1919), 144–155 (no. 37 of the *Papers*)].

For the divergence of the Hardy-Ramanujan series see Lehmer (**2**, **4**, **5**). The first of these papers settles the question of divergence, while in the two latter he proves more precise results.

§8.10. For the relevant properties of Farey series see, for example, Hardy and Wright, ch. III, or Landau, *Vorlesungen*, i, 98–100.

§8.17. See Watson, *Bessel functions*, ch. VI, especially p. 176.

§8.18. For (8.18.1) see Lehmer (**4**), 292.

IX

THE REPRESENTATION OF NUMBERS AS SUMS OF SQUARES

9.1. The problem of the representation of an integer n as the sum of a given number k of integral squares is one of the most celebrated in the theory of numbers. Its history may be traced back to Diophantus, but begins effectively with Girard's (or Fermat's) theorem that a prime $4m+1$ is the sum of two squares. Almost every arithmetician of note since Fermat has contributed to the solution of the problem, and it has its puzzles for us still.

We denote the number of representations of n by k squares, i.e. the number of integral solutions of

$$x_1^2+x_2^2+\ldots+x_k^2 = n,$$

by $r_k(n)$. We are to pay attention to the sign and order of $x_1, x_2, \ldots, x_k$. Thus

$$1 = (\pm 1)^2+0^2 = 0^2+(\pm 1)^2, \quad 5 = (\pm 2)^2+(\pm 1)^2 = (\pm 1)^2+(\pm 2)^2,$$

and $$r_2(1) = 4, \quad r_2(5) = 8.$$

The problem is that of determining $r_k(n)$ in terms of simpler arithmetical functions of n, such as the number or the sum of its divisors. It is a good deal easier if k is an *even* number $2s$, and I shall suppose this throughout the lecture.

Jacobi solved the problem for $2s = 2$, 4, 6 and 8. Thus he proved that

$$(9.1.1) \qquad r_2(n) = 4 \sum_{d \text{ odd},\, d|n} (-1)^{\frac{1}{2}(d-1)} = 4\{d_1(n)-d_3(n)\};$$

and that

$$(9.1.2) \qquad r_4(n) = 8\sum_{d|n} d = 8\sigma(n),$$

or

$$(9.1.3) \qquad r_4(n) = 24 \sum_{d \text{ odd},\, d|n} d = 24\sigma^o(n),$$

according as n is odd or even. Here the sums extend over all divisors d of n, or all odd divisors; $d_1(n)$ and $d_3(n)$ are the numbers of the divisors of n of the forms $4m+1$ and $4m+3$ respectively; $\sigma(n)$ is the sum of the divisors of n, and $\sigma^o(n)$ the sum of its odd divisors. The formula for $r_6(n)$ is a little more complicated, but of the same general character, and that for $r_8(n)$ will occur later.

9.2. Jacobi found his formulae from the theory of the elliptic theta-functions. If

(9.2.1) $$\vartheta(x) = 1+2x+2x^4+\ldots = \sum_{-\infty}^{\infty} x^{n^2},$$

then

(9.2.2.) $$\vartheta^2(x) = 1+4\left(\frac{x}{1-x}-\frac{x^3}{1-x^3}+\frac{x^5}{1-x^5}-\frac{x^7}{1-x^7}+\ldots\right)$$

and

(9.2.3) $$\vartheta^4(x) = 1+8\left(\frac{x}{1-x}+\frac{2x^2}{1+x^2}+\frac{3x^3}{1-x^3}+\frac{4x^4}{1+x^4}+\ldots\right),$$

and (9.1.1), (9.1.2) and (9.1.3) are the results of equating coefficients. Thus the right-hand side of (9.2.2) is (ignoring the 1)

$$4\sum_{d\text{ odd}}(-1)^{\frac{1}{2}(d-1)}\frac{x^d}{1-x^d} = 4\sum_{d\text{ odd}}\sum_{i=1}^{\infty}(-1)^{\frac{1}{2}(d-1)}x^{id};$$

and the coefficient of x^n is

$$4\sum_{d\text{ odd},\, d\mid n}(-1)^{\frac{1}{2}(d-1)}.$$

In this way the formulae for $r_2(n)$ and $r_4(n)$ appear as corollaries of an analytical theory.

It is possible to reverse this procedure, to prove the arithmetical formulae directly and deduce the analytical identities; and here there is some difference between $r_2(n)$ and $r_4(n)$. It is easy to deduce (9.1.1) from the theory of the Gaussian complex integers, but (9.1.2) and (9.1.3) present greater difficulties. For this reason it is interesting to have an elementary deduction of (9.2.3) from (9.2.2), and Ramanujan found one. I repeat it here because, though it has very little bearing on the substance of my lecture, it is a very characteristic specimen of Ramanujan's work.

It is easily verified that

$$\frac{x}{1-x}+\frac{2x^2}{1+x^2}+\frac{3x^3}{1-x^3}+\frac{4x^4}{1+x^4}+\ldots$$
$$= \frac{x}{1-x}+\frac{2x^2}{1-x^2}+\frac{3x^3}{1-x^3}+\frac{5x^5}{1-x^5}+\ldots = \Sigma' m u_m,$$

where $$u_m = \frac{x^m}{1-x^m}$$

and the dash indicates the omission of multiples of 4. We have therefore to prove that

(9.2.4) $$(\tfrac{1}{4}+u_1-u_3+u_5-\ldots)^2 = \tfrac{1}{16}+\tfrac{1}{2}\Sigma' m u_m.$$

Ramanujan proves, more generally, that

(9.2.5) $$S^2 = T_1+T_2,$$

where $S = \frac{1}{4}\cot\frac{1}{2}\theta + u_1 \sin\theta + u_2 \sin 2\theta + \dots,$

$$T_1 = (\tfrac{1}{4}\cot\tfrac{1}{2}\theta)^2 + u_1(1+u_1)\cos\theta + u_2(1+u_2)\cos 2\theta + \dots,$$

$$T_2 = \tfrac{1}{2}\{u_1(1-\cos\theta) + 2u_2(1-\cos 2\theta) + 3u_3(1-\cos 3\theta) + \dots\}.$$

This reduces to (9.2.4) when $\theta = \frac{1}{2}\pi$. For then

$$S = \tfrac{1}{4} + u_1 - u_3 + u_5 - \dots,$$

$$T_1 = \tfrac{1}{16} + \sum_1^\infty (-1)^m u_{2m}(1+u_{2m})$$

$$= \tfrac{1}{16} + \sum_1^\infty (-1)^m \frac{x^{2m}}{(1-x^{2m})^2} = \tfrac{1}{16} + \sum_1^\infty (-1)^m \sum_1^\infty n x^{2mn}$$

$$= \tfrac{1}{16} - \sum_1^\infty \frac{n x^{2n}}{1+x^{2n}} = \tfrac{1}{16} - \sum_1^\infty \left(\frac{n x^{2n}}{1-x^{2n}} - \frac{2n x^{4n}}{1-x^{4n}}\right)$$

$$= \tfrac{1}{16} - u_2 - 3u_6 - 5u_{10} - \dots,$$

and $$T_2 = \tfrac{1}{2}(u_1 + 3u_3 + 5u_5 + \dots) + 2u_2 + 6u_6 + 10u_{10} + \dots,$$

so that $$T_1 + T_2 = \tfrac{1}{16} + \tfrac{1}{2}\Sigma' m u_m.$$

To prove (9.2.5), we write

$$S^2 = \left(\tfrac{1}{4}\cot\tfrac{1}{2}\theta + \sum_1^\infty u_m \sin m\theta\right)^2$$

$$= (\tfrac{1}{4}\cot\tfrac{1}{2}\theta)^2 + \tfrac{1}{2}\cot\tfrac{1}{2}\theta \sum_1^\infty u_m \sin m\theta + \left(\sum_1^\infty u_m \sin m\theta\right)^2$$

$$= (\tfrac{1}{4}\cot\tfrac{1}{2}\theta)^2 + S_1 + S_2,$$

say. We express S_1 and S_2 in the forms

$$S_1 = \sum_1^\infty \{\tfrac{1}{2} + \cos\theta + \cos 2\theta + \dots + \cos(m-1)\theta + \tfrac{1}{2}\cos m\theta\} u_m,$$

$$S_2 = \sum_1^\infty u_m \sin m\theta \sum_1^\infty u_n \sin n\theta = \tfrac{1}{2}\sum_1^\infty\sum_1^\infty \{\cos(m-n)\theta - \cos(m+n)\theta\} u_m u_n,$$

and rearrange $S_1 + S_2$ as

$$S_1 + S_2 = \sum_{k=0}^\infty C_k \cos k\theta.$$

(i) The contribution of S_1 to C_0 is $\frac{1}{2}\Sigma u_m$, and that of S_2 is $\frac{1}{2}\Sigma u_m^2$. Hence

$$C_0 = \tfrac{1}{2}\sum_1^\infty u_m(1+u_m) = \tfrac{1}{2}\sum_1^\infty \frac{x^m}{(1-x^m)^2} = \tfrac{1}{2}\sum_1^\infty\sum_1^\infty n x^{mn}$$

$$= \tfrac{1}{2}\sum_1^\infty \frac{n x^n}{1-x^n} = \tfrac{1}{2}\sum_1^\infty n u_n.$$

(ii) If $k > 0$, then the contribution of S_1 to C_k is

$$\tfrac{1}{2}u_k + \sum_{k+1}^\infty u_m = \tfrac{1}{2}u_k + \sum_{l=1}^\infty u_{k+l}.$$

That of S_2 is

$$\tfrac{1}{2}\sum_{m-n=k} u_m u_n + \tfrac{1}{2}\sum_{n-m=k} u_m u_n - \tfrac{1}{2}\sum_{m+n=k} u_m u_n = \sum_{l=1}^{\infty} u_l u_{k+l} - \tfrac{1}{2}\sum_{l=1}^{k-1} u_l u_{k-l}.$$

Hence
$$C_k = \tfrac{1}{2}u_k + \sum_{l=1}^{\infty} u_{k+l} + \sum_{l=1}^{\infty} u_l u_{k+l} - \tfrac{1}{2}\sum_{l=1}^{k-1} u_l u_{k-l}.$$

It is easily verified that

$$u_{k+l}(1+u_l) = u_k(u_l - u_{k+l}), \quad u_l u_{k-l} = u_k(1+u_l+u_{k-l}),$$

so that

$$C_k = u_k\left\{\tfrac{1}{2} + \sum_{l=1}^{\infty}(u_l - u_{k+l}) - \tfrac{1}{2}\sum_{l=1}^{k-1}(1+u_l+u_{k-l})\right\}$$
$$= u_k\{\tfrac{1}{2} - u_1 + u_2 + \dots + u_k - \tfrac{1}{2}(k-1) - (u_1+u_2+\dots+u_{k-1})\}$$
$$= u_k(1 - u_k - \tfrac{1}{2}k).$$

Hence finally

$$S^2 = (\tfrac{1}{4}\cot\tfrac{1}{2}\theta)^2 + \tfrac{1}{2}\sum_{n=1}^{\infty} nu_n + \sum_{k=1}^{\infty} u_k(1+u_k-\tfrac{1}{2}k)\cos k\theta$$
$$= (\tfrac{1}{4}\cot\tfrac{1}{2}\theta)^2 + \sum_{m=1}^{\infty} u_m(1+u_m)\cos m\theta + \tfrac{1}{2}\sum_{m=1}^{\infty} mu_m(1-\cos m\theta)$$
$$= T_1 + T_2.$$

The identity is equivalent to

$$\left\{\zeta(u) - \frac{\eta_1 u}{\omega_1}\right\}^2 - \wp(u) = \left(\frac{2\pi}{\omega_1}\right)^2\left\{-\frac{1}{24} + \sum_1^{\infty}\frac{q^{2m}}{(1-q^{2m})^2}\cos\frac{m\pi u}{\omega_1}\right\},$$

in the ordinary notation of elliptic functions.

9.3. Jacobi did not attempt to determine $r_{2s}(n)$ when $2s$ exceeds 8, and the first results in this direction, for $2s = 10$ and $2s = 12$, were found by Liouville and Eisenstein. In these cases $r_{2s}(n)$ is not usually expressible as a simple "divisor-function" of n. For example, in Glaisher's notation,

$$r_{10}(n) = \tfrac{4}{5}\{E_4(n) + 16E_4'(n) + 8\chi_4(n)\},$$

where
$$E_4(n) = \sum_{d\ \mathrm{odd},\ d|n} (-1)^{\frac{1}{2}(d-1)} d^4,$$
$$E_4'(n) = \sum_{d'\ \mathrm{odd},\ d|n} (-1)^{\frac{1}{2}(d'-1)} d^4$$

(d' being n/d, the divisor of n "conjugate" to d), and

$$\chi_4(n) = \tfrac{1}{4}\sum_{a^2+b^2=n} (a+bi)^4$$

(a sum extended over the Gaussian complex divisors of n). There are similar formulae for $2s = 12, 14, \dots$, $r_{2s}(n)$ being in each case the sum of a "divisor-function" and one or more supplementary functions. The complexity of these supplementary functions increases with s, and it is only the simplest of them, such as $\chi_4(n)$, which can be defined in an illuminating arithmetical way. The majority can only be recognised as coefficients in the expansions of modular functions.

We shall find, however, that there is always a divisor-function which "dominates" $r_{2s}(n)$. In all cases

$$r_{2s}(n) = \delta_{2s}(n) + e_{2s}(n),$$

where $\delta_{2s}(n)$ is a divisor-function and $e_{2s}(n)$ is much smaller than $\delta_{2s}(n)$ for large n, so that

$$r_{2s}(n) \sim \delta_{2s}(n)$$

when n tends to infinity.

9.4. I propose to work out two special cases in detail here, $2s = 8$ and $2s = 24$. The analysis is a little simpler than usual when $2s$ is a multiple of 8, but these two cases are quite typical of the general theory. In the first case I shall obtain the classical formulae of Jacobi, in the second the most characteristic of Ramanujan's new theorems. I do not think that any complete proof of this has been published before.

I shall not use Jacobi's or Ramanujan's methods, but the later "function-theoretic" method devised by Mordell and myself, which shows up the foundations of the theory much more clearly. I must however begin with a few general remarks about Ramanujan's contributions to the subject, which are set out in two substantial papers in the *Transactions of the Cambridge Philosophical Society*.[1]

It is always difficult to say how much Ramanujan owed to other writers, and the difficulty is at its maximum when he is developing work which he began before he came to England, and when he is concerned, as he is here, with some side of the theory of elliptic functions. There is no book on elliptic functions which he could have seen in India and which says anything about the arithmetical applications of the theory. I believe therefore that Ramanujan had rediscovered Jacobi's formulae, which certainly lay well within his powers.

By the time he published these papers Ramanujan had read a great deal more, and knew all about Jacobi's and much later work. In particular he had read Glaisher's papers, and treats their content as known; and a reader who takes his acknowledgements at their face value will be liable to underestimate his originality. The papers are highly original, whatever deductions we may make: they are characteristic of Ramanujan at his best. They contain many remarkable theorems which are undeniably new, and conjectures still more remarkable, which were confirmed later by Mordell; and

[1] Nos. 18 and 21 of the *Papers*.

the general level of the analysis is astonishingly high. In particular the second paper contains all the formal theory of "Ramanujan's sum", which is fundamental for my purposes here. I shall have to start from formulae of Ramanujan even when I am developing the theory on lines entirely different from his.

Ramanujan's sum $c_q(n)$

9.5. Ramanujan's sum is

$$c_q(n) = \sum_{p(q)} e^{-2np\pi i/q},$$

where the notation indicates a sum over values of p less than and prime to q.[1] It may also be written as

$$c_q(n) = \sum_{p(q)} \cos \frac{2np\pi}{q}$$

(a form which shows that it is real), or again as

$$c_q(n) = \Sigma \rho_q^n,$$

where ρ_q is a primitive q-th root of 1.

It is easy to evaluate $c_q(n)$ in terms of the Möbius function $\mu(n)$. This function is defined by

(i) $$\mu(1) = 1,$$

(ii) $$\mu(n) = (-1)^\nu$$

if $n = p_1 p_2 \ldots p_\nu$ is the product of ν different prime factors, and

(iii) $$\mu(n) = 0$$

if n has a repeated factor.[2] The chief properties of $\mu(n)$ are (*a*) that

$$\sum_{d|q} \mu(d) = 0$$

for all $q > 1$, and (*b*) that the identities

(9.5.1) $$g(q) = \sum_{d|q} f(d)$$

and

(9.5.2) $$f(q) = \sum_{d|q} \mu\left(\frac{q}{d}\right) g(d)$$

are equivalent to one another. The last theorem is usually described as the "Möbius inversion formula".

Ramanujan proved that

(9.5.3) $$c_q(n) = \sum_{d|q,\, d|n} \mu\left(\frac{q}{d}\right) d$$

[1] Or any complete system of residues prime to q.

[2] $\mu(n)$ has occurred already in Lectures II and IV. See p. 23, footnote.

(a sum extended over all common divisors of q and n). To prove this he observes that

$$\eta_q(n) = \sum_{h=0}^{q-1} e^{-2nh\pi i/q}$$

is q if $q \mid n$ and 0 otherwise. But it is plain that

$$\eta_q(n) = \sum_{d \mid q} c_d(n),$$

and therefore, by the Möbius formula,

$$c_q(n) = \sum_{d \mid q} \mu\left(\frac{q}{d}\right) \eta_d(n),$$

which is (9.5.3).

There is another proof which is a little longer but depends on principles which will be useful later. We say that $f(q)$ is *multiplicative* if

$$f(qq') = f(q)\,f(q') \tag{9.5.4}$$

whenever $(q, q') = 1$. In particular this involves $f(1) = 1$. It is plain that, if we want to prove that

$$f(q) = F(q), \tag{9.5.5}$$

and know both $f(q)$ and $F(q)$ to be multiplicative, then it is enough to prove the result when q is a power of a prime.

Now, if $(q, q') = 1$, we have

$$c_q(n)\,c_{q'}(n) = \sum_{p(q)} e^{-2np\pi i/q} \sum_{p'(q')} e^{-2np'\pi i/q'}$$
$$= \sum_{p(q),\,p'(q')} e^{-2nP\pi i/qq'},$$

where $$P = pq' + p'q;$$

and P runs over the range $P(qq')$ when p and p' run over $p(q)$ and $p'(q')$. Hence

$$\sum_{p(q),\,p'(q')} e^{-2nP\pi i/qq'} = \sum_{P(qq')} e^{-2nP\pi i/qq'} = c_{qq'}(n),$$

and Ramanujan's sum is multiplicative.

Again, if we denote the right-hand side of (9.5.3) by $C_q(n)$, we have

$$C_q(n)\,C_{q'}(n) = \Sigma\mu\left(\frac{q}{d}\right)\mu\left(\frac{q'}{d'}\right)dd' = \Sigma\mu\left(\frac{qq'}{dd'}\right)dd'. \tag{9.5.6}$$

Here $d \mid q$, $d \mid n$, $d' \mid q'$, $d' \mid n$, and these relations are equivalent to

$$dd' \mid qq', \quad dd' \mid n$$

(since q and q' are coprime). Hence the right-hand side of (9.5.6) is $C_{qq'}(n)$, and $C_q(n)$ also is multiplicative.

We have therefore only to prove that

$$c_{\varpi^k}(n) = C_{\varpi^k}(n)$$

when ϖ is prime. The values of p in $p(\varpi^k)$ are

$$p = \varpi^{k-1}z + p_1,$$

where $z = 0, 1, 2, \ldots, \varpi - 1$ and p_1 runs through $p_1(\varpi^{k-1})$. Hence

$$c_{\varpi^k}(n) = \sum_{p_1(\varpi^{k-1})} e^{-2np_1\pi i/\varpi^k} \sum_{z=0}^{\varpi-1} e^{-2nz\pi i/\varpi},$$

and the inner sum is ϖ when $\varpi \mid n$ and zero otherwise. It follows that

(9.5.7) $$c_{\varpi^k}(n) = \varpi \sum_{p_1(\varpi^{k-1})} e^{-2n_1p_1\pi i/\varpi^{k-1}} = \varpi c_{\varpi^{k-1}}(n_1)$$

if $\varpi \mid n$ and $n = \varpi n_1$; and that $c_{\varpi^k}(n) = 0$ otherwise.

Now $$c_\varpi(n) = \sum_1^{\varpi-1} e^{-2np\pi i/\varpi},$$

so that

(9.5.8) $$c_\varpi(n) = -1 \quad (\varpi \nmid n), \quad c_\varpi(n) = \varpi - 1 \quad (\varpi \mid n);$$

and we can now calculate $c_{\varpi^k}(n)$ by (9.5.7) for every k. We find that

(9.5.9) $$c_{\varpi^2}(n) = 0 \quad (\varpi \nmid n), \quad -\varpi \quad (\varpi \mid n, \varpi^2 \nmid n), \quad \varpi(\varpi - 1) \quad (\varpi^2 \mid n),$$

and generally

(9.5.10)

$$c_{\varpi^k}(n) = 0 \quad (\varpi^{k-1} \nmid n), \quad -\varpi^{k-1} \quad (\varpi^{k-1} \mid n, \varpi^k \nmid n), \quad \varpi^{k-1}(\varpi - 1) \quad (\varpi^k \mid n);$$

and we can verify at once that these are also the values of $C_{\varpi^k}(n)$. Thus $c_q(n) = C_q(n)$ whenever $q = \varpi^k$, and therefore for all q.

The series $\Sigma q^{-s}c_q(n)$

9.6. Ramanujan summed a large number of series of the form $\Sigma a_q c_q(n)$. The simplest is

(9.6.1) $$U(n) = \sum_{q=1}^{\infty} \frac{c_q(n)}{q^s}.$$

There are a number of interesting ways of summing this series. It is absolutely convergent when $s > 1$, since $|c_q(n)| \leqq d(n)$ for every q.

(i) The shortest way is due to Estermann. We can write $c_q(n)$ in the form

$$c_q(n) = \sum_{lm=q, m \mid n} \mu(l)\, m,$$

so that $$\frac{c_q(n)}{q^s} = \sum_{lm=q, m \mid n} \mu(l)\, l^{-s} m^{1-s}.$$

When we sum with respect to q we remove the restriction on l, which now assumes all positive integral values; and so

$$(9.6.2)\qquad U(n) = \sum_{l,m\mid n} \mu(l)\, l^{-s} m^{1-s}$$

$$= \sum_{m\mid n} m^{1-s} \sum_{l=1}^{\infty} \frac{\mu(l)}{l^s} = \frac{\sigma_{1-s}(n)}{\zeta(s)} = \frac{n^{1-s}\sigma_{s-1}(n)}{\zeta(s)},$$

where $\sigma_\nu(n)$ is the sum of the ν-th powers of the divisors of n.

(ii) Ramanujan argued as follows. We suppose that $F(x,y)$ is any function of the two variables x and y, and that

$$D(n) = \sum_{d\mid n} F\left(d, \frac{n}{d}\right);$$

and define $\eta_\nu(n)$ as in § 9.5, so that $\eta_\nu(n)$ is ν or 0 according as ν is or is not a divisor of n. Then

$$D(n) = \sum_{\nu=1}^{t} \frac{\eta_\nu(n)}{\nu} F\left(\nu, \frac{n}{\nu}\right)$$

for any value of t not less than n; and so

$$D(n) = \sum_{\nu=1}^{t} \frac{1}{\nu} F\left(\nu, \frac{n}{\nu}\right) \sum_{d\mid \nu} c_d(n).$$

Now $c_j(n)$ occurs in this series when $j \mid \nu$, or $\nu = j\mu$, in which case $\mu \leqq t/j$; and therefore

$$(9.6.3.)\quad D(n) = c_1(n) \sum_{1}^{t} \frac{1}{\mu} F\left(\mu, \frac{n}{\mu}\right)$$

$$+ c_2(n) \sum_{1}^{\frac{1}{2}t} \frac{1}{2\mu} F\left(2\mu, \frac{n}{2\mu}\right) + c_3(n) \sum_{1}^{\frac{1}{3}t} \frac{1}{3\mu} F\left(3\mu, \frac{n}{3\mu}\right) + \dots.$$

Suppose in particular that $F(x,y) = x^{1-s}$. Then

$$D(n) = \sum_{d\mid n} d^{1-s} = \sigma_{1-s}(n)$$

and
$$\sigma_{1-s}(n) = \frac{c_1(n)}{1^s} \sum_{1}^{t} \mu^{-s} + \frac{c_2(n)}{2^s} \sum_{1}^{\frac{1}{2}t} \mu^{-s} + \frac{c_3(n)}{3^s} \sum_{1}^{\frac{1}{3}t} \mu^{-s} + \dots.$$

Finally, if $s > 1$, and we make $t \to \infty$, we obtain (9.6.2).

(iii) A third proof proceeds on lines like those of the second proof of § 9.5. We start from the identity

$$\sum_{q=1}^{\infty} \frac{f(q)}{q^s} = \prod_{\varpi} \left\{1 + \frac{f(\varpi)}{\varpi^s} + \frac{f(\varpi^2)}{\varpi^{2s}} + \dots\right\} = \prod_{\varpi} \chi_\varpi,$$

where $f(q)$ is any multiplicative function and ϖ runs through the primes. The identity reduces to "Euler's product" when $f(q) = 1$.

In this case $$\chi_\varpi = 1 + \frac{c_\varpi(n)}{\varpi^s} + \frac{c_{\varpi^2}(n)}{\varpi^{2s}} + \dots,$$

and we can quote the formulae (9.5.10). Suppose that ϖ^a is the highest power of ϖ which divides n. Then

$$\chi_\varpi = 1 - \varpi^{-s}$$

if $a = 0$, and generally

$$\chi_\varpi = 1 + \frac{\varpi - 1}{\varpi^s} + \frac{\varpi(\varpi-1)}{\varpi^{2s}} + \dots + \frac{\varpi^{a-1}(\varpi-1)}{\varpi^{as}} - \frac{\varpi^a}{\varpi^{(a+1)s}}$$

$$= (1 - \varpi^{-s})\frac{1 - \varpi^{(a+1)(1-s)}}{1 - \varpi^{1-s}}.$$

Hence $$\sum_q \frac{c_q(n)}{q^s} = \prod_\varpi (1 - \varpi^{-s}) \prod_{\varpi \mid n} \frac{1 - \varpi^{(a+1)(1-s)}}{1 - \varpi^{1-s}} = \frac{\sigma_{1-s}(n)}{\zeta(s)},$$

by the ordinary formula for a sum of powers of divisors of n.

In particular, when $s = 2$, we obtain the formula

$$\sigma(n) = \tfrac{1}{6}\pi^2 n\left\{1 + \frac{(-1)^n}{2^2} + \frac{2\cos\frac{2}{3}n\pi}{3^2} + \frac{2\cos\frac{1}{2}n\pi}{4^2} + \frac{2(\cos\frac{2}{5}n\pi + \cos\frac{4}{5}n\pi)}{5^2} + \dots\right\}$$

for the sum of the divisors of n. The formula shows in a very striking way the oscillations of $\sigma(n)$ about its "average" $\frac{1}{6}\pi^2 n$.

We have supposed that $s > 1$, when our series are absolutely convergent. We can write (9.6.2) as

(9.6.4) $$\sum_q \frac{c_q(n)}{q^s} = \sum_{d \mid n} \frac{d}{d^s} \sum_m \frac{\mu(m)}{m^s}.$$

The first factor on the right is a finite, and therefore absolutely convergent, Dirichlet's series, and the second a Dirichlet's series convergent for $s \geqq 1$; and therefore, by a familiar theorem on the multiplication of Dirichlet's series, the series on the left is convergent, and (9.6.4) is true, for $s \geqq 1$. Putting $s = 1$, we obtain

$$c_1(n) + \frac{c_2(n)}{2} + \frac{c_3(n)}{3} + \dots = 0$$

(a theorem of the same depth as the prime number theorem).

Ramanujan made a number of similar summations, among which

$$c_1(n)\log 1 + \frac{c_2(n)}{2}\log 2 + \frac{c_3(n)}{3}\log 3 + \dots = -d(n)$$

and $$\pi\left\{c_1(n) - \frac{c_3(n)}{3} + \frac{c_5(n)}{5} - \dots\right\} = r_2(n)$$

are two of the most striking.

The series $\Sigma \epsilon_q q^{-s} c_q(n)$

9.7. The series which is important for our present purpose is not (9.6.1) but

(9.7.1) $$V(n) = \sum_{q=1}^{\infty} \epsilon_q \frac{c_q(n)}{q^s},$$

where $$\epsilon_q = 1 \;\; (q \equiv 1, 3), \quad 0 \;\; (q \equiv 2), \quad 2^s \;\; (q \equiv 0),$$

the congruences being to modulus 4. This series may be summed by any of the methods of § 9.6; I select the first as the shortest. I define $\sigma_\nu^*(n)$ by

$$\sigma_\nu^*(n) = \sigma_\nu(n) \quad (n \text{ odd}),$$

$$\sigma_\nu^*(n) = \sigma_\nu^e(n) - \sigma_\nu^o(n) \quad (n \text{ even}),$$

$\sigma_\nu^e(n)$ and $\sigma_\nu^o(n)$ being the sums of the ν-th powers of the even and odd divisors of n; and I prove that

(9.7.2) $$V(n) = \frac{n^{1-s}}{(1-2^{1-s})\,\zeta(s)} \sigma_{s-1}^*(n).$$ [1]

We have

$$V(n) = \sum_{q=1,3,\ldots} q^{-s} c_q(n) + 2^s \sum_{q=4,8,\ldots} q^{-s} c_q(n) = V_1(n) + V_2(n),$$

say. Here, first,

$$V_1(n) = \sum_{q=1,3,\ldots} (lm)^{-s} \sum_{lm=q,\, m|n} \mu(l)\, m$$

$$= \sum_{lm=1,3,\ldots,\, m|n} \mu(l)\, l^{-s} m^{1-s} = \sigma_{1-s}^o(n) \sum_{l=1,3,\ldots} \frac{\mu(l)}{l^s}$$

$$= \sigma_{1-s}^o(n) \prod_{\varpi > 2} \left(1 - \frac{1}{\varpi^s}\right) = \frac{\sigma_{1-s}^o(n)}{(1-2^{-s})\,\zeta(s)}.$$

Next,

$$V_2(n) = 2^s \sum_{q=4,8,\ldots} q^{-s} \sum_{lm=q,\, m|n} \mu(l)\, m = 2^s \sum_{lm=4,8,\ldots} \mu(l)\, l^{-s} m^{1-s}.$$

If $4 \mid l$, then $\mu(l) = 0$. Hence we may suppose that either (i) l is odd and $4 \mid m$ or (ii) $l = 2l_1$, where l_1 is odd, and $2 \mid m$. The terms of type (i) give

$$2^s \sum_{l=1,3,\ldots} \frac{\mu(l)}{l^s} \sum_{4|m,\, m|n} m^{1-s} = \frac{2^s \sigma_{1-s}^{ee}(n)}{(1-2^{-s})\,\zeta(s)},$$

where the double index indicates a sum over doubly even divisors; and the terms of type (ii) give

$$-\sum_{l_1=1,3,\ldots} \frac{\mu(l_1)}{l_1^s} \sum_{2|m,\, m|n} m^{1-s} = -\frac{\sigma_{1-s}^e(n)}{(1-2^{-s})\,\zeta(s)}.$$

[1] We have not now two alternative forms like those in (9.6.2), since

$$n^{1-s}\sigma_{s-1}^*(n) \neq \sigma_{1-s}^*(n)$$

when n is even.

Collecting our results, we find that

$$(1-2^{-s})\,\zeta(s)\,V(n) = \sigma^{o}_{1-s}(n) - \sigma^{e}_{1-s}(n) + 2^s\sigma^{ee}_{1-s}(n);$$

and we have only to verify that

$$(9.7.3) \qquad \sigma^{o}_{1-s}(n) - \sigma^{e}_{1-s}(n) + 2^s\sigma^{ee}_{1-s}(n) = n^{1-s}\sigma^{*}_{s-1}(n).$$

This is obvious if n is odd. If $n = 2N$, where N is odd, and δ runs through the divisors of N, then the left-hand side of (9.7.3) is

$$\Sigma\delta^{1-s} - \Sigma(2\delta)^{1-s} = (1-2^{1-s})\,\Sigma\delta^{1-s} = (1-2^{1-s})\,N^{1-s}\,\Sigma\delta^{s-1}$$
$$= (2^{s-1}-1)\,n^{1-s}\,\Sigma\delta^{s-1} = n^{1-s}\{\Sigma(2\delta)^{s-1} - \Sigma\delta^{s-1}\} = n^{1-s}\sigma^{*}_{s-1}(n).$$

Finally, if $n = 2^{\alpha}N$, where $\alpha > 1$, then it is

$$\Sigma\delta^{1-s} - \{2^{1-s} + 4^{1-s} + \ldots + 2^{\alpha(1-s)}\}\,\Sigma\delta^{1-s} + 2^s\{4^{1-s} + 8^{1-s} + \ldots + 2^{\alpha(1-s)}\}\,\Sigma\delta^{1-s}$$
$$= \{1 + 2^{1-s} + 4^{1-s} + \ldots + 2^{(\alpha-1)(1-s)} - 2^{\alpha(1-s)}\}\,\Sigma\delta^{1-s}$$
$$= N^{1-s}\{1 + 2^{1-s} + 4^{1-s} + \ldots + 2^{(\alpha-1)(1-s)} - 2^{\alpha(1-s)}\}\,\Sigma\delta^{s-1}$$
$$= n^{1-s}\{2^{\alpha(s-1)} + 2^{(\alpha-1)(s-1)} + \ldots + 2^{s-1} - 1\}\,\Sigma\delta^{s-1}$$
$$= n^{1-s}\{\sigma^{e}_{s-1}(n) - \sigma^{o}_{s-1}(n)\} = n^{1-s}\sigma^{*}_{s-1}(n).$$

This completes the proof of (9.7.2).

The singular series in the problem of 2s squares

9.8. I must now introduce ideas which are not to be found (at any rate explicitly) in Ramanujan's work.[1] They are the ideas from which Littlewood and I started in our work on Waring's problem.

It is easy to find asymptotic formulae for the behaviour of

$$f(s) = \vartheta^{2s}(x) = (1 + 2x + 2x^4 + \ldots)^{2s},$$

where x tends radially to a "rational point" $e^{2p\pi i/q}$ on the unit circle. We may suppose that $q = 1$, $p = 0$ or that $q > 1$, $0 < p < q$ and $(p, q) = 1$. If

$$x = re^{2p\pi i/q}$$

and $r \to 1$, then

$$\vartheta(x) = 1 + 2\sum_{1}^{\infty} r^{n^2} e^{2n^2 p\pi i/q}$$
$$= 1 + 2\sum_{j=1}^{q}\sum_{l=0}^{\infty} r^{(lq+j)^2}\, e^{2(lq+j)^2 p\pi i/q}$$
$$= 1 + 2\sum_{j=1}^{q} e^{2j^2 p\pi i/q} \sum_{l=0}^{\infty} r^{(lq+j)^2}.$$

If $r = e^{-\delta}$, so that $\delta \to 0$, then

$$\sum_{l=0}^{\infty} r^{(lq+j)^2} = \sum_{l=0}^{\infty} e^{-\delta(lq+j)^2} \sim \int_0^{\infty} e^{-\delta(xq+j)^2}\,dx$$
$$\sim \int_0^{\infty} e^{-\delta x^2 q^2}\,dx = \frac{1}{2q}\sqrt{\left(\frac{\pi}{\delta}\right)} = \frac{\sqrt{\pi}}{2q}\left(\log\frac{1}{r}\right)^{-\frac{1}{2}};$$

[1] Except in our joint work on partitions.

and hence

(9.8.1) $$\vartheta(x) \sim \frac{\sqrt{\pi}}{q} S_{p,q}\left(\log\frac{1}{r}\right)^{-\frac{1}{2}},$$

where

(9.8.2) $$S_{p,q} = \sum_{j=1}^{q} e^{2j^2 p\pi i/q}$$

is one of "Gauss's sums". If $S_{p,q} = 0$, as happens when $q \equiv 2 \pmod 4$, then (9.8.1) is to be interpreted as

$$\vartheta(x) = o\left\{\left(\log\frac{1}{r}\right)^{-\frac{1}{2}}\right\}$$

It follows that

(9.8.3) $$f(x) \sim \pi^{-s}\left(\frac{S_{p,q}}{q}\right)^{2s}\left(\log\frac{1}{r}\right)^{-s}$$

We now form an auxiliary function which mimics the behaviour of $f(x)$ when x approaches the unit circle in this way. It is known that if

$$F_s(x) = \sum_1^{\infty} n^{s-1} x^n,$$

then $$F_s(x) - \Gamma(s)\left(\log\frac{1}{x}\right)^{-s}$$

is regular at $x = 1$. Hence, if

$$f_{p,q}(x) = \frac{\pi^s}{\Gamma(s)}\left(\frac{S_{p,q}}{q}\right)^{2s} F_s(xe^{-2p\pi i/q}),$$

then $$f_{p,q}(x) \sim \pi^s\left(\frac{S_{p,q}}{q}\right)^{2s}\left(\log\frac{1}{r}\right)^{-s} \sim f(x)$$

when x tends to $e^{2p\pi i/q}$. Thus $f_{p,q}(x)$ "mimics" $f(x)$ near this point; and, if we write

$$\Theta_{2s}(x) = 1 + \sum_{p,q} f_{p,q}(x),$$

then we may expect that $\Theta_{2s}(x)$ will mimic $f(x)$ near *all* rational points $e^{2p\pi i/q}$. To say this is to say that $\Theta_{2s}(x)$ mimics $f(x)$ very comprehensively, so comprehensively that there should be a very close relation between the coefficients of the two functions. If this be so, then the way will be open at any rate to an approximate determination of $r_{2s}(n)$.

Now $$\Theta_{2s}(x) = 1 + \frac{\pi^s}{\Gamma(s)} \sum_{p,q}\left(\frac{S_{p,q}}{q}\right)^{2s} \sum_{n=1}^{\infty} n^{s-1} e^{-2np\pi i/q} x^n$$

$$= 1 + \sum_{n=1}^{\infty} \rho_{2s}(n)\, x^n,$$

where $$\rho_{2s}(n) = \frac{\pi^s}{\Gamma(s)} n^{s-1} \sum_{p,q}\left(\frac{S_{p,q}}{q}\right)^{2s} e^{-2np\pi i/q}.$$

We can write this as

$$(9.8.4) \qquad \rho_{2s}(n) = \frac{\pi^s}{\Gamma(s)} n^{s-1} \sum_{q=1}^{\infty} A_q(n),$$

where $A_1(n) = 1$ and

$$(9.8.5) \qquad A_q(n) = q^{-2s} \sum_{p(q)} S_{p,q}^{2s} e^{-2np\pi i/q}$$

when $q > 1$. We are entitled to expect a pretty close relation between $r_{2s}(n)$ and $\rho_{2s}(n)$. It is only an expectation, since our analysis has been entirely "heuristic"; but it is plainly one worth pursuing.

We call (9.8.4) the *singular series*. Our construction is typical of that of the "singular series" in the general Waring problem.

Summation of the singular series when $2s \equiv 0 \pmod 8$

9.9. The singular series can be summed for all s. The analysis is simplest when $2s \equiv 0 \pmod 8$, as I shall suppose.

The Gaussian sum $S_{p,q}$ (or simply S_q, if we omit the explicit reference to p) can be calculated, for all p, q, from the formulae

$$S_{p,qq'} = S_{pq',q} S_{pq,q'},$$

$$S_1 = 1, \quad S_2 = 0, \quad S_{2^{2\mu}} = 2^{\mu}(1 + i^p), \quad S_{2^{2\mu+1}} = 2^{\mu+1} e^{\frac{1}{4}p\pi i},$$

$$S_{\varpi} = \left(\frac{p}{\varpi}\right) i^{\frac{1}{4}(\varpi-1)^2} \sqrt{\varpi}, \quad S_{\varpi^{2\mu}} = \varpi^{\mu}, \quad S_{\varpi^{2\mu+1}} = \varpi^{\mu} S_{\varpi}.$$

Here q and q' are coprime, and ϖ is an odd prime. The formulae for the 8-th powers become much simpler, and we find that

$$S_{p,q}^{2s} = \epsilon_q q^s$$

when $2s \equiv 0 \pmod 8$, ϵ_q being the symbol of § 9.7. It follows that

$$A_q = \epsilon_q q^{-s} \sum_{p(q)} e^{-2np\pi i/q} = \epsilon_q q^{-s} c_q(n),$$

and that, in the notation of § 9.7,

$$(9.9.1) \qquad \rho_{2s}(n) = \frac{\pi^s n^{s-1}}{\Gamma(s)} V(n).$$

Thus the singular series is (apart from the outside factor) Ramanujan's series (9.7.1), and so

$$(9.9.2) \qquad \rho_{2s}(n) = \frac{\pi^s}{\Gamma(s)(1-2^{-s})\zeta(s)} \sigma_{s-1}^{*}(n).$$

In particular, if $2s = 8$, then

$$(1 - 2^{-s})\zeta(s) = \frac{\pi^4}{96},$$

$$(9.9.3) \qquad \rho_8(n) = 16\sigma_3^{*}(n).$$

If $2s = 24$, then

$$(1-2^{-12})\zeta(12) = (1-2^{-12})\,2^{11}\pi^{12}\frac{B_6}{12!}, \quad B_6 = \frac{691}{2730},$$

and

(9.9.4) $$\rho_{24}(n) = \tfrac{16}{691}\sigma_{11}^{*}(n).$$

The modular functions

9.10. We have strong reasons for expecting to find a fairly close resemblance between $r_{2s}(n)$ and $\rho_{2s}(n)$, and further analysis does more than would be required to justify our hopes, the correspondence being extremely close. Indeed when $2s \leqq 8$ the two functions are identical; in particular

$$r_8(n) = \rho_8(n).$$

The proof of this depends on arguments of a quite different character, first applied to this problem by Mordell.

9.11. We need the elements of the theory of the *modular group* and the functions associated with it. The modular group Γ has two forms. In the homogeneous form it is defined as the group of substitutions

(9.11.1) $$\omega_1' = a\omega_1 + b\omega_2, \quad \omega_2' = c\omega_1 + d\omega_2,$$

where a, b, c, d are integers and

(9.11.2) $$ad - bc = 1.$$

In the non-homogeneous form we write

(9.11.3) $$\tau = \frac{\omega_2}{\omega_1},$$

and the group is defined by the substitutions

$$\tau' = \frac{c+d\tau}{a+b\tau}.$$

We shall use any of the symbols

$$S, \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad \left(\frac{c+d\tau}{a+b\tau}\right)$$

to denote the substitution (9.11.1) or (9.11.3). Γ is generated by the repeated application of the two substitutions

$$\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \quad \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

or

(9.11.4) $$\tau' = \tau + 1, \quad \tau' = -\frac{1}{\tau}.$$

In what follows we shall always suppose that

(9.11.5) $$\mathfrak{J}(\tau) > 0,$$

so that

(9.11.6) $$|x| = |e^{\pi i\tau}| < 1.$$[1]

If $\tau = u+iv$, $\tau' = u'+iv'$, then

$$v' = \frac{(ad-bc)v}{(a+bu)^2+b^2v^2} > 0,$$

so that one point in the upper half-plane is transformed into another, and only such points are relevant.

We call the region D defined by

$$-\tfrac{1}{2} < u < \tfrac{1}{2}, \quad u^2+v^2 > 1$$

the *fundamental region* of Γ. Each substitution of Γ transforms D into a curvilinear triangle, whose sides are circles and whose angles are $(\frac{1}{3}\pi, \frac{1}{3}\pi, 0)$. These triangles cover up the half-plane without overlapping.

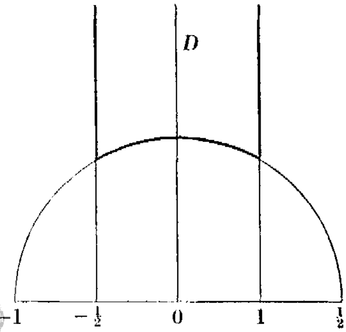


Fig. 3.

The fundamental function associated with the modular group is Klein's "absolute invariant" $J(\tau)$, which is defined as follows. We write

$$g_2 = g_2(\omega_1, \omega_2) = \frac{1}{12}\left(\frac{\pi}{\omega_1}\right)^4 \left\{1+240\left(\frac{1^3x^2}{1-x^2}+\frac{2^3x^4}{1-x^4}+\dots\right)\right\},$$

$$g_3 = g_3(\omega_1, \omega_2) = \frac{1}{216}\left(\frac{\pi}{\omega_1}\right)^6 \left\{1-504\left(\frac{1^5x^2}{1-x^2}+\frac{2^5x^4}{1-x^4}+\dots\right)\right\}$$

(these being the ordinary invariants of the Weierstrassian theory),

$$\Delta = \Delta(\omega_1, \omega_2) = g_2^3 - 27g_3^2 = \left(\frac{\pi}{\omega_1}\right)^{12} x^2\{(1-x^2)(1-x^4)\dots\}^{24},$$

and $$J(\tau) = \frac{g_2^3}{\Delta}.$$

Then $J(\tau)$ is a function of τ only, invariant for the substitutions of Γ. It assumes every value just once in D (when proper conventions have been laid down about the boundary of D), and represents D conformally on the whole complex plane (regarded as bounded by the points 0, 1, and ∞).

We call a function which, like $J(\tau)$, is invariant for Γ a *modular invariant*: the phrase "modular function" is used more vaguely. The function

$$J = J(\tau)$$

plays, for modular invariants, the part played by z in the ordinary theory of one-valued functions $f(z)$. Thus a modular invariant, with properly restricted singularities, is a one-valued function of J. In particular, *if a modular*

[1] I use x instead of the usual q, as in Lecture VIII, q being required for other purposes.

invariant is regular and bounded in D, then it is a one-valued function of J bounded in the whole plane of J, and accordingly *it is constant.*

Functions associated with the sub-group Γ_3

9.12. We shall be concerned now with functions which are not modular invariants in the full sense just defined, but which are invariant, or "all but" invariant, for the substitutions of a certain sub-group of Γ.

It is easily verified that the substitutions of Γ which satisfy the congruence conditions[1]

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \pmod 2$$

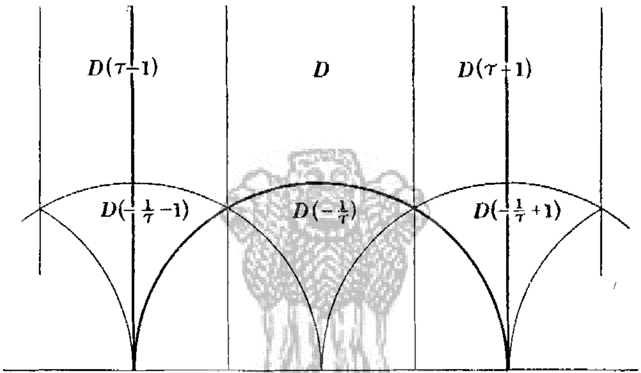


Fig. 4.

form a group, a sub-group of Γ which we call Γ_3. Γ_3 is generated by

(9.12.1) $$\tau' = \tau + 2, \quad \tau' = -\frac{1}{\tau}.$$

It has a "fundamental region" D_3 defined by

$$-1 < u < 1, \quad u^2 + v^2 > 1;$$

and the substitutions of Γ_3 transform D_3 into a system of triangles, all of whose angles are 0, which just fill up the half-plane.[2]

There is a principal invariant $J_3(\tau)$ of Γ_3, which is related to Γ_3 as $J(\tau)$ is to Γ (but whose expression we shall not require). A one-valued function, invariant for Γ_3, is a one-valued function of J_3 and a three-valued function of J. Finally *a function invariant for Γ_3, and regular and bounded in D_3, is constant.*

[1] Either a and d are odd and b and c even, or conversely.

[2] D_3 may be described roughly as formed by fitting together three regions congruent to D (i.e. transforms of D for substitutions of Γ). More strictly, it is formed by D and one-half of each of the four regions

$$D(\tau+1), \quad D(\tau-1), \quad D\left(-\frac{1}{\tau}+1\right), \quad D\left(-\frac{1}{\tau}-1\right)$$

(the transforms of D by $\tau' = \tau + 1$, etc.). In Fig. 4, D_3 is bounded by the thicker lines.

Proof that $r_8(n) = \rho_8(n)$

9.13. The functions

$$\vartheta^8 = \vartheta^8(x) = \vartheta^8(0,\tau) = (1+2x+2x^4+\dots)^8$$

and

$$\Theta_8 = \Theta_8(x) = 1+\sum_1^\infty \rho_8(n)\,x^n,$$

where $x = e^{\pi i\tau}$, are one-valued functions of τ. If we can prove that

(A) $\vartheta^{-8}\Theta_8$ *is invariant for* Γ_3,

(B) $\vartheta^{-8}\Theta_8$ *is regular and bounded in* D_3,

then it will follow from the general theorem of § 9.12 that $\vartheta^{-8}\Theta_8$ is a constant, which is plainly 1; and from this it will follow that

$$r_8(n) = \rho_8(n) = 16\sigma_3^*(n).$$

Proof of (A)

9.14. It is sufficient to prove that $\vartheta^{-8}\Theta_8$ is invariant for the two substitutions

$$S_1\ (\tau+2),\quad S_2\left(-\frac{1}{\tau}\right)$$

which generate Γ_3. We can prove, quite generally, that $\vartheta^{-2s}\Theta_{2s}$ is invariant.[1]

In the first place

(9.14.1) $$\vartheta^{2s}(0,\tau+2) = \vartheta^{2s}(0,\tau),$$

(9.14.2) $$\vartheta^{2s}\left(0,-\frac{1}{\tau}\right) = \tau^{s}\vartheta^{2s}(0,\tau),$$

by the familiar formulae for the linear transformation of the ϑ-functions.[2] It remains to determine the behaviour of Θ_{2s}. It is obvious that Θ_{2s} is

[1] We are supposing $2s \equiv 0 \pmod 8$, but the proof is very much the same in other cases.

[2] We shall require the full table for the functions

$$\vartheta_2(0,\tau) = 2x^{\frac14}+2x^{\frac94}+2x^{\frac{25}{4}}+\dots,$$

$$\vartheta_3(0,\tau) = 1+2x+2x^4+2x^9+\dots,$$

$$\vartheta_4(0,\tau) = 1-2x+2x^4-2x^9+\dots,$$

which is

$$\vartheta_2(0,\tau+1) = \sqrt{i}\,\vartheta_2(0,\tau),\quad \vartheta_3(0,\tau+1) = \vartheta_4(0,\tau),\quad \vartheta_4(0,\tau+1) = \vartheta_3(0,\tau),$$

$$\vartheta_2\left(0,-\frac{1}{\tau}\right) = \frac{\sqrt{\tau}}{\sqrt{i}}\vartheta_4(0,\tau),$$

$$\vartheta_3\left(0,-\frac{1}{\tau}\right) = \frac{\sqrt{\tau}}{\sqrt{i}}\vartheta_3(0,\tau),$$

$$\vartheta_4\left(0,-\frac{1}{\tau}\right) = \frac{\sqrt{\tau}}{\sqrt{i}}\vartheta_2(0,\tau).$$

Here $\sqrt{i} = e^{\frac14\pi i}$ and $\sqrt{\tau}$ has its real and imaginary parts positive. The notation is Tannery and Molk's, and $\vartheta_3(0,\tau) = \vartheta(0,\tau)$.

invariant for S_1, since x is invariant for S_1 (and so that $\vartheta^{-2s}\Theta_{2s}$ is invariant). This will also appear incidentally from the analysis which follows.

The function $F_s(x)$ of § 9.8 is elementary. In fact, if $x = e^{-y}$, so that $y = -\pi i\tau$,

$$F_s(x) = \sum n^{s-1}x^n = \sum n^{s-1}e^{-ny} = \left(\frac{d}{dy}\right)^{s-2}\sum ne^{-ny}$$

$$= \left(\frac{d}{dy}\right)^{s-2}\frac{e^{-y}}{(1-e^{-y})^2} = \left(\frac{d}{dy}\right)^{s-2}\tfrac{1}{4}\operatorname{cosech}^2\tfrac{1}{2}y$$

$$= \left(\frac{d}{dy}\right)^{s-2}\sum_{-\infty}^{\infty}\frac{1}{(y+2n\pi i)^2} = \Gamma(s)\sum_{-\infty}^{\infty}\frac{1}{(y+2n\pi i)^s} = \frac{\Gamma(s)}{\pi^s}\sum_{-\infty}^{\infty}\frac{1}{(2n-\tau)^s}.$$

Also
$$xe^{-2p\pi i/q} = e^{\pi i\{\tau-(2p/q)\}},$$

and so
$$F_s(xe^{-2p\pi i/q}) = \frac{\Gamma(s)}{\pi^s}\sum_{-\infty}^{\infty}\frac{1}{\{2n-\tau+(2p/q)\}^s},$$

$$f_{p,q}(x) = \frac{\pi^s}{\Gamma(s)}\left(\frac{S_{p,q}}{q}\right)^{2s}F_s(xe^{-2p\pi i/q})$$

$$= \frac{\pi^s}{\Gamma(s)}\frac{\epsilon_q}{q^s}F_s(xe^{-2p\pi i/q}) = \epsilon_q\sum_{-\infty}^{\infty}\frac{1}{\{2(nq+p)-q\tau\}^s}.$$

Hence

$$(9.14.3)\qquad \Theta_{2s}(x) = 1+\sum_{p,q}f_{p,q}(x) = 1+\sum_{p,q,n}\frac{\epsilon_q}{\{2(nq+p)-q\tau\}^s},$$

where the range of summation is defined by

$$q = 1, 2, 3, \ldots;\quad 0<p<q,\ (p,q)=1;\quad -\infty<n<\infty$$

(except that $p=0$ when $q=1$).

We can write (9.14.3) in the form

$$(9.14.4)\qquad \Theta_{2s} = 1+\sum_{p,q}\frac{\epsilon_q}{(2p-q\tau)^s},$$

where $q = 1, 2, \ldots$ and p runs through all values (positive or negative) prime to q; and this is

$$(9.14.5)\qquad \Theta_{2s} = 1+\sum_{q=1,3,\ldots}\frac{1}{(2p-q\tau)^s}+\sum_{q=4,8,\ldots}\frac{2^s}{(2p-q\tau)^s}$$

$$= 1+\sum_{q=1,3,\ldots}\frac{1}{(2p-q\tau)^s}+\sum_{q=2,4,\ldots}\frac{1}{(p-q\tau)^s}$$

$$= 1+\sum_{p,q}\frac{1}{(p-q\tau)^s},$$

where now $q = 1, 2, \ldots$ and p runs through all values prime to and of opposite parity to q.

We wish to remove the restriction that p should be prime to q. We can do this by multiplying both sides of (9.14.5) by

$$\eta(s) = (1-2^{-s})\zeta(s) = 1+3^{-s}+5^{-s}+\ldots.$$

We thus obtain[1]

(9.14.6) $$\eta(s)\Theta_{2s} = \eta(s)+\sum_{p,q}\frac{1}{(p-q\tau)^s},$$

where $q = 1, 2, 3, \ldots$, and p runs through all values of opposite parity to q. We may also write this in either of the forms

(9.14.7) $$\eta(s)\Theta_{2s} = -\eta(s)+\sum_{p,q}\frac{1}{(p-q\tau)^s} \qquad (q=0,1,2,\ldots;\ p+q\equiv 1),$$

(9.14.8) $$\eta(s)\Theta_{2s} = \tfrac{1}{2}\Sigma'\frac{1}{(p-q\tau)^s} \qquad (p+q\equiv 1).$$

Here the congruences are to modulus 2, and q, in (9.14.8), runs through all integral values.

If we write $$\eta(s)\Theta_{2s} = \chi(\tau),$$

then it follows at once from any of (9.14.5)–(9.14.8) that

$$\chi(\tau+2) = \chi(\tau).$$

This, as I pointed out, is obvious from the beginning (but a useful check on our analysis).

We use (9.14.8) in investigating the effect of the substitution S_2. It gives

$$\chi\left(-\frac{1}{\tau}\right) = \tfrac{1}{2}\tau^s \sum_{p+q\equiv 1}\frac{1}{(p\tau+q)^s}$$

$$= \tfrac{1}{2}\tau^s \sum_{p+q\equiv 1}\frac{1}{(p-q\tau)^s} = \tau^s\chi(\tau),$$

on replacing p, q by $-q, p$.

Thus $\chi(\tau)$ is affected by S_1 and S_2 in just the same way as ϑ^{2s}; and $\vartheta^{-2s}\Theta_{2s}$ *is invariant for S_1 and S_2 and therefore for Γ_3.*

We have thus proved (A). More generally, we have proved the invariance of $\vartheta^{-2s}\Theta_{2s}$ whenever $2s\equiv 0 \pmod 8$.

Proof of (B)

9.15. The functions ϑ^8 and Θ_8 are defined by power series in $x = e^{\pi i\tau}$ convergent in the circle $|x| < 1$ or the half-plane $\mathfrak{I}\tau > 0$. Also

$$\vartheta(x) = \prod_1^\infty \{(1-x^{2n})(1+x^{2n-1})^2\}$$

[1] A pair (p, q) of opposite parity is of one of the forms (P, Q), $(3P, 3Q, \ldots)$, where P and Q are coprime and of opposite parity.

has no zeros in the circle. Hence $\vartheta^{-8}\Theta_8$ is regular for $|x|<1$ or $v>0$. It is bounded in any closed part of D_3 which excludes the two points in which D_3 abuts on the real axis; and therefore it is bounded in D_3 if it is bounded in the neighbourhood of the two points $\tau=\pm 1$. It is plain that we need consider only the point $\tau = 1$.

We write

(9.15.1) $$\tau = 1-\frac{1}{\mathrm{T}},\quad \mathrm{T}=\frac{1}{1-\tau},$$

and suppose that $\tau\to 1$ inside D_3, so that $0<u<1$ and $u^2+v^2>1$. If $\mathrm{T}=U+iV$ then

$$U=\frac{1-u}{(1-u)^2+v^2},\quad V=\frac{v}{(1-u)^2+v^2},$$

so that $0<U<1$ and $V\to\infty$. Hence $\mathrm{T}\to\infty$ inside D_3, and

$$X=e^{\pi i\mathrm{T}}\to 0.$$

Now $$\vartheta^8(x)=\vartheta_3^8(0,\tau)=\vartheta_3^8\left(0,1-\frac{1}{\mathrm{T}}\right)=\mathrm{T}^4\vartheta_2^8(0,\mathrm{T})$$
$$=\mathrm{T}^4(2X^{\frac{1}{4}}+2X^{\frac{9}{4}}+\ldots)^8,$$

so that

(9.15.2) $$\vartheta^8\sim 256\mathrm{T}^4X^2$$

when $\mathrm{T}\to\infty$ and $X\to 0$. We require a similar formula for Θ_8.

We have from (9.14.8)

(9.15.3) $$\frac{\pi^4}{96}\Theta_8=\chi(\tau)=\chi\left(1-\frac{1}{\mathrm{T}}\right)$$
$$=\tfrac{1}{2}\mathrm{T}^4\Sigma\frac{1}{\{q+(p-q)\mathrm{T}\}^4}\qquad (p+q\equiv 1)$$
$$=\tfrac{1}{2}\mathrm{T}^4\Sigma\frac{1}{(Q+P\mathrm{T})^4},$$

where now Q runs through all integers and P through all odd integers. If

$$|P|\mathrm{T}=\zeta,\quad \xi=e^{\pi i\zeta}=X^{|P|},$$

then $$\sum_Q\frac{1}{(Q+P\mathrm{T})^4}=\sum_Q\frac{1}{(Q+\zeta)^4}=\frac{1}{6}\left(\frac{d}{d\zeta}\right)^2\Sigma\frac{1}{(Q+\zeta)^2}$$
$$=\frac{1}{6}\left(\frac{d}{d\zeta}\right)^2\pi^2\operatorname{cosec}^2\pi\zeta=-\frac{1}{6}\left(\frac{d}{d\zeta}\right)^2\frac{4\pi^2e^{2\pi i\zeta}}{(1-e^{2\pi i\zeta})^2}$$
$$=-\tfrac{2}{3}\pi^2\left(\frac{d}{d\zeta}\right)^2(e^{2\pi i\zeta}+2e^{4\pi i\zeta}+3e^{6\pi i\zeta}+\ldots)$$
$$=\tfrac{8}{3}\pi^4(e^{2\pi i\zeta}+2^3e^{4\pi i\zeta}+3^3e^{6\pi i\zeta}+\ldots)$$
$$=\tfrac{8}{3}\pi^4(X^{2|P|}+2^3X^{4|P|}+3^3X^{6|P|}+\ldots).$$

We have to sum this with respect to P, but we need only the lowest powers of X, and X^2 occurs only for $P = \pm 1$. Hence (9.15.3) gives

$$\frac{\pi^4}{96}\Theta_8 \sim \tfrac{1}{2}.2.\tfrac{8}{3}\pi^4 \mathrm{T}^4 X^2,$$

or

(9.15.4) $$\Theta_8 \sim 256\mathrm{T}^4 X^2.$$

Finally (9.15.2) and (9.15.4) show that $\vartheta^{-8}\Theta_8$ is bounded when $\tau \to 1$, and therefore bounded in D_3.

It follows that $\vartheta^{-8}\Theta_8$ is a constant, which must be 1, and that $r_8(n) = \rho_8(n)$.

24 squares

9.16. We proved in § 9.14 that $\vartheta^{-2s}\Theta_{2s}$ is invariant for Γ_3 whenever $2s \equiv 0 \pmod 8$, and this is true in particular when $2s = 24$. If $\vartheta^{-24}\Theta_{24}$ were bounded in D_3, it would follow that $\vartheta^{24} = \Theta_{24}$ and $r_{24}(n) = \rho_{24}(n)$; but this is untrue. The correct formula, which was found first by Ramanujan, and which I shall proceed to prove, is

(9.16.1) $$\vartheta^{24}(x) = \Theta_{24}(x) - \tfrac{33152}{691}g(-x) - \tfrac{65536}{691}g(x^2),$$

where

(9.16.2) $$g(x) = x\{(1-x)(1-x^2)(1-x^3)\ldots\}^{24},$$

so that $$g(x^2) = x^2\{(1-x^2)(1-x^4)(1-x^6)\ldots\}^{24} = h^{24}(\tau),$$

in Tannery and Molk's notation. This last function is, apart from a factor of homogeneity, the discriminant $\Delta(\omega_1, \omega_2)$.

We require the formulae for the linear transformation of $h(\tau)$. These are

(9.16.3) $$h(\tau+1) = e^{\frac{1}{12}\pi i}h(\tau),$$

(9.16.4) $$h\left(-\frac{1}{\tau}\right) = e^{-\frac{1}{4}\pi i}\sqrt{\tau}\,h(\tau).$$

We shall also use one other formula, viz.

(9.16.5) $$h^2\left(\frac{\tau+1}{2}\right) = e^{\frac{1}{12}\pi i}h(\tau)\vartheta_3(0,\tau).$$

This belongs to the theory of *quadratic* transformation, but is a simple corollary of the product formulae for $h(\tau)$ and $\vartheta_3(0,\tau)$.

The three functions

(9.16.6) $$\vartheta^{-24}\Theta_{24}, \quad \vartheta^{-24}g(-x), \quad \vartheta^{-24}g(x^2)$$

are all invariant for Γ_3. We have already proved the first invariant. The invariance of the third follows from the formulae

$$g(x^2) = h^{24}(\tau), \quad h^{24}(\tau+2) = h^{24}(\tau), \quad h^{24}\left(-\frac{1}{\tau}\right) = \tau^{12}h^{24}(\tau).$$

Finally
$$g(-x) = h^{24}\left(\frac{\tau+1}{2}\right) = -h^{12}(\tau)\vartheta_3^{12}(0,\tau),$$

by (9.16.5), and the invariance of the second of the functions (9.16.6) follows from the formulae (9.14.2) and (9.16.4).

Hence
$$\vartheta^{-24}\Theta_{24}^{*} = \vartheta^{-24}\{\Theta_{24}+\alpha g(-x)+\beta g(x^2)\}$$

is invariant, for any α and β. We shall prove that it is possible to choose α and β so that $\vartheta^{-24}\Theta_{24}^{*}$ is bounded in D_3. For this, we use the substitution (9.15.1) of § 9.15, and examine the behaviour of all the functions concerned when $\mathrm{T}\to\infty$.

(i) First, by (9.15.2),

(9.16.7)
$$\vartheta^{24}\left(0, 1-\frac{1}{\mathrm{T}}\right) \sim 2^{24}\mathrm{T}^{12}X^{6}.$$

(ii) Secondly
$$g(-x) = -h^{12}(\tau)\vartheta^{12}(0,\tau) = -h^{12}\left(1-\frac{1}{\mathrm{T}}\right)\vartheta_3^{12}\left(0, 1-\frac{1}{\mathrm{T}}\right)$$
$$= h^{12}\left(-\frac{1}{\mathrm{T}}\right)\vartheta_4^{12}\left(0,-\frac{1}{\mathrm{T}}\right) = \left(\frac{\mathrm{T}}{i}\right)^{12} h^{12}(T)\vartheta_2^{12}(0,T)$$
$$= \mathrm{T}^{12}X\{(1-X^2)(1-X^4)\ldots\}^{12}(2X^{\frac{1}{4}}+2X^{\frac{9}{4}}+\ldots)^{12}.$$

Hence

(9.16.8)
$$g(-x) = 2^{12}\mathrm{T}^{12}\{X^4+O(X^6)\}.$$

(iii) Thirdly
$$g(x^2) = h^{24}(\tau) = h^{24}\left(1-\frac{1}{\mathrm{T}}\right) = h^{24}\left(-\frac{1}{\mathrm{T}}\right)$$
$$= \mathrm{T}^{12}h^{24}(\mathrm{T}) = \mathrm{T}^{12}X^2\{(1-X^2)(1-X^4)\ldots\}^{24},$$

(9.16.9)
$$g(x^2) = \mathrm{T}^{12}\{X^2-24X^4+O(X^6)\}.$$

(iv) Finally we have to determine the behaviour of Θ_{24}, which can be done by calculations like those of § 9.15. We have now

$$\eta(12)\,\Theta_{24} = \tfrac{1}{2}\mathrm{T}^{12}\sum_{Q}\sum_{P}\frac{1}{(Q+P\mathrm{T})^{12}},$$

$$\sum_{Q}\frac{1}{(Q+P\mathrm{T})^{12}} = \sum_{Q}\frac{1}{(Q+\zeta)^{12}} = \frac{1}{11!}\left(\frac{d}{d\zeta}\right)^{10}\sum_{Q}\frac{1}{(Q+\zeta)^2}$$
$$= \frac{(2\pi)^{12}}{11!}(e^{2\pi i\zeta}+2^{11}e^{4\pi i\zeta}+\ldots)$$
$$= \frac{(2\pi)^{12}}{11!}(X^{2|P|}+2^{11}X^{4|P|}+\ldots).$$

It is again only the terms with $P = \pm 1$ that matter, since $P = \pm 3$ yields $O(X^6)$. Hence we obtain

$$(9.16.10) \qquad \Theta_{24} = \frac{(2\pi)^{12}}{11!\,\eta(12)} \mathrm{T}^{12}\{X^2 + 2^{11}X^4 + O(X^6)\}$$

$$= \frac{2^{14}}{691} \mathrm{T}^{12}\{X^2 + 2^{11}X^4 + O(X^6)\},$$

on inserting the value of $\eta(12)$.

We have to choose α and β so that

$$\frac{2^{14}}{691}\{X^2 + 2^{11}X^4 + O(X^6)\} + \alpha 2^{12}\{X^4 + O(X^6)\} + \beta\{X^2 - 24X^4 + O(X^6)\} = O(X^6).$$

It will then follow from (9.16.7), (9.16.8), (9.16.9), and (9.16.10) that $\vartheta^{-24}\Theta_{24}^*$ is bounded. Equating coefficients, we find that

$$\alpha = -\tfrac{33152}{691}, \quad \beta = -\tfrac{65536}{691},$$

and (9.16.1) follows.

The function $\tau(n)$

9.17. We define $\tau(n)$[1] as the coefficient of x^n in

$$g(x) = x\{(1-x)(1-x^2)\ldots\}^{24} = \sum_1^\infty \tau(n)\,x^n.$$

Then, after § 9.9,

$$\Theta_{24} = 1 + \sum_1^\infty \rho_{24}(n)\,x^n = 1 + \tfrac{16}{691}\sum_1^\infty \sigma_{11}^*(n)\,x^n.$$

Also
$$g(-x) = \sum_1^\infty (-1)^n \tau(n)\,x^n,$$

and
$$g(x^2) = \sum_1^\infty \tau(\tfrac{1}{2}n)\,x^n,$$

if we agree that $\tau(y)$ means 0 when y is not an integer. Hence finally we obtain Ramanujan's formula

$$(9.17.1) \qquad r_{24}(n) = \tfrac{16}{691}\,\sigma_{11}^*(n) + e_{24}(n),$$

where

$$(9.17.2) \qquad e_{24}(n) = \tfrac{128}{691}\{(-1)^{n-1}\,259\tau(n) - 512\tau(\tfrac{1}{2}n)\}.$$

9.18. The function $\tau(n)$ has been defined only as a coefficient, and it is natural to ask whether there is any reasonably simple "arithmetical" definition; but none has yet been found. In the next lecture I shall discuss some of the most remarkable properties of the function. I must however

[1] I retain Ramanujan's notation. The collision of the τ in $\tau(n)$ and the τ in $e^{\pi i \tau}$ is a little unfortunate but is not likely to cause confusion.

prove something here about the order of $\tau(n)$, since I have to justify my assertion, in § 9.3, that $r_{24}(n)$ is "dominated" by $\rho_{24}(n)$.

It follows from a formula of Jacobi which I have quoted several times already that

$$\Sigma\tau(n)\,x^n = x\{(1-x)(1-x^2)\ldots\}^{24} = x(1-3x+5x^3-7x^6+\ldots)^8,$$

the exponents in the series being the triangular numbers. Now $(1-3x+\ldots)^8$ is majorised by

$$\left\{\sum_{n=0}^{\infty}(2n+1)\,x^{\frac{1}{2}n(n+1)}\right\}^8,$$

which is of order $(1-x)^{-8}$ when $x\to 1$.[1] Hence

$$|\,\tau(n)\,|\,x^n < \Sigma\,|\,\tau(n)\,|\,x^n < A(1-x)^{-8},$$

where A is a constant, for all n and x. Taking $x = 1-n^{-1}$, when x^n is about e^{-1}, we find that

(9.18.1) $$\tau(n) = O(n^8).$$

On the other hand

$$\frac{16}{691}\sigma_{11}^*(n) = \rho_{24}(n) = \frac{\pi^{12}n^{11}}{11!}\sum_{q=1}^{\infty}\epsilon_q\frac{c_q(n)}{q^{12}};$$

and the series is greater than[2]

$$1-\frac{3}{3^{12}}-\frac{2^{12}.4}{4^{12}}-\frac{5}{5^{12}}-\frac{7}{7^{12}}-\frac{2^{12}.8}{8^{12}}-\ldots > \tfrac{1}{2}.$$

Hence $\sigma_{11}^*(n)$ is greater than a constant multiple of n^{11}, and

$$r_{24}(n) = \frac{16}{691}\sigma_{11}^*(n)\left\{1+O\left(\frac{1}{n^3}\right)\right\}$$

is dominated heavily by its leading term.

The order of $\tau(n)$ is really a good deal smaller than is shown by (9.18.1). Ramanujan showed, by a more sophisticated method, that

(9.18.2) $$\tau(n) = O(n^7);$$

and I showed later, by a function-theoretic method, that

(9.18.3) $$\tau(n) = O(n^6).$$

I shall prove a better result, due to Rankin, in Lecture X. It is very plausible to suppose that (as Ramanujan conjectured)

$$\tau(n) = O(n^{\frac{11}{2}+\epsilon})$$

for every positive ϵ; but these questions I must postpone.

[1] That of $(\Sigma n x^{\frac{1}{2}n^2})^8$ or of $\left(\int_0^\infty t e^{-\frac{1}{2}yt^2}\,dt\right)^8$, where $e^{-y}=x$. This is that of y^{-8} or $(1-x)^{-8}$.

[2] Using the crude inequality $|\,c_q(n)\,| \leqq n$.

9.19. I conclude by repeating that the results which we have proved are typical of those in the general problem of $2s$ squares. We can always express $e_{2s}(n)$ as the sum of a number, fixed by Ramanujan and Mordell, of terms defined as modular coefficients; and each of the coefficients gives rise to a series of problems resembling those arising from $\tau(n)$. In some cases it is possible to define them fairly simply in arithmetical terms. In all cases the number of representations is dominated by the divisor function $\rho_{2s}(n)$.

NOTES ON LECTURE IX

§9.1. There is a full account of the history of the classical theorems concerning representation by two or four squares in Dickson, *History*, ii, chs. VI and VIII.

Jacobi's results concerning 2, 4, 6 and 8 squares are quoted by Smith on p. 307 of his *Report on the theory of numbers* (*Collected papers*, i, 38–364). They are contained implicitly in §§40–42 and 65–66 of the *Fundamenta nova*. Liouville gave formulae for 10 and 12 squares in the *Journal de math.* (2), 11 (1866), 1–8 and 9 (1864), 296–298.

Gauss, *Disquisitiones arithmeticae*, §182, stated a theorem equivalent to (9.1.1).

Glaisher, *Proc. London Math. Soc.* (2), 5 (1907), 479–490 (480), gives a systematic table of formulae for $r_{2s}(n)$ up to $2s = 18$. He had obtained these formulae in a series of papers in vols. 36–39 of the *Quarterly Journal of Math.* The formulae for 14 and 18 squares contain functions defined only as the coefficients in certain modular functions and not 'arithmetically'. Ramanujan, in no. 18 of the *Papers*, continues Glaisher's table up to $2s = 24$, and gives a general identity for $\vartheta^{2s}(x)$ which was proved afterwards by Mordell (in the first paper mentioned in the note on §9.4).

Boulyguine gave general formulae for $r_{2s}(n)$ in which every function which occurs has in a sense an arithmetical definition. Thus the formula for $r_{2s}(n)$ contains functions of the type

$$\Sigma\phi(x_1, x_2, \ldots, x_t),$$

where ϕ is a polynomial, t has one of the values $2s-8$, $2s-16$, ..., and the summation extends over all solutions of $x_1^2+x_2^2+\ldots+x_t^2 = n$. There are references to Boulyguine's work in Dickson's *History*, ii, 317, and the papers of Uspensky quoted below.

Uspensky has developed the elementary methods which seem to have been used by Liouville in a series of papers published in the *Bulletin de l'Acad. des Sciences de l'URSS* and other Russian periodicals: references will be found in a later paper of his in *Trans. Amer. Math. Soc.* 30 (1928), 385–404. He carries his analysis up to 12 squares, and states that his methods enable him to prove Boulyguine's general formulae. They can also be applied to many other problems concerning representations by quadratic forms.

H. Bessel (*Dissertation*, Königsberg, 1929) has developed Liouville's methods independently, and gives formulae up to $2s = 16$.

I am not concerned in this lecture with *odd* values of k, but it may be useful to add a short note on the subject. The functions $r_3(n)$, $r_5(n)$, and $r_7(n)$ may be expressed as finite sums involving symbols of quadratic reciprocity. Thus $r_3(n)$ was evaluated in this form by Dirichlet, and $r_5(n)$ and $r_7(n)$ by Eisenstein, Smith, and Minkowski. When $k = 3$ the problem is, as had been shown long before by Gauss, much the same as that of finding the number of classes of binary quadratic forms of determinant $-n$.

The results do not seem to have been worked out so systematically as those for even k, and it is not possible to refer to a comprehensive statement of them, though many formulae can be found in chs. VII and IX of the second volume of Dickson's *History* and ch. X of Bachmann's *Die Arithmetik von quadratischen Formen* (I Abtheilung).

There are two completely different solutions of the 5 and 7 square problems, the 'arithmetic' solution of Minkowski and Smith and the 'function-theoretic' solution of Hardy and Mordell. Bachmann gives an account of the first and Dickson, *Studies in the theory of numbers*, ch. XIII, of the second. References to the work of Hardy and Mordell are given in the note to § 9.4.

The formulae are generally stated, when k is odd, in terms of *primitive* representation (in which $x_1, x_2, \ldots, x_k$ have no common factor). The simplest formulae are due to Dirichlet and Eisenstein; thus the number of primitive representations of an odd n by 3 squares is

$$24 \sum_{s \leq \frac{1}{4}n} \left(\frac{s}{n}\right) \quad (n \equiv 1), \qquad 8 \sum_{s \leq \frac{1}{2}n} \left(\frac{s}{n}\right) \quad (n \equiv 3).$$

Here $\left(\frac{s}{n}\right)$ is Jacobi's generalisation of Legendre's symbol, and the congruences are to modulus 4. Eisenstein's formulae for 5 and 7 squares are proved by Bachmann. For references see Dickson, *History*, ii, 263 and 305.

§ 9.2. The formulae (9.2.2) and (9.2.3) occur in Gauss's posthumous work. See the references given by Dickson, *History*, ii, 283.

There is a proof of (9.1.1), by means of Gaussian integers, in Hardy and Wright, 240–242; and one of (9.1.2) and (9.1.3), by means of integral quaternions, in Dickson, *Algebren und ihre Zahlentheorie*, Kap. IX (see in particular 181–182, Satz 22). Landau, *Vorlesungen*, i, 110–113, gives an elementary deduction of (9.1.2) and (9.1.3) from (9.1.1).

Ramanujan's proof of (9.2.3) occurs in no. 18 of the *Papers*; the formula is (17) on p. 139. The proof is reproduced by Hardy and Wright, 311–314. Ramanujan seems to have used similar arguments frequently; Watson (10) points out that the familiar formulae

$$\mathrm{cn}^2 u + \mathrm{sn}^2 u = 1, \quad \mathrm{dn}^2 u + k^2 \mathrm{sn}^2 u = 1$$

appear, in the note-books, as identities between q-series to be proved by elementary calculation. Formula (18) on p. 139 of the *Papers* may be reduced to the familiar form

$$\wp''(u) = 6\wp^2(u) - \tfrac{1}{2}g_2.$$

Thus Ramanujan deduces the differential equation satisfied by $\wp(u)$, by direct algebra, from its expansion as a trigonometrical series.

§ 9.3. The formulae for $r_{10}(n)$ is Liouville's: see the note on § 9.1.

It may happen that $e_{2s}(n) = 0$ for special forms of n. Thus

$$e_{10}(n) = \tfrac{32}{5}\chi_4(n) = 0$$

if n is not a sum of two squares, and $e_{12}(n) = 0$ if n is even. These results, due to Liouville, were rediscovered by Glaisher.

In order that $e_{2s}(n) = 0$ for all n, it is necessary and sufficient that $2s \leq 8$. In these cases only $r_{2s}(n)$ is a 'divisor function', represented by the 'singular series' of § 9·8. The 'reasons' for this have been shown in a new light recently by Siegel, *Annals of Math.* (2), 36 (1935), 527–606.

The theory of the representation of n by the special form

(1) $$x_1^2 + x_2^2 + \ldots + x_k^2$$

can be extended to general definite quadratic forms in k variables. There are a certain number of genera of forms of a given discriminant, each containing a certain number of classes. If we select one representative of each class of a given genus, and define $N(n)$ as the total number of representations of n by one or other of these representatives, then $N(n)$ is the sum of a singular series like those of § 9.8.

If the discriminant is 1, then (1) is a representative of a class in the principal genus. In order that there should be only one such class, it is necessary and sufficient that $k \leqq 8$. In this case only is $r_k(n)$ the sum of the singular series.

I follow Ramanujan's notation except for a factor 2. He writes

$$\vartheta^{2s} = 1 + 2\Sigma r_{2s}(n)\, x^n,$$

so that his r_{2s}, δ_{2s}, and e_{2s} are half mine.

§ 9.4. The work of Mordell and myself is contained in

Mordell, *Quarterly Journal of Math.* 48 (1917), 93–104 and *Trans. Camb. Phil. Soc.* 22 (1919), 361–372;

Hardy, *Proc. Nat. Acad. of Sciences*, 4 (1918), 189–193; *Trans. Amer. Math. Soc.* 21 (1920), 255–284.

§ 9.5. 'Ramanujan's sum' occurs in earlier writings, and (9.5.3) seems to be due to Kluyver: see *Papers*, 343. But Ramanujan was the first to appreciate the importance of the sum and to use it systematically. The proofs of (9.5.3) given here are taken from the *Papers*, 180, and from Hardy (7).

For the Möbius inversion formulae see, for example, Hardy and Wright, 234–237, or Landau, *Handbuch*, 577–582.

§ 9.6. The three ways of summing (9.6.1) are due to Estermann, *Proc. London Math. Soc.* (2), 34 (1932), 194–195; Ramanujan, *Papers*, 180–185; and Hardy (7). Ramanujan and Hardy prove many other formulae of the same kind.

For the formula for $\sigma_{1-s}(n)$ see, for example, Hardy and Wright, 238.

The theorem on multiplication of Dirichlet's series referred to near the end of the section is proved in Landau, *Handbuch*, 671–673, and on pp. 63–64 of the tract of Hardy and Riesz quoted on p. 69.

§ 9.8. Ramanujan formed 'singular series': thus the series (11.11)–(11.41) of no. 21 of the *Papers* are the singular series relevant in this problem. But his approach to them is quite different; he determines the 'divisor-function' $\delta_{2s}(n)$, as an approximation to $r_{2s}(n)$, independently, and then expands it as a singular series. Here the singular series comes first, and $\delta_{2s}(n)$ appears as its sum.

The phrase 'singular series' has been used differently by different writers. Thus Littlewood and I have sometimes called $\Sigma A_q(n)$, without the outside factor, the singular series.

§ 9.9. The formulae for $S_{p,q}$ will be found in Bachmann, *Analytische Zahlentheorie*, ch. VII.

If k (here $2s$) were odd, then $S^k_{p,q}$ would involve a Legendre or Jacobi symbol, which disappears when k is even. This is the origin of the Legendre or Jacobi symbols in the formulae for $r_3(n)$, $r_5(n)$, ...

§ 9.11. The standard treatise on the elliptic modular functions is Klein-Fricke, *Theorie der elliptischen Modulfunktionen*, 2 vols, Leipzig, 1890–1892. This is very long and elaborate. Vivanti's *Fonctions polyédriques et modulaires* (French translation by A. Cahen, Paris, 1910) is a more elementary book designed "permettre au lecteur d'aborder sans difficultés les leçons classiques de MM. Klein et Fricke".

Hurwitz has given a self-contained account of the theory in two papers in the *Math. Annalen*, 18 (1881), 528–592 and 58 (1904), 343–360. This contains proofs of the theorems quoted here.

There is a clear account of the elementary geometry of the modular group in Copson's *Theory of functions of a complex variable*, ch. XV.

Dr Heilbronn has shown me the following simple and direct proof of the theorem quoted at the end of the section. Suppose that $f(\tau)$ is regular and bounded for $\Im(\tau) > 0$, and that

$$f(\tau+1) = f(\tau), \quad f\left(-\frac{1}{\tau}\right) = f(\tau).$$

Then $g(x) = g(e^{\pi i \tau}) = f(\tau)$ cannot have an essential singularity at the origin (since then it would assume arbitrarily large values near the origin), nor a pole (since then it would tend to infinity); and so it is regular at the origin. We may therefore suppose (subtracting a constant if necessary) that $f(\tau) \to 0$ when $\Im(\tau) \to \infty$.

Now $|f(\tau)|$ attains its upper bound M in D at a point on the boundary of D (and not at infinity), so that there is a finite τ_0 on the boundary of D for which $|f(\tau_0)| = M$. If $f(\tau)$ were not constant, there would be a τ_1 near τ_0 for which $|f(\tau_1)| > |f(\tau_0)| = M$. But there is a τ_2 in D for which $f(\tau_2) = f(\tau_1)$, and so $|f(\tau_2)| > M$; a contradiction.

§§ 9.14–15. The proof follows the second paper of Hardy quoted in the note on § 9.4.

§ 9.17. (9.17.2) is formula (148) on p. 159 of the *Papers*. There is an error of sign in formula 7 of Table VI: Ramanujan seems to forget momentarily that $g(-x)$ begins with $-x$ and not x. In any case, as I remarked in the note on § 9·3, his $r_{24}(n)$ and $e_{24}(n)$ are half mine.

§ 9·18. Ramanujan's proof of (9.18.2) is to be found (as a special case of more general theorems) in the *Papers*, 146–148 and 153, or 160; and Hardy's proof of (9.18.3), which is reproduced in a slightly different form in the next lecture (§ 10.8), in Hardy (9).

X

RAMANUJAN'S FUNCTION $\tau(n)$

10.1. I proved in Lecture IX that

$$(10.1.1)\quad r_{24}(n) = \frac{16}{691}\sigma_{11}^{*}(n) + \frac{128}{691}\{(-1)^{n-1}259\tau(n) - 512\tau(\tfrac{1}{2}n)\},$$

where $\sigma_{11}^{*}(n)$ is a simple "divisor function" of n, and $\tau(n)$ is defined by

$$(10.1.2)\qquad g(x) = x\{(1-x)(1-x^2)\dots\}^{24} = \sum_{1}^{\infty}\tau(n)\,x^n.$$

I shall devote this lecture to a more intensive study of some of the properties of $\tau(n)$, which are very remarkable and still very imperfectly understood. We may seem to be straying into one of the backwaters of mathematics, but the genesis of $\tau(n)$ as a coefficient in so fundamental a function compels us to treat it with respect.

The multiplicative property of $\tau(n)$

10.2. The coefficients in many modular expansions have simple arithmetical meanings; thus the coefficient in

$$\vartheta^{s}(x) = (1 + 2x + 2x^4 + \dots)^s$$

is $r_s(n)$. But $\tau(n)$ has no such obvious interpretation,[1] and its arithmetical properties are still very obscure.

Ramanujan conjectured that

$$(10.2.1)\qquad \tau(nn') = \tau(n)\,\tau(n')$$

if $(n, n') = 1$, i.e. that $\tau(n)$ *is multiplicative*; and this was proved a little later by Mordell. Mordell's proof is very instructive, and sufficiently simple for insertion here. It depends on the identity

$$(10.2.2)\qquad \sum_{1}^{\infty}\tau(pn)\,x^n = \tau(p)\sum_{1}^{\infty}\tau(n)\,x^n - p^{11}\sum_{1}^{\infty}\tau(n)\,x^{pn},$$

where p is prime.

[1] Jacobi's identity shows that

$$\sum_{1}^{\infty}\tau(n)\,x^n = x(1 - 3x + 5x^3 - 7x^6 + \dots)^8.$$

Hence $\qquad \tau(n) = \Sigma(-1)^{n_1+n_2+\dots+n_8}(2n_1+1)(2n_2+1)\dots(2n_8+1),$

the summation being extended over all representations of $n-1$ as a sum

$$\sum_{i=1}^{8}\tfrac{1}{2}n_i(n_i+1) = \sum_{i=1}^{8} t_i$$

of 8 triangular numbers t_i; but this interpretation is not illuminating.

We write, as usual

$$\Delta(\omega_1, \omega_2) = \left(\frac{2\pi}{\omega_1}\right)^{12} x^2\{(1-x^2)(1-x^4)\ldots\}^{24}, \tag{10.2.3}$$

where $x = e^{\pi i\tau}$, $\tau = \omega_2/\omega_1$. Then $\Delta(\omega_1, \omega_2)$ is invariant for the substitutions

$$S_1\begin{pmatrix}1 & 0\\ 1 & 1\end{pmatrix},\quad S_2\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix},$$

which replace ω_1, ω_2 by ω_1, $\omega_1+\omega_2$ and by $-\omega_2$, ω_1 respectively. We prove first that

$$P = \Delta(\omega_1, p\omega_2) + \sum_{\kappa=0}^{p-1} \Delta(p\omega_1, \kappa\omega_1+\omega_2) = u_0 + \sum_{\kappa=0}^{p-1} v_\kappa \tag{10.2.4}$$

is invariant for S_1 and S_2, and therefore for all the substitutions of the modular group Γ.

First, S_1 leaves u_0 unaltered, and permutes the v_κ, so that P is invariant for S_1.

Next, S_2 changes P into

$$\Delta(-\omega_2, p\omega_1) + \Delta(-p\omega_2, \omega_1) + \sum_{\kappa=1}^{p-1} \Delta(-p\omega_2, \omega_1-\kappa\omega_2)$$
$$= \Delta(p\omega_1, \omega_2) + \Delta(\omega_1, p\omega_2) + \sum_{\kappa=1}^{p-1} \Delta(-p\omega_2, \omega_1-\kappa\omega_2);$$

and, in order to prove P invariant, it is enough to show that

$$\Delta(-p\omega_2, \omega_1-\kappa\omega_2) = \Delta(p\omega_1, \kappa'\omega_1+\omega_2), \tag{10.2.5}$$

where κ' runs through the values $1, 2, \ldots, p-1$ with κ, or through these residues (mod p).

We take
$$a = -\kappa,\quad b = p,$$

and determine c and d so that

$$ad-bc = -\kappa d - pc = 1.$$

Then $\kappa' = d$ runs through the residues required. Also

$$-ap\omega_2 + b(\omega_1-\kappa\omega_2) = p\omega_1,\quad -cp\omega_2 + d(\omega_1-\kappa\omega_2) = \kappa'\omega_1+\omega_2,$$

and so
$$\Delta(-p\omega_2, \omega_1-\kappa\omega_2) = \Delta(p\omega_1, \kappa'\omega_1+\omega_2),$$

which is (10.2.5).

It follows that P is invariant for Γ, and therefore that

$$Q = \frac{P}{\Delta(\omega_1, \omega_2)} \tag{10.2.6}$$

is invariant.

Now

(10.2.7)

$$\Delta(\omega_1, \omega_2) = \left(\frac{\pi}{\omega_1}\right)^{12} \sum_1^\infty \tau(n)\, x^{2n}, \quad \Delta(\omega_1,\, p\omega_2) = \left(\frac{\pi}{\omega_1}\right)^{12} \sum_1^\infty \tau(n)\, x^{2pn},$$

and

$$(10.2.8) \quad \sum_{\kappa=0}^{p-1} \Delta(p\omega_1,\, \kappa\omega_1 + \omega_2) = \left(\frac{\pi}{p\omega_1}\right)^{12} \sum_{n=1}^\infty \tau(n)\, x^{2n/p} \sum_{\kappa=0}^{p-1} e^{2n\kappa\pi i/p}$$

$$= \left(\frac{\pi}{p\omega_1}\right)^{12} p \sum_{n=1}^\infty \tau(pn)\, x^{2n},$$

since the sum with respect to κ gives p if $p \mid n$ and 0 otherwise. Hence the expansion of P in powers of x begins with

$$\left(\frac{\pi}{\omega_1}\right)^{12} p^{-11}\, \tau(p)\, x^2,$$

while that of $\Delta(\omega_1, \omega_2)$ begins with

$$\left(\frac{\pi}{\omega_1}\right)^{12} x^2.$$

It follows that $Q = P/\Delta$ is bounded in D, and therefore constant, so that

$$(10.2.9) \qquad P = p^{-11}\tau(p)\,\Delta.$$

Finally, substituting from (10.2.7)–(10.2.9) into (10.2.4), we obtain (10.2.2), with x^2 in place of x.

We can now prove (10.2.1). It is sufficient to prove that

$$(10.2.10) \qquad \tau(p^\lambda n) = \tau(p^\lambda)\,\tau(n)$$

for every λ, p being prime and $(n, p) = 1$. This is obvious when $\lambda = 0$. If we equate the coefficients of x^n in (10.2.2), we obtain

$$\tau(pn) = \tau(p)\,\tau(n),$$

which is (10.2.10) for $\lambda = 1$. If we equate those of $x^{p^{\lambda-1}n}$, where $\lambda > 1$, we obtain

$$(10.2.11) \qquad \tau(p^\lambda n) = \tau(p)\,\tau(p^{\lambda-1}n) - p^{11}\tau(p^{\lambda-2}n),$$

and in particular

$$(10.2.12) \qquad \tau(p^\lambda) = \tau(p)\,\tau(p^{\lambda-1}) - p^{11}\tau(p^{\lambda-2}).$$

Hence, if $\qquad u_\lambda = \tau(np^\lambda) - \tau(n)\,\tau(p^\lambda),$

we have $\qquad u_\lambda = \tau(p)\,u_{\lambda-1} - p^{11}u_{\lambda-2}.$

But $u_\lambda = 0$ when $\lambda = 0$ and $\lambda = 1$, and therefore $u_\lambda = 0$ for all λ.

The function $F(s) = \Sigma \frac{\tau(n)}{n^s}$

10.3. From (10.2.1) we can deduce a remarkable formula for the Dirichlet's series

(10.3.1) $$F(s) = \sum_1^\infty \frac{\tau(n)}{n^s}.$$

I proved in § 9.18 that $\tau(n) = O(n^8)$,

so that (10.3.1) is absolutely convergent for $\sigma = \Re(s) > 9$. We shall see later that much more than this is true, but this imperfect result is sufficient to show that the transformations which follow are valid for sufficiently large σ.

Since $\tau(n)$ is multiplicative, we have

(10.3.2) $$F(s) = \prod_p \chi_p,$$

where

(10.3.3) $$\chi_p = 1 + \frac{\tau(p)}{p^s} + \frac{\tau(p^2)}{p^{2s}} + \ldots.$$

We can calculate $\tau(p^\lambda)$ in terms of $\tau(p)$ from (10.2.11), and so determine χ_p.

We write

(10.3.4) $$\cos\theta_p = \tfrac{1}{2}p^{-\frac{11}{2}}\tau(p),$$

(10.3.5) $$a_\lambda = p^{-\frac{11}{2}\lambda}\tau(p^\lambda).$$

Then (10.2.11) gives $a_\lambda - 2\cos\theta_p a_{\lambda-1} + a_{\lambda-2} = 0.$

But $$a_0 = 1 = \frac{\sin\theta_p}{\sin\theta_p}, \quad a_1 = 2\cos\theta_p = \frac{\sin 2\theta_p}{\sin\theta_p};$$

and it follows by induction that

$$a_\lambda = \frac{\sin(\lambda+1)\theta_p}{\sin\theta_p}$$

and

(10.3.6) $$\tau(p^\lambda) = p^{\frac{11}{2}\lambda}\frac{\sin(\lambda+1)\theta_p}{\sin\theta_p}.$$

Hence

(10.3.7) $$\chi_p = \frac{1}{\sin\theta_p}\sum_{\lambda=0}^\infty p^{(\frac{11}{2}-s)\lambda}\sin(\lambda+1)\theta_p = \frac{1}{1-2p^{\frac{11}{2}-s}\cos\theta_p + p^{11-2s}}$$

$$= \frac{1}{1-\tau(p)p^{-s}+p^{11-2s}},$$

and

(10.3.8) $$F(s) = \prod_p \chi_p = \prod_p \frac{1}{1-\tau(p)p^{-s}+p^{11-2s}}.$$

This is the analogue of Euler's product for $\zeta(s)$. The series and product are absolutely convergent for sufficiently large σ.

From (10.2.1) and (10.3.6) it follows that

(10.3.9) $$\tau(n) = n^{\frac{11}{2}} \Pi \frac{\sin(\lambda+1)\theta_p}{\sin\theta_p}$$

if $n = \Pi p^{\lambda}$.

The function $F(s)$ also satisfies a functional equation of the same type as that satisfied by $\zeta(s)$. It will be convenient to defer the proof of this to § 10.9.

Congruence properties of $\tau(n)$

10.4. The results of § 10.2 enable us to prove a number of curious arithmetical properties of $\tau(n)$. These have a certain resemblance to those of $p(n)$, which I spoke of in an earlier lecture; but they are naturally more numerous, since $\tau(n)$ is multiplicative.

Suppose that p is prime and that

(10.4.1) $$\tau(p) \equiv 0 \pmod p.$$

Then (10.2.12) and (10.2.10) show that

(10.4.2) $$\tau(pn) \equiv 0 \pmod p$$

for every n. Ramanujan tabulated $\tau(n)$ up to $n = 30$, and his table shows that (10.4.1) is true for

(10.4.3) $$p = 2, 3, 5, 7, 23;$$

so that these primes have the property (10.4.2).

The last two primes yield further properties of $\tau(n)$, which do not depend upon the multiplicative property. We can write $g(x)$ in the form

$$g(x) = \Sigma\tau(n)x^n = x\Pi(1-x^n)^{21}\Pi(1-x^n)^3.$$

Now $$(1-x^n)^7 \equiv 1-x^{7n} \pmod 7,$$

$$(1-x^n)^{21} \equiv 1-3x^{7n}+3x^{14n}-x^{21n} \pmod 7,$$

and $$\Pi(1-x^n)^{21} = \Sigma c_\mu x^\mu,$$

c_μ being divisible by 7 whenever μ is not. Also

$$\Pi(1-x^n)^3 = \Sigma(-1)^\nu(2\nu+1)x^{\frac{1}{2}\nu(\nu+1)},$$

by Jacobi's identity. Hence

$$\Sigma\tau(n)x^n = \Sigma\Sigma(-1)^\nu(2\nu+1)c_\mu x^{\frac{1}{2}\nu(\nu+1)+\mu+1}$$

and $$\tau(n) = \Sigma(-1)^\nu(2\nu+1)c_\mu,$$

the summation being over all μ and ν for which

$$n = \tfrac{1}{2}\nu(\nu+1)+\mu+1.$$

Now $$\tfrac{1}{2}\nu(\nu+1) \equiv 0, 1, 3, \text{ or } 6 \pmod 7,$$

so that $$n \equiv \mu,\ \mu+1,\ \mu+2, \text{ or } \mu+4 \pmod 7.$$

If $n \equiv 3$, 5, or 6, μ cannot be a multiple of 7, and then c_μ is a multiple of 7. Hence

$$\tau(7m+k) \equiv 0 \pmod 7$$

for $k = 3, 5, 6$ as well as for $k = 0$.

We can prove similarly (using Euler's identity instead of Jacobi's) that

$$\tau(23m+k) \equiv 0 \pmod{23}$$

when k is any quadratic non-residue of 23.

10.5. The congruences of § 10.4 are satisfied by all n of certain arithmetical progressions. There are also congruences satisfied by "almost all" n. For example

$$\tau(n) \equiv 0 \pmod 5 \qquad (10.5.1)$$

for almost all n (in the sense of § 3.4).

We begin by proving that

$$\tau(n) \equiv n\sigma(n) \pmod 5, \qquad (10.5.2)$$

where $\sigma(n)$ is the sum of the divisors of n, for all n. This depends on two identities in the theory of the modular functions, viz.

$$Q^3 - R^2 = 1728g(x), \qquad (10.5.3)$$

and

$$Q - P^2 = 288\Sigma\frac{n^2x^n}{(1-x^n)^2}, \qquad (10.5.4)$$

where

$$P = 1-24\left(\frac{x}{1-x}+\frac{2x^2}{1-x^2}+\frac{3x^3}{1-x^3}+\ldots\right), \qquad (10.5.5)$$

$$Q = 1+240\left(\frac{x}{1-x}+\frac{2^3x^2}{1-x^2}+\frac{3^3x^3}{1-x^3}+\ldots\right) \qquad (10.5.6)$$

$$R = 1-504\left(\frac{x}{1-x}+\frac{2^5x^2}{1-x^2}+\frac{3^5x^3}{1-x^3}+\ldots\right). \qquad (10.5.7)$$

The identity (10.5.3) is familiar, but I have not seen (10.5.4) anywhere except in Ramanujan's work.

We can deduce (10.5.4) from (9.2.5). We have

$$\tfrac{1}{4}\cot\tfrac{1}{2}\theta = \frac{1}{2\theta}-\frac{\theta}{24}-\frac{\theta^3}{1440}-\ldots,$$

$$(\tfrac{1}{4}\cot\tfrac{1}{2}\theta)^2 = \frac{1}{4\theta^2}-\frac{1}{24}+\frac{\theta^2}{960}+\ldots,$$

$$u_1\sin\theta+u_2\sin 2\theta+\ldots = -\frac{P-1}{24}\theta-\frac{Q-1}{1440}\theta^3+\ldots,$$

$$u_1(1+u_1)\cos\theta+u_2(1+u_2)\cos 2\theta+\ldots$$
$$=\frac{x}{(1-x)^2}+\frac{x^2}{(1-x^2)^2}+\ldots-\frac{1}{2}\left\{\frac{x}{(1-x)^2}+\frac{2^2x^2}{(1-x^2)^2}+\ldots\right\}\theta^2+\ldots,$$

$$\tfrac{1}{2}\{u_1(1-\cos\theta)+2u_2(1-\cos 2\theta)+\ldots\}=\frac{Q-1}{960}\theta^2+\ldots,$$

$$\left(\frac{1}{2\theta}-\frac{P}{24}\theta-\frac{Q}{1440}\theta^3-\ldots\right)^2=\frac{1}{4\theta^2}-\frac{1}{24}+\frac{x}{(1-x)^2}+\frac{x^2}{(1-x^2)^2}+\ldots$$
$$+\left[\frac{Q}{960}-\frac{1}{2}\left\{\frac{x}{(1-x)^2}+\frac{2^2x^2}{(1-x^2)^2}+\ldots\right\}\right]\theta^2+\ldots;$$

and (10.5.4) follows by equating the coefficients of θ^2.[1]

It is now easy to prove (10.5.2). For $n^5\equiv n \pmod 5$ for all n, and so

$$R\equiv 21P\equiv P \pmod 5,$$

while
$$Q\equiv 1 \pmod 5.$$

Hence
$$1728\,\Sigma\tau(n)x^n=Q^3-R^2\equiv Q-P^2 \pmod 5.$$

But
$$Q-P^2=288\sum_{\nu=1}^{\infty}\frac{\nu^2x^\nu}{(1-x^\nu)^2}=288\sum_{\mu=1}^{\infty}\sum_{\nu=1}^{\infty}\mu\nu^2x^{\mu\nu}$$
$$=288\sum_1^\infty n\sigma(n)x^n,$$

so that
$$\tau(n)\equiv 6\tau(n)\equiv n\sigma(n) \pmod 5.$$

In order to prove (10.5.1), then, it is sufficient to prove that

(10.5.8) $$\sigma(n)\equiv 0 \pmod 5$$

for almost all n.

10.6. The congruence (10.5.8) expresses a special case of the more general theorem that

(10.6.1) $$\sigma(n)\equiv 0 \pmod k$$

for every k and almost all n.[2] We may plainly suppose that k is prime.

We denote a general prime by p and a prime of the special form $km-1$ by ϖ, and write
$$n=\prod_{p\neq\varpi}p^a\prod_{\varpi}\varpi^\alpha.$$

[1] Equating the constant terms gives
$$P=1-24\left\{\frac{x}{(1-x)^2}+\frac{x^2}{(1-x^2)^2}+\ldots\right\},$$
which is easily verified directly.

[2] Still more generally, $\sigma_s(n)\equiv 0 \pmod k$ for all odd s, all k, and almost all n.

Then
$$\sigma(n) = \prod_{p \neq \varpi} \frac{p^{\alpha+1}-1}{p-1} \prod_{\varpi} \frac{\varpi^{\alpha+1}-1}{\varpi-1}.$$
If α is odd then
$$\frac{\varpi^{\alpha+1}-1}{\varpi-1} = (\varpi+1)\frac{\varpi^{\alpha+1}-1}{\varpi^2-1} \equiv 0 \pmod{k},$$

since $\varpi+1 \equiv 0 \pmod{k}$ and the second factor is integral. It follows that (10.6.1) is true unless every special prime ϖ occurs in n with an even exponent α. It is therefore sufficient to prove that *α is odd, for at least one ϖ, in almost all n.*

We define b_n by
$$b_n = 1 \quad \text{(if all } \alpha \text{ are even)},$$
$$b_n = 0 \quad \text{(otherwise)};$$

and we have to prove that

(10.6.2) $$B(n) = \sum_{n \leq x} b_n = o(x).$$

We can in fact prove more, viz. that

(10.6.3) $$B(x) \sim \frac{Cx}{(\log x)^{1-\kappa}},$$

where $\kappa = 1/(k-1)$ and C depends on k only. The proof is very much like Landau's proof quoted in §§ 4.5–4.6.

Let
$$F(s) = \sum \frac{b_n}{n^s}.$$

Then
$$F(s) = \prod_{p \neq \varpi}\left(1+\frac{1}{p^s}+\frac{1}{p^{2s}}+\dots\right)\prod_{\varpi}\left(1+\frac{1}{\varpi^{2s}}+\frac{1}{\varpi^{4s}}+\dots\right)$$
$$= \prod_{p \neq \varpi}\frac{1}{1-p^{-s}}\prod_{\varpi}\frac{1}{(1-\varpi^{-s})(1+\varpi^{-s})} = \frac{\zeta(s)}{\prod (1+\varpi^{-s})}$$

for $\sigma = \Re(s) > 1$. Hence
$$\log F(s) = \log\zeta(s) - G(s),$$
where
$$G(s) = \sum \log(1+\varpi^{-s}) = \sum \varpi^{-s} + R(s),$$

and $R(s)$ is regular for $\sigma > \frac{1}{2}$.

I must now take for granted some of the standard theory of the primes of an arithmetical progression. The function $\sum p^{-s}$ behaves, in the ordinary theory of primes, "sufficiently like" $\log\zeta(s)$; and $\sum \varpi^{-s}$ behaves, in the theory of the primes of the arithmetical progression, sufficiently like $\kappa\log\zeta(s)$. Hence $\log F(s)$ behaves sufficiently like $(1-\kappa)\log\zeta(s)$, and $F(s)$ sufficiently like $\{\zeta(s)\}^{1-\kappa}$. In fact
$$F(s) = \{\zeta(s)\}^{1-\kappa} H(s),$$

where $H(s)$ is regular at $s = 1$ and does not behave too badly for large s whose real part is nearly 1.[1]

If we accept all this, then the road to (10.6.3) is clear, since we have only to repeat Landau's argument with comparatively trivial changes; and we have seen that (10.6.1) is a corollary. Here k is arbitrary; but the passage to (10.5.1) depends upon the special properties of 5.

There are similar congruences for other moduli, of which the most noteworthy is 691. Ramanujan proved that

(10.6.4) $$\tau(n) \equiv \sigma_{11}(n) \pmod{691};$$

and Watson has deduced[2] that (as Ramanujan conjectured) $\tau(n)$ is divisible by 691 for almost all n.

I may observe that (10.6.4), for *odd n*, follows from (10.1.1). For then

$$16\tau(n) \equiv -128.259\tau(n) = 16\sigma_{11}(n) - 691r_{24}(n) \equiv 16\sigma_{11}(n) \pmod{691}.$$

The order of $\tau(n)$

10.7. I return to the problem of the order of $\tau(n)$, which I referred to at the end of Lecture IX. This is the most fundamental of the unsolved problems presented by the function.

Ramanujan conjectured that

(10.7.1) $$\tau(p) \leqq 2p^{\frac{11}{2}}$$

for every prime p, or, what is the same thing, that all the angles θ_p of § 10.3 are real. I shall call this the "Ramanujan hypothesis". If the hypothesis is true then

(10.7.2) $$|\tau(n)| = n^{\frac{11}{2}} \prod_{p|n} \left|\frac{\sin(\lambda+1)\theta_p}{\sin\theta_p}\right| \leqq n^{\frac{11}{2}} \prod_{p|n} (\lambda+1) = n^{\frac{11}{2}} d(n),$$

and so

(10.7.3) $$\tau(n) = O(n^{\frac{11}{2}+\epsilon})$$

for every positive ϵ. It is easy to prove, in the other direction, (i) that

(10.7.4) $$\tau(n) \geqq n^{\frac{11}{2}}$$

[1] Actually, when $k = 5$,

$$F(s) = \{\zeta(s)\}^{\frac{1}{4}} \left\{\frac{L_2(s)\, L_3(s)}{L_4(s)}\right\}^{\frac{1}{4}} h(s),$$

where $h(s)$ is regular for $\sigma > \frac{1}{2}$, and $L_2(s)$, $L_3(s)$, $L_4(s)$ are the three Dirichlet's L-functions associated with the characters

$$(1, i, -i, -1), \quad (1, -i, i, -1), \quad (1, -1, -1, 1).$$

[2] Using the theorem referred to in footnote 2, p. 167.

for an infinity of n, so that the $\frac{11}{2}$ of (10.7.3) cannot be replaced by any smaller number; and (ii) that the truth of (10.7.3), for every positive n, involves the truth of the Ramanujan hypothesis.[1]

We saw in § 9.18 that

(10.7.5) $$\tau(n) = O(n^8).$$

Ramanujan gave a more complicated proof of

(10.7.6) $$\tau(n) = O(n^7),$$

and this is the most that has been proved by "elementary" methods. I proved in 1918, by the method used by Littlewood and myself in our work on Waring's problem, that

(10.7.7) $$\tau(n) = O(n^6).$$

Kloosterman proved in 1927 that

(10.7.8) $$\tau(n) = O(n^{\frac{47}{8}+\epsilon})$$

for every positive ϵ; Davenport and Salié proved independently in 1933 that

(10.7.9) $$\tau(n) = O(n^{\frac{35}{6}+\epsilon});$$

and finally Rankin proved in 1939 that

(10.7.10) $$\tau(n) = O(n^{\frac{29}{5}}),$$

the best result yet known. The indices here are (apart from the ϵ's) less than 6 by $\frac{1}{8}$, $\frac{1}{6}$, and $\frac{1}{5}$ respectively.

Proof that $\tau(n) = O(n^6)$

10.8. I prove (10.7.7) by showing that

(10.8.1) $$g(z) = z\{(1-z)(1-z^2)\dots\}^{24} = O\left\{\frac{1}{(1-r)^6}\right\},$$

uniformly in θ. Here $z = re^{i\theta}$ and $0 < r < 1$. Assuming this for the moment, we have

$$\tau(n) = \frac{1}{2\pi i}\int_C \frac{g(z)}{z^{n+1}}\,dz,$$

[1] For (i) take $n = p^\lambda$, and observe that

$$\left|\frac{\sin(\lambda+1)\theta_p}{\sin\theta_p}\right| \geqq 1$$

for an infinity of λ. For (ii) take the same n and suppose θ_p complex. Then $\theta_p = k\pi \pm i\eta_p$, where η_p is positive (since $\cos\theta_p$ is real), and

$$\left|\frac{\sin(\lambda+1)\theta_p}{\sin\theta_p}\right| = \frac{\sinh(\lambda+1)\eta_p}{\sinh\eta_p}$$

is greater than a constant multiple of $e^{\lambda\eta_p}$ or n^δ, where

$$\delta = \frac{\eta_p}{\log p}.$$

where C is the circle $r = e^{-1/n}$. On this circle $|z^{-n}| = e$, and therefore

$$\tau(n) = O\left\{\operatorname*{Max}_{r=e^{-1/n}} |g(re^{i\theta})|\right\} = O(n^6),$$

so that (10.7.7) is a corollary of (10.8.1).

We divide up C, by the Farey dissection[1] of order

$$\nu = [\sqrt{n}]+1,$$

into arcs $\xi_{p,q}$. It is sufficient to prove that (10.8.1) is true on $\xi_{p,q}$, with uniformity in p and q.

If $z = x^2 = e^{2\pi i\tau}$, then

$$f(z) = h^{24}(\tau),$$

in Tannery and Molk's notation; and

$$h^{24}(\mathrm{T}) = (a+b\tau)^{12} h^{24}(\tau) \tag{10.8.2}$$

if

$$\mathrm{T} = \frac{c+d\tau}{a+b\tau}$$

is any substitution of the modular group Γ. We take

$$a = p, \quad b = -q, \quad c = \frac{1+pp'}{q}, \quad d = -p',$$

where p' is defined by $\qquad 1+pp' \equiv 0 \pmod{q}$.

We write $\qquad \theta = \dfrac{2p\pi}{q}+\phi, \quad \tau = \dfrac{p}{q}+\dfrac{i\zeta}{q}.$

Then

$$z = e^{2\pi i\tau} = \exp\left(-\frac{2\pi\zeta}{q}+\frac{2p\pi i}{q}\right) \tag{10.8.3}$$

and

$$-\frac{2\pi\zeta}{q}+\frac{2p\pi i}{q} = -\frac{1}{n}+i\theta,$$

so that

$$\zeta = \frac{q}{2\pi}\left(\frac{1}{n}-i\phi\right). \tag{10.8.4}$$

Also $\qquad \mathrm{T} = \dfrac{c+d\tau}{a+b\tau} = \dfrac{d}{b}-\dfrac{1}{b(a+b\tau)} = \dfrac{p'}{q}+\dfrac{1}{q(p-q\tau)} = \dfrac{p'}{q}+\dfrac{i}{q\zeta},$

and

$$Z = e^{2\pi i\mathrm{T}} = \exp\left(-\frac{2\pi}{q\zeta}+\frac{2p'\pi i}{q}\right). \tag{10.8.5}$$

Finally, (10.8.2), when stated in terms of $g(z)$ and $g(Z)$, is

$$g(Z) = \zeta^{12} g(z). \tag{10.8.6}$$

[1] See § 8.10.

Now

(10.8.7) $$|Z| = e^{-2\pi\Im(T)} = \exp\left\{-2\pi\Re\left(\frac{1}{q\zeta}\right)\right\}$$
$$= \exp\left\{-\frac{4\pi^2 n^{-1}}{q^2(n^{-2}+\phi^2)}\right\}.$$

Also $$|\phi| < \frac{2\pi}{q\nu}, \quad q^2\phi^2 < \frac{4\pi^2}{\nu^2} < \frac{4\pi^2}{n}$$

on $\xi_{p,q}$, and $q^2n^{-2} < \nu^2 n^{-2} < 2n^{-1}$. Hence there are constants $A > 0$ and $\delta < 1$ such that
$$|Z| < e^{-A} = \delta$$
on $\xi_{p,q}$; and

(10.8.8) $$|g(Z)| \leqq |Z| \sum_1^\infty |\tau(n)|\,|Z|^{n-1} < B|Z|,$$

where B is another constant.

It follows from (10.8.4), (10.8.6), (10.8.7) and (10.8.8) that
$$|g(z)| = |\zeta|^{-12}|g(Z)| < B|\zeta|^{-12}|Z|$$
$$\leqq \frac{B}{\{q^2(n^{-2}+\phi^2)\}^6}\exp\left\{-\frac{4\pi^2 n^{-1}}{q^2(n^{-2}+\phi^2)}\right\} = Bn^6\mu^6 e^{-4\pi^2\mu},$$

where $$\mu = \frac{n^{-1}}{q^2(n^{-2}+\phi^2)}.$$

Since $\mu^6 e^{-4\pi^2\mu}$ is bounded for positive μ, it follows that
$$g(z) = O(n^6) = O\{(1-r)^{-6}\}$$
uniformly in p and q. This proves (10.8.1), first on the circle $r = e^{-1/n}$, and then, by the maximum modulus principle, generally.[1]

We have thus proved (10.7.7), and indeed a little more. We have
$$\Sigma\tau^2(m)\,r^{2m} = \frac{1}{2\pi}\int_0^{2\pi}|g(re^{i\theta})|^2\,d\theta = O\{(1-r)^{-12}\}$$

and *a fortiori* $$\sum_1^n \tau^2(m)\,r^{2m} = O\{(1-r)^{-12}\} = O(n^{12}).$$

Also $$r^{2m} = e^{-2m/n} \geqq e^{-2}$$
if $m \leqq n$, so that

(10.8.9) $$\sum_1^n \tau^2(m) = O(n^{12}).$$

This includes (10.7.7), and shows that $\tau(n)$ is $O(n^{\frac{11}{2}})$ "in mean square".

[1] It would be sufficient that (10.8.1) should be true when $r = e^{-1/n}$.

Further properties of $F(s) = \Sigma \dfrac{\tau(n)}{n^s}$

10.9. Rankin's proof of (10.7.10) depends upon a functional equation satisfied by

(10.9.1) $$f(s) = \Sigma \frac{\tau^2(n)}{n^s}.$$

The simpler function $F(s)$ of § 10.3 also satisfies a functional equation, and it will be convenient to investigate this here.

We saw in § 10.3 that

(10.9.2) $$F(s) = \Sigma \frac{\tau(n)}{n^s} = \Pi \frac{1}{1-\tau(p)\,p^{-s}+p^{11-2s}}$$

for sufficiently large σ. It follows from (10.8.9) and Cauchy's inequality that

$$\sum_1^n |\tau(\nu)| = O(n^{\frac{13}{2}}),$$

and that the series and product are absolutely convergent for $\sigma > \frac{13}{2}$. The abscissa of non-absolute convergence depends upon the order of the sum function

(10.9.3) $$\mathrm{T}(x) = \sum_{n \leq x} \tau(n).$$

Now $$\Gamma(s)\,F(s) = \int_0^\infty y^{s-1} g(e^{-y})\,dy,$$

where $g(x)$ is the function (10.1.2), for $\sigma > \frac{13}{2}$.[1] But

$$g(e^{-y}) = \left(\frac{2\pi}{y}\right)^{12} g(e^{-4\pi^2/y})$$ [2]

tends exponentially to zero both when $y \to 0$ and when $y \to \infty$, so that the integral is convergent for all s, and $F(s)$ is an integral function of s. Also

$$\Gamma(s)\,F(s) = (2\pi)^{12} \int_0^\infty y^{s-13} g(e^{-4\pi^2/y})\,dy$$
$$= (2\pi)^{2s-12} \int_0^\infty z^{11-s} g(e^{-z})\,dz = (2\pi)^{2s-12}\,\Gamma(12-s)\,F(12-s),$$

so that $F(s)$ satisfies

(10.9.4) $$(2\pi)^{-s}\,\Gamma(s)\,F(s) = (2\pi)^{s-12}\,\Gamma(12-s)\,F(12-s).$$

[1] When $$\Sigma |\tau(n)| \int_0^\infty y^{\sigma-1} e^{-ny}\,dy = \Gamma(\sigma)\,\Sigma \frac{|\tau(n)|}{n^\sigma} < \infty.$$

[2] This is the special case $q = 1$, $p = 0$ of (10.8.6).

The behaviour of $F(s)$ is "trivial" in the half-plane of absolute convergence; in particular, it has no zeros whose real part exceeds $\frac{13}{2}$. The functional equation then defines its behaviour for $\sigma < \frac{11}{2}$. It has "trivial" zeros for $s = 0, -1, -2, \ldots$, but no other zeros whose real part is less than $\frac{11}{2}$.

The "critical strip", corresponding to the strip $0 \leqq \sigma \leqq 1$ for $\zeta(s)$, is $\frac{11}{2} \leqq \sigma \leqq \frac{13}{2}$; and there are problems concerning its behaviour in this strip corresponding to all the familiar problems about $\zeta(s)$. It has been shown, for example, that there are no zeros on the lines $\sigma = \frac{11}{2}$ and $\sigma = \frac{13}{2}$, and an infinity on $\sigma = 6$. The "Riemann hypothesis" for $F(s)$ is that all the zeros, other than the "trivial" zeros, lie on $\sigma = 6$.

There is another generating function which may be made useful in the study of $\tau(n)$, viz.

(10.9.5) $$\mathfrak{F}(s) = \sum_1^\infty \tau(n) e^{-s\sqrt{n}}.$$

It is easy to prove that

(10.9.6) $$\mathfrak{F}(s) = (2\pi)^{\frac{25}{2}} \Gamma(\tfrac{25}{2})\, s \sum_1^\infty \frac{\tau(n)}{(s^2+4n\pi^2)^{\frac{25}{2}}},$$

but the properties of $\mathfrak{F}(s)$ are not so interesting as those of $F(s)$.

Properties of $f(s) = \Sigma \dfrac{\tau^2(n)}{n^s}$

10.10. We have now to investigate the corresponding properties of the function $f(s)$ of (10.9.1); and we begin by finding an integral representation of $f(s)$, valid for $\sigma > 12$, viz.

(10.10.1) $$2(4\pi)^{-s}\Gamma(s)\zeta(s-22)f(s) = \iint_D y^{s-1} |\Delta(\tau)|^2 \xi(s,x,y)\,dx\,dy.$$

Here $$\tau = x+iy, \quad y>0,$$

$$\Delta(\tau) = e^{2\pi i\tau}\{(1-e^{2\pi i\tau})(1-e^{4\pi i\tau})\ldots\}^{24},[1]$$

(10.10.2) $$\xi(s) = \xi(s,x,y) = \Sigma\Sigma' \frac{1}{|m\tau+n|^{2s-22}},$$

when the dash has its usual meaning, and D is the fundamental region

$$-\tfrac{1}{2} \leqq x \leqq \tfrac{1}{2}, \quad |x+iy| \geqq 1$$

of the modular group.

It follows from (10.8.9) that the series for $f(s)$ is absolutely convergent when $\sigma > 12$. From

$$\Delta(\tau) = \sum_1^\infty \tau(n) e^{2n\pi ix - 2n\pi y}$$

[1] So that $\Delta(\tau) = g(e^{2\pi i\tau})$ in the notation of § 10.1.

it follows, by Parseval's theorem, that

$$\int_{-\frac{1}{2}}^{\frac{1}{2}} |\Delta(\tau)|^2\,dx = \sum_1^\infty \tau^2(n)\,e^{-4n\pi y}.$$

Further, if we suppose $\sigma > 12$, we have

(10.10.3) $$(4\pi)^{-s}\Gamma(s)f(s) = \sum_1^\infty \tau^2(n)\int_0^\infty y^{s-1}e^{-4n\pi y}\,dy$$
$$= \int_0^\infty y^{s-1}\left(\sum_1^\infty \tau^2(n)\,e^{-4n\pi y}\right)dy$$
$$= \int_0^\infty y^{s-1}dy\int_{-\frac{1}{2}}^{\frac{1}{2}} |\Delta(\tau)|^2\,dx = \iint_S y^{s-1}|\Delta(\tau)|^2\,dx\,dy,$$

where S is the strip $-\frac{1}{2} \leqq x \leqq \frac{1}{2}, \quad y > 0.$

Since $\Delta(\tau) = O(e^{-2\pi y})$

for large y, and $\Delta(\tau) = O(y^{-6})$

for small y,[1] uniformly in x, the integral

$$\iint_S y^{\sigma-1}|\Delta(\tau)|^2\,dx\,dy$$

is convergent for $\sigma > 12$, and this justifies the transformations.

10.11. We call a modular substitution

(10.11.1) $$\tau = T(\tau') = \frac{-c+a\tau'}{d-b\tau'},$$

or T, a substitution T_S if it transforms points τ' of the fundamental region D into points τ of a triangle D_T lying in S. Since a, b, c, d and $-a, -b, -c, -d$ give the same substitution, we may suppose that

(10.11.2) $$b \leqq 0, \quad d > 0 \text{ if } b = 0, \quad (b,d) = 1.$$

Any T transforms D into a triangle lying in a strip

$$n-\tfrac{1}{2} \leqq x \leqq n+\tfrac{1}{2},$$

and then $$\tau = T(\tau')-n$$

is a T_S; so that there is a T_S corresponding to any pair (b,d) which satisfies (10.11.1). Further, if (b,d) is such a pair, and (a_0,c_0) a pair which, with (b,d), gives a T_S, then the general solution of

$$ad-bc = 1$$

is $$a = a_0+nb, \quad c = c_0+nd;$$

and then $$T(\tau') = \frac{-c_0+a_0\tau'}{d-b\tau'}-n$$

[1] By (10.8.1).

gives a D_T outside S except when $n = 0$. Hence there is just one T_S corresponding to any pair (b, d) satisfying (10.11.2).

Now

$$(10.11.3) \qquad (4\pi)^{-s}\Gamma(s)f(s) = \sum_{T_S}\iint_{D_T} y^{s-1}|\Delta(\tau)|^2\,dx\,dy.$$

If (10.11.1) is the T_S which transforms D into D_T, then elementary calculations show that

$$y = \frac{y'}{|d-b\tau'|^2}, \quad \left|\frac{d\tau}{d\tau'}\right| = \frac{1}{|d-b\tau'|^2},$$

$$dx\,dy = \left|\frac{d\tau}{d\tau'}\right|^2 dx'dy' = \frac{dx'dy'}{|d-b\tau'|^4}.$$

Also $$\Delta(\tau) = (d-b\tau')^{12}\Delta(\tau').$$

Hence, substituting from (10.11.1) into (10.11.3), writing m for $-b$, n for d, and then dropping the dashes, we obtain

$$(10.11.4) \qquad (4\pi)^{-s}\Gamma(s)f(s) = \sum_{T_S}\iint_D \frac{y^{s-1}}{|d-b\tau|^{2s-22}}|\Delta(\tau)|^2\,dx\,dy$$
$$= \iint_D y^{s-1}|\Delta(\tau)|^2 F(s,\tau)\,dx\,dy,$$

where

$$(10.11.5) \qquad F(s,\tau) = \Sigma\frac{1}{|m\tau+n|^{2s-22}}$$

and the summation is defined by

$$(10.11.6) \qquad 0 \leqq m < \infty, \quad -\infty < n < \infty, \quad (m,n) = 1, \quad n = 1 \text{ if } m = 0.$$

Finally we multiply both sides of (10.11.4) by

$$2\zeta(2s-22) = 2\sum_1^\infty \frac{1}{k^{2s-22}} = \sum_k{}' \frac{1}{|k|^{2s-22}},$$

where the dash excludes the value $k = 0$. It is plain that

$$2\zeta(2s-22)\,F(s,\tau) = \Sigma\Sigma'\frac{1}{|m\tau+n|^{2s-22}} = \xi(s),$$

where the dash now excludes the pair $m = 0$, $n = 0$; and so we obtain (10.10.1).

10.12. We now require certain properties of the function

$$(10.12.1) \qquad K(w) = \Sigma\Sigma'\exp\left(-\frac{\pi w}{y}|m\tau+n|^2\right),$$

where $w > 0$. The most essential is the functional equation

$$(10.12.2) \qquad 1+K(w) = \frac{1}{w}\left\{1+K\left(\frac{1}{w}\right)\right\}.$$

This is a simple corollary of Poisson's summation formula for functions of two variables, viz.

$$\Sigma\Sigma\,\phi(m,n) = \Sigma\Sigma\iint \phi(\xi,\eta)\,e^{2\pi i(m\xi+n\eta)}\,d\xi\,d\eta, \tag{10.12.3}$$

where all summations and integrations are from $-\infty$ to ∞. In this case

$$1+K(w) = \Sigma\Sigma\,\phi(m,n)$$

with

$$\phi(\xi,\eta) = \exp\left\{-\frac{\pi w}{y}\,|\,\xi(x+iy)+\eta\,|^2\right\}.$$

If we substitute this value of ϕ into (10.12.3), and use the formula

$$\iint e^{-\alpha\xi^2-2\beta\xi\eta-\gamma\xi^2+2\pi i(m\xi+n\eta)}\,d\xi\,d\eta$$

$$= \frac{\pi}{\sqrt{(\alpha\gamma-\beta^2)}}\exp\left\{-\frac{\pi^2}{\alpha\gamma-\beta^2}(\alpha n^2-2\beta nm+\gamma n^2)\right\},$$

where α, γ, and $\alpha\gamma-\beta^2$ are positive, we obtain (10.12.2).

We shall also need an upper bound for $K(w)$ for large w. This is

$$K(w) = O(ye^{-\frac{1}{4}\pi w/y}), \tag{10.12.4}$$

and holds uniformly for $w>1$ and $\tau = x+iy$ in D.

We have
$$K(w) = \Sigma\Sigma'\exp\left(-\frac{\pi w}{y}\{(mx+n)^2+m^2y^2\}\right).$$

The combination $m = 0$, $n = 0$ is excluded. Also $y \geqq \frac{1}{2}\sqrt{3}$, since τ is in D.

(i) The terms $m = 0$, $n \neq 0$ give

$$2\sum_1^\infty e^{-n^2\pi w/y} = 2(e^{-\pi w/y}+e^{-4\pi w/y}+\dots) < \frac{2e^{-\pi w/y}}{1-e^{-\pi w/y}},$$

which is $O(y/w)$ when w/y is small and $O(e^{-\pi w/y})$ when w/y is large; and so

$$O\left\{\left(1+\frac{y}{w}\right)e^{-\pi w/y}\right\} = O(ye^{-\frac{1}{4}\pi w/y})$$

in any case.

(ii) There remain the terms for which $m \neq 0$. If we fix m, then there is at most one n for which

$$|\,mx+n\,| < \tfrac{1}{2}.$$

This m and n give $O(e^{-m^2\pi yw})$, and summation with respect to m then gives

$$O(e^{-\pi wy}) = O(ye^{-\frac{1}{4}\pi w/y}).^{1}$$

The other n give two sequences whose numerical values exceed $\frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \dots$; and so (for the fixed m) we get

$$O\{(e^{-\frac{1}{4}\pi w/y}+e^{-\frac{9}{4}\pi w/y}+\dots)\,e^{-m^2\pi wy}\},$$

which, as in (i), is

$$O\{ye^{-\frac{1}{4}\pi w/y}\,.\,e^{-m^2\pi wy}\}.$$

[1] $y > \frac{1}{2}$ and $y > \frac{1}{4}y^{-1}$.

Finally, summing with respect to m, we get

$$O\{ye^{-\frac{1}{4}\pi w/y} . e^{-\pi w y}\} = O(ye^{-\frac{1}{4}\pi w/y}).$$

This completes the proof of (10.12.4).

If y is fixed, then

$$K(w) = O(e^{-Aw}) \tag{10.12.5}$$

for large w and a positive A.

10.13. We can now determine the principal analytical properties of $\xi(s)$. It is defined by (10.10.2) when $\sigma > 12$. Also

$$\left(\frac{y}{\pi}\right)^{s-1} \frac{\Gamma(s-11)}{|m\tau+n|^{2s-22}} = \int_0^\infty w^{s-12} \exp\left(-\frac{\pi w}{y} |m\tau+n|^2\right) dw$$

for $\sigma > 11$, and so

$$\left(\frac{y}{\pi}\right)^{s-1} \Gamma(s-11)\,\xi(s) = \int_0^\infty w^{s-12} \Sigma\Sigma' \exp\left(-\frac{\pi w}{y} |m\tau+n|^2\right) dw \tag{10.13.1}$$
$$= \int_0^\infty w^{s-12} K(w)\, dw,$$

if the summation under the integral sign is legitimate. This will be so if

$$\int_0^\infty w^{\sigma-12} K(w)\, dw < \infty.$$

This integral is convergent at infinity, whatever σ, because of (10.12.5). Near the origin

$$K(w) = -1 + \frac{1}{w} + \frac{1}{w} K\left(\frac{1}{w}\right)$$

behaves like w^{-1}, and the integral is convergent for $\sigma > 12$. Thus (10.13.1) is true for all such σ. Also

$$\left(\frac{y}{\pi}\right)^{s-11} \Gamma(s-11)\,\xi(s) \tag{10.13.2}$$
$$= \int_1^\infty w^{s-12} K(w)\, dw + \int_0^1 w^{s-12} \left\{-1 + \frac{1}{w} + \frac{1}{w} K\left(\frac{1}{w}\right)\right\} dw$$
$$= \int_1^\infty w^{s-12} K(w)\, dw - \frac{1}{s-11} + \frac{1}{s-12} + \int_1^\infty w^{11-s} K(w)\, dw$$
$$= \frac{1}{(s-11)(s-12)} + \int_1^\infty (w^{s-12} + w^{11-s})\, K(w)\, dw.$$

The integral here is an integral function of s, by (10.12.5), for any τ of D; and the right-hand side is unchanged by the substitution of $23-s$ for s.

Thus $\xi(s)$ is regular over the whole plane, except for a simple pole at $s = 12$; and

$$(10.13.3)\qquad \left(\frac{y}{\pi}\right)^{s-11} \Gamma(s-11)\,\xi(s) = \left(\frac{y}{\pi}\right)^{12-s} \Gamma(12-s)\,\xi(23-s).$$

10.14. We deduce the properties of $f(s)$ from those of $\xi(s)$. It follows from (10.10.1) and (10.13.2) that

$$(10.14.1)\qquad 2(4\pi)^{-s}\Gamma(s)\,\Gamma(s-11)\,\zeta(2s-22)f(s)$$

$$= \pi^{s-11}\iint_D y^{10}\,|\Delta(\tau)|^2 \left\{\Gamma(s-11)\left(\frac{y}{\pi}\right)^{s-1}\xi(s)\right\} dx\,dy$$

$$= \frac{\pi^{s-11}}{(s-11)(s-12)}\iint_D y^{10}\,|\Delta(\tau)|^2\,dx\,dy$$

$$+\pi^{s-11}\iint_D y^{10}\,|\Delta(\tau)|^2\,dx\,dy\int_1^\infty (w^{s-12}+w^{11-s})\,K(w)\,dw.$$

Now for any real c, and τ in D, the integral

$$\int_1^\infty w^c K(w)\,dw$$

is majorised, after (10.12.4), by a multiple of

$$y\int_1^\infty w^c e^{-\frac{1}{4}\pi w/y}\,dw.$$

If $c \geqq 0$, we replace the lower limit 1 by 0; if $c < 0$, we omit the w^c and make the same change in the limits. In either case the integral is less than

$$Ay^\gamma,$$

where A and γ depend only on c.[1] Hence the treble integral in (10.14.1) is majorised by

$$A\pi^{\sigma-12}\iint_D y^\delta\,|\Delta(\tau)|^2\,dx\,dy,$$

where A and δ depend only on σ; and this integral is convergent for every σ, because

$$\Delta(\tau) = O(e^{-2\pi y})$$

for large y.

It follows that the treble integral is absolutely and uniformly convergent throughout any bounded domain of values of s, and represents an integral function of s; and that

$$2(4\pi)^{-s}\Gamma(s)\,\Gamma(s-11)\,\zeta(2s-22)f(s)$$

[1] $\gamma = c+2$ if $c \geqq 0$, $\gamma = 2$ if $c < 0$.

is regular for all s except for simple poles at $s = 11$ and $s = 12$. Hence $f(s)$ is a meromorphic function of s regular except for

(i) a simple pole at $s = 12$, with residue

(10.14.2) $$\alpha = 12\frac{(4\pi)^{11}}{\Gamma(12)}\iint_D y^{10}\,|\Delta(\tau)|^2\,dx\,dy.$$

(ii) poles corresponding to the complex zeros of $\zeta(2s-22)$, all lying to the left of $\sigma = 12$.[1]

Finally, it follows from (10.14.1) that

(10.14.3) $$(2\pi)^{-2s}\Gamma(s)\Gamma(s-11)\zeta(2s-22)f(s) = (2\pi)^{2s-46}\Gamma(23-s)\Gamma(12-s)\zeta(24-2s)f(23-s),$$

i.e. that

(10.14.4) $$\phi(s) = \phi(23-s),$$

where

(10.14.5) $$\phi(s) = (2\pi)^{-2s}\Gamma(s)\Gamma(s-11)\zeta(2s-22)f(s).$$

It also follows from these properties of $f(s)$, and a well-known theorem of Ikehara, that

(10.14.6) $$\tau^2(1)+\tau^2(2)+\ldots+\tau^2(n) \sim \tfrac{1}{12}\alpha n^{12}.$$

This was a new theorem; but Rankin has shown that it is possible to go much further by using a theorem of Landau.

10.15. Landau's theorem[2] is as follows. *Suppose that*

(1) $$c_n \geqq 0;$$

(2) $$Z(s) = \Sigma\frac{c_n}{n^s}$$

is absolutely convergent for $\sigma > 1$;

(3) *$Z(s)$ is regular all over the plane, except for a simple pole at $s = 1$, with residue β;*

(4) $$Z(s) = O(e^{\gamma|t|})$$

for large $|t|$ and some $\gamma = \gamma(\sigma_1, \sigma_2)$, in any strip $\sigma_1 \leqq \sigma < \sigma_2$;

(5) $$\Gamma(s)\Gamma(s+11)Z(s) = \Gamma(1-s)\Gamma(12-s)\Sigma e_n(An)^s,$$

the last series being absolutely convergent for $\sigma < 0$;

(6) $$\sum_{n\leqq x} n\,|e_n| = O(x).$$

Then

(10.15.1) $$C(x) = \sum_{c_n\leqq x} c_n = \beta x + O(x^{3/5}).$$

[1] The "trivial" zeros of the zeta-function are cancelled by poles of $\Gamma(s-11)$.

[2] Particularised for the application.

We show that

(10.15.2) $$Z(s) = \zeta(2s) f(s+11) = \Sigma \frac{c_n}{n^s}$$

satisfies these conditions.

(i) Since

(10.15.3) $$Z(s) = \Sigma \frac{1}{n^{2s}} \Sigma \frac{\tau^2(n)}{n^{s+11}},$$

we have

(10.15.4) $$c_n = \sum_{\mu\nu^2 = n} \mu^{-11} \tau^2(\mu) \geqq 0.$$

(ii) Both series in (10.15.3) are absolutely convergent for $\sigma > 1$,[1] and therefore the product series is so.

(iii) We proved that $Z(s)$ satisfies condition (3) in § 10.14. The value of β is

$$\beta = \tfrac{1}{6}\pi^2 \alpha.$$

(iv) It is plain from (10.14.1) that $Z(s)$ satisfies (4).

(v) The functional equation (10.14.3) may be written as

$$\Gamma(s)\,\Gamma(s+11)\,Z(s) = (2\pi)^{4s-2}\,\Gamma(1-s)\,\Gamma(12-s)\,Z(1-s)$$
$$= \Gamma(1-s)\,\Gamma(12-s)\,\Sigma e_n (16\pi^4 n)^s,$$

where $$e_n = \frac{c_n}{4\pi^2 n}.$$

The series is absolutely convergent for $\sigma < 0$.

(vi) It remains only to verify condition (6), which is

$$C(x) = \sum_{n \leqq x} c_n = O(x).$$

Now $$\sum_{\mu \leqq x} \mu^{-11} \tau^2(\mu) = O(x),$$

by (10.8.9) and a partial summation; and, by (10.15.4),

$$C(x) = \sum_{\mu\nu^2 \leqq x} \mu^{-11}\tau^2(\mu) = \sum_{\mu \leqq x} \mu^{-11}\tau^2(\mu) + \sum_{\mu \leqq \frac{1}{4}x} \mu^{-11}\tau^2(\mu) + \sum_{\mu \leqq \frac{1}{9}x} \mu^{-11}\tau^2(\mu) + \dots$$
$$= O(x) + O(\tfrac{1}{4}x) + O(\tfrac{1}{9}x) + \dots = O(x).$$

Hence $Z(s)$ satisfies all the conditions (1)–(6), and (10.15.1) is true.

The formula which we actually use, however, is not (10.15.1) but

(10.15.5) $$D(x) = \sum_{n \leqq x} n^{11} c_n = \tfrac{1}{12}\beta x^{12} + O(x^{12-\frac{2}{5}}).$$

[1] The second by (10.8.9).

This follows from (10.15.1) by partial summation. In fact, if $[x] = \xi$, we have

$$\begin{aligned} D(x) &= 1^{11}C(1) + \sum_{2}^{\xi} \nu^{11}\{C(\nu) - C(\nu-1)\} \\ &= \xi^{11}C(\xi) - \sum_{1}^{\xi-1} \{(\nu+1)^{11} - \nu^{11}\}\, C(\nu) \\ &= \{x^{11} + O(x^{10})\}\{\beta x + O(x^{\frac{3}{5}})\} - \sum_{1}^{\xi-1} \{11\nu^{10} + O(\nu^{9})\}\{\beta\nu + O(\nu^{\frac{3}{5}})\} \\ &= \beta x^{12}(1 - \tfrac{11}{12}) + O(x^{12-\frac{2}{5}}) = \tfrac{1}{12}\beta x^{12} + O(x^{12-\frac{2}{5}}). \end{aligned}$$

10.16. It is now easy to prove that

(10.16.1) $$\tau^2(1) + \tau^2(2) + \ldots + \tau^2(n) = \tfrac{1}{12}\alpha n^{12} + O(n^{12-\frac{2}{5}}),$$

and to deduce Rankin's theorem (10.7.10).

It follows from (10.15.3) that

$$\sum \frac{\tau^2(n)}{n^s} = \frac{Z(s-11)}{\zeta(2s-22)} = \sum \frac{n^{11}c_n}{n^s} \sum \frac{n^{22}\mu(n)}{n^{2s}},$$

$$\tau^2(n) = \sum_{kl^2=n} k^{11}c_k \,.\, l^{22}\mu(l),$$

$$\begin{aligned} \sum_{n \leq x} \tau^2(n) &= \sum_{kl^2 \leq x} k^{11}c_k \,.\, l^{22}\mu(l) \\ &= \sum_{l \leq x^{\frac{1}{2}}} l^{22}\mu(l) \sum_{k \leq x/l^2} k^{11}c_k = \sum_{l \leq x^{\frac{1}{2}}} l^{22}\mu(l)\, D\left(\frac{x}{l^2}\right) \\ &= \sum_{l \leq x^{\frac{1}{2}}} l^{22}\mu(l) \left\{\frac{\pi^2\alpha}{72}\left(\frac{x}{l^2}\right)^{12} + O\left(\frac{x}{l^2}\right)^{12-\frac{2}{5}}\right\}, \end{aligned}$$

by (10.15.5). The main term here gives

$$\frac{\pi^2\alpha}{72} x^{12} \sum_{l \leq x^{\frac{1}{2}}} \frac{\mu(l)}{l^2} = \frac{\pi^2\alpha}{72} x^{12} \left\{\frac{1}{\zeta(2)} + O(x^{-\frac{1}{2}})\right\} = \tfrac{1}{12}\alpha x^{12} + O(x^{12-\frac{1}{2}}),$$

and the error term gives

$$O(x^{12-\frac{2}{5}}) \sum_{l \leq x^{\frac{1}{2}}} l^{-\frac{6}{5}} = O(x^{12-\frac{2}{5}}).$$

Hence we obtain (10.16.1).

Finally, replacing n by $n-1$ in (10.16.1), and subtracting, we find that

$$\tau^2(n) = O(n^{11}) + O(n^{12-\frac{2}{5}}) = O(n^{12-\frac{2}{5}}),$$

$$\tau(n) = O(n^{6-\frac{1}{5}}),$$

which is (10.7.10).

The sum function $\mathrm{T}(n)$

10.17. There is another set of problems connected with the "sum function"

(10.17.1) $$\mathrm{T}(n) = \sum_{n \leq x}{}' \tau(n)$$

of $\tau(n)$. The dash over the Σ implies, as usual, that when x is an integer the last term $\tau(x)$ is to be replaced by $\frac{1}{2}\tau(x)$.

I can explain these problems best by stating their analogues for a more familiar arithmetical function. If $r(n) = r_2(n)$ is the number of representations of n as the sum of two squares, then the sum function

(10.17.2) $$R(x) = \sum_{n \leqq x}' r(n)$$

is the number of lattice points in the circle

$$u^2+v^2 \leqq x,$$

with the convention that we count $\frac{1}{2}$ instead of 1 for a lattice point on the circumference of the circle. It is familiar that

(10.17.3) $$R(x) = \pi x + \mathrm{P}(x),$$

where $$\mathrm{P}(x) = O(x^{\frac{1}{2}});$$

but the true order of $\mathrm{P}(x)$ is still obscure. Sierpinski proved in 1906 that

$$\mathrm{P}(x) = O(x^{\frac{1}{3}}),$$

and this result has been improved by van der Corput and later writers. In the other direction, Landau and I proved that

$$\mathrm{P}(x) \neq O(x^{\frac{1}{4}}).$$

Another problem is that of the "identity" for $\mathrm{P}(x)$. It was stated by Voronoi in 1905 that

(10.17.4) $$\mathrm{P}(x) = x^{\frac{1}{2}} \sum_{1}^{\infty} \frac{r(n)}{n^{\frac{1}{2}}} J_1\{2\pi(nx)^{\frac{1}{2}}\},$$

where J_1 is the Bessel function of order 1. I proved this in 1915, and many other proofs have been published since.

There are similar "order" and "identity" problems for $\mathrm{T}(x)$. They are naturally less important, and any order results are bound to be imperfect so long as we are uncertain about the true order of $\tau(n)$ itself. The best that we can prove at present is that

(10.17.5) $$\mathrm{T}(x) = O(x^{\frac{59}{10}}).$$

If the Ramanujan hypothesis is true, then

(10.17.6) $$\mathrm{T}(x) = O(x^{\frac{35}{6}+\epsilon}).$$

It is certainly not true that

(10.17.7) $$\mathrm{T}(x) = o(x^{\frac{23}{4}}).$$

In this case the identity is perhaps more interesting. It is

(10.17.8) $$\mathrm{T}(x) = x^6 \sum_{1}^{\infty} \frac{\tau(n)}{n^6} J_{12}\{4\pi(nx)^{\frac{1}{2}}\}.$$

Wilton proved that

$$(10.17.9)\quad \mathrm{T}_\alpha(x) = \frac{1}{\Gamma(\alpha+1)}\sum_{n\leq x}(x-n)^\alpha\tau(n) = x^{6+\frac{1}{2}\alpha}\sum_1^\infty \frac{\tau(n)}{n^{6+\frac{1}{2}\alpha}}J_{12+\alpha}\{4\pi(nx)^{\frac{1}{2}}\}$$

for $\alpha > 0$, and I proved more recently that this result is still true for $\alpha = 0$.

NOTES ON LECTURE X

§§ 10.1–3. Ramanujan, in no. 18 of the *Papers*, considers the function $\psi_\alpha(n)$ defined by

$$x\{(1-x^{24/\alpha})(1-x^{48/\alpha})\ldots\}^\alpha = \Sigma\psi_\alpha(n)x^n,$$

where $\alpha \mid 24$. When $\alpha = 24$, $\psi_\alpha(n)$ is $\tau(n)$.

When α is 1 or 3, $\psi_\alpha(n)$ is given by the identities of Euler and Jacobi. When $\alpha = 12$, $\psi_\alpha(n)$ is a multiple of the $e_{12}(n)$ of § 9.3. It is 0 if n is even, and can be defined when n is odd by a sum extended over the representations of n as a sum of 4 squares; the formula will be found in Glaisher's paper referred to in the note on § 9.1.

Ramanujan conjectured that $\psi_\alpha(n)$ is multiplicative in every case, so that $\Sigma n^{-s}\psi_\alpha(n)$ has a product expression analogous to (10.3.8); and he deduced formulae for $\psi_\alpha(n)$ when α is 2, 4, 6 and 8. All these conjectures were confirmed in Mordell's paper **2**. Mordell says there that the multiplicative property of $\psi_{12}(n)$ had been proved before by Glaisher, but this seems to be incorrect: see Glaisher, *Quarterly Journal of Math.* 37 (1906), 36–38.

Rankin, in an unpublished manuscript, has found elementary proofs of the multiplicative property of $\psi_{12}(n)$ and $\psi_{16}(n)$. These are substantially Glaisher's $\Omega(n)$ and $\Theta(n)$.

§ 10.4. The properties discussed in §§ 10.4–6 were all enunciated by Ramanujan in a manuscript "Properties of $p(n)$ and $\tau(n)$" now in Prof. Watson's possession (see also no. 28 of the *Papers*). This manuscript has never been published in full, but no. 30 of the *Papers* is substantially an extract from it, and Watson, in his papers **23** and **24**, has since rewritten and completed a good deal more.

The proofs in § 10.4 were given by Mordell (**1**), but stand also in Ramanujan's manuscript: Ramanujan of course assumes the multiplicative property of $\tau(n)$.

§§ 10.5–6. The proofs here are derived partly from my old notes on Ramanujan's manuscript and partly from Watson's paper **23**. Watson works out in detail the proof sketched in § 10.6.

If we replace x by q^2, then P, Q, R are

$$\frac{12\eta_1\omega_1}{\pi^2},\quad \frac{12g_2\omega_1^4}{\pi^4},\quad \frac{216g_2\omega_1^6}{\pi^6},$$

in the ordinary notation of elliptic functions; and (10.5.3) is equivalent to

$$\Delta = g_2^3 - 27g_3^2.$$

See § 9.11.

§§ 10.7–8. See Hardy (**9**); Hecke, *Hamburg math. Abhandlungen*, 5 (1927), 199–224; Kloosterman, *ibid.* 337–352; Salié, *Math. Zeitschrift*, 36 (1932), 263–278; Davenport, *Journal für Math.* 169 (1933), 158–176; Rankin (**2**). Davenport does not mention $\tau(n)$ explicitly, but (10.7.9) is, after Kloosterman's work, an immediate corollary of what he proves.

Everything that has been proved about the order of $\tau(n)$ has its analogue for the coefficients c_n in the expansion

$$\sum_1^\infty c_n e^{2n\pi i\tau/N}$$

of any modular form of 'Stufe' N and dimension $-k$ which vanishes at all the rational points corresponding to cusps on the real axis in the modular figure. Here 'vanishing' means "having the limit 0 for approach along a perpendicular to the axis". The special function $g(e^{2\pi i\tau})$ is of Stufe 1 and weight -12. The general ideas of the theory of such modular functions were laid down by Hecke, who proves the result for general modular forms which corresponds to (10.7.7).

The proof of (10.7.7) in Hardy (9) is arranged differently, the underlying Farey dissection not being mentioned explicitly.

I show there that there are constants A and B such that

$$An^{12} < \tau^2(1) + \tau^2(2) + \ldots + \tau^2(n) < Bn^{12}.$$

This result is now superseded by Rankin's theorems (10.14.6) and (10.16.1).

§ 10.9. The functional equation (10.9.4) must have been familiar to Ramanujan, but I cannot find an explicit statement of it either in the *Papers* or in the note-books. It was first stated in print by Wilton (1), who proves it as a special case of a functional equation for

$$F(s, p, q) = \Sigma \frac{\tau(n)}{n^s} e^{2np\pi i/q}.$$

Wilton (1) proved that $F(s)$ has an infinity of zeros on $\sigma = 6$, and Rankin (1) that it has none on $\sigma = \frac{13}{2}$ or $\sigma = \frac{11}{2}$.

§§ 10.10–16. The proof is that given by Rankin (2), with a few small simplifications. His proof of the corresponding theorems for functions of Stufe N is more complex.

§ 10.12. For Poisson's formula see, for example, Bochner, *Vorlesungen über Fourier-sche Integrale*, 33–38 and 203–208, or Titchmarsh, *Fourier integrals*, 60–68. The case here is a simple one.

§ 10.14. Ikehara's theorem, in the form relevant for its application here, runs: *if $a_n \geqq 0$, and $\Sigma a_n n^{-s}$ is convergent for $\sigma > 1$ and represents a function regular for $\sigma \geqq 1$, except for a simple pole, with residue C, at $s = 1$, then*

$$\sum_{n \leq x} a_n \sim Cx.$$

The theorem (which has been referred to already in § 2.8) is based entirely on the ideas of Wiener. There is a simple proof of the theorem, in a rather more general form, in a paper by Bochner in *Math. Zeitschrift*, 37 (1933), 1–9.

§ 10.15. See Landau, *Göttinger Nachrichten* (1915), 209–243.

§ 10.17. The 'circle' problem has been referred to already in Lecture V (§ 5.1 and the note on that section).

My first proof of the identity (10.17.4) occurs in a paper in the *Quarterly Journal of Math.* 48 (1915), 263–283. A number of other proofs have been given since; for references see Hardy and Landau, *Proc. Royal Soc.* (A), 105 (1923), 244–258. Perhaps the best proof is that in Landau, *Vorlesungen*, ii, 221–232.

Of the theorems about $T(x)$ stated at the end of the section, (10.17.5) is Rankin's, (10.17.6), (10.17.7), and (10.17.8) my own. For (10.17.5) see Rankin (3), for (10.17.9), Wilton (1), and for (10.17.8), Hardy (10). No proofs of (10.17.6) and (10.17.7) have been published. My own proof of (10.17.7) depended on the identity (10.9.6).

XI

DEFINITE INTEGRALS

11.1. In this lecture I propose to speak about some theorems of Ramanujan which have not attracted very much attention, which are, as I said in my opening lecture, "inevitably less impressive" than much of his work, but which are still very interesting and will repay a careful analysis.

Ramanujan was occupied, during most of the last year before he came to England, with general formulae in the theory of definite integrals. He then held a research scholarship in Madras, and submitted three quarterly reports to the University on the progress of his researches. It is from these that most of the formulae which I quote are taken. I owe my knowledge of the contents of these reports to Professor Watson, who included them in his copy of Ramanujan's notebooks; but most of the formulae which I quote had been shown to me by Ramanujan himself.

11.2. The formulae which I take as my text are as follows.

(A) $$\int_0^\infty x^{s-1}\{\phi(0)-x\phi(1)+x^2\phi(2)-\ldots\}\,dx = \frac{\pi}{\sin s\pi}\phi(-s).$$

(B) $$\int_0^\infty x^{s-1}\left\{\lambda(0)-\frac{x}{1!}\lambda(1)+\frac{x^2}{2!}\lambda(2)-\ldots\right\}dx = \Gamma(s)\,\lambda(-s).$$

(C) $$\phi(0)+\phi(1)+\phi(2)+\ldots$$
$$=\int_0^\infty \phi(x)\,dx+\int_0^\infty \frac{\phi(0)-x\phi(1)+x^2\phi(2)-\ldots}{x\{\pi^2+(\log x)^2\}}\,dx.$$

(D 1) $$\int_0^\infty\left\{\lambda(0)-\frac{x}{1!}\lambda(1)+\frac{x^2}{2!}\lambda(2)-\ldots\right\}\cos yx\,dx$$
$$=\lambda(-1)-\lambda(-3)\,y^2+\lambda(-5)\,y^4-\ldots.$$

(D 2) $$\int_0^\infty\left\{\lambda(0)-\frac{x}{1!}\lambda(1)+\frac{x^2}{2!}\lambda(2)-\ldots\right\}\sin yx\,dx$$
$$=\lambda(-2)\,y-\lambda(-4)\,y^3+\lambda(-6)\,y^5-\ldots.$$

(E) $$\int_0^\infty\left\{\lambda(0)-\frac{x}{1!}\lambda(1)+\ldots\right\}\left\{\mu(0)-\frac{x}{1!}\mu(1)+\ldots\right\}dx$$
$$=\lambda(-1)\,\mu(0)-\lambda(-2)\,\mu(1)+\ldots.$$

(F) If

(F1) $$\int_0^\infty F(\alpha x)\,G(\beta x)\,dx = \frac{1}{\alpha+\beta}$$

and

(F2) $$f(s) = \int_0^\infty F(x)\,x^{s-1}\,dx, \quad g(s) = \int_0^\infty G(x)\,x^{s-1}\,dx,$$

then

(F3) $$f(s)\,g(1-s) = \frac{\pi}{\sin s\pi}.$$

And if

(F4) $$\int_0^\infty A(x)\,\tfrac{1}{2}\{F(yxi)+F(-yxi)\}\,dx = B(y),$$

then

(F5) $$\int_0^\infty B(x)\,\tfrac{1}{2}\{G(yxi)+G(-yxi)\}\,dx = \tfrac{1}{2}\pi A(y).$$

I shall say something about each of these formulae in turn, but I must begin with some general remarks about Ramanujan's treatment of all of them. He had no real proofs of any of the formulae; and here I am using the phrase "real proof" not quite in its ordinary sense but in one which I must explain.

It is reasonable to say that we now know, roughly, the conditions for the truth of most analytical formulae. We can say, for example, that a formula like (A) is true for certain ϕ and certain s, and that the conditions which we impose on ϕ and s are "natural" conditions; they do not intrude merely on account of the weakness of our analysis, but are genuine limitations corresponding broadly to the facts. Our theorems will not cover all cases in which the formula is true, and it may be interesting and profitable to do what we can to extend them; but the conditions under which we have proved them would become insufficient if widened in any really drastic way.

A mathematician may have stated a formula and advanced reasons for its truth which are inadequate as they stand, in which case he cannot be said to have "proved" it. But it often happens that his method, when restated and developed by a modern analyst, leads to a proof valid under "natural" conditions, and in that case we may fairly say that he has "really" proved the theorem. Thus Euler "really" proved large parts of the classical analysis, and there are a great many theorems which Ramanujan had "really" proved; but he had not "really" proved any of the formulae which I have quoted. It was impossible that he should have done so because the "natural" conditions involve ideas of which he knew nothing in 1914, and which he had hardly absorbed before his death. He had also, as Littlewood says, no clear-cut conception of proof: "if a significant piece of reasoning occurred somewhere, and the total mixture of evidence and intuition gave him certainty, he looked no further". In this case any "real" proof was inevitably beyond his grasp, and the "significant pieces of reasoning" which are indicated in the notebooks and reports, though we shall find them curious and interesting, are quite inadequate for the occasion.

Formulae (A) *and* (B)

11.3. The first two formulae are the most important, and I shall discuss them in more detail than the rest. Ramanujan was particularly fond of them, and used them as one of his commonest tools. They are variants of one another, (A) becoming (B) when we write

$$\phi(u) = \frac{\lambda(u)}{\Gamma(1+u)};$$

and we may take (A) as the standard form of the formulae, though we shall sometimes find (B) more convenient.

It is easy to give examples of both the truth and the falsity of (A). Thus

$$\phi(u) = 1, \quad \phi(u) = \frac{1}{\Gamma(1+u)}$$

give the formulae

$$\int_0^\infty \frac{x^{s-1}}{1+x}\,dx = \frac{\pi}{\sin s\pi}, \quad \int_0^\infty e^{-x}x^{s-1}\,dx = \frac{\pi}{\sin s\pi\,\Gamma(1-s)} = \Gamma(s),$$

true for $0<s<1$ and for $s>0$ respectively. On the other hand the formula is plainly false when

$$\phi(u) = \sin \pi u,$$

the integral then vanishing identically. The formula is an "interpolation formula", which defines $\phi(-s)$ in terms of $\phi(0), \phi(1), \ldots$. If it has been proved for all $\phi(u)$ of a certain class, then any $\phi(u)$ of that class vanishes identically if it vanishes for all non-negative integral values of u. There is a well-known theorem of Carlson which says that if (i) $\phi(u)$ is regular, and

$$|\phi(u)| < Ce^{A|u|},$$

where $A<\pi$, in the right-hand half-plane of complex values of u, and (ii) $\phi(0) = \phi(1) = \ldots = 0$, then $\phi(u)$ vanishes identically; and it is natural to suppose that (A) should be true for all such $\phi(u)$ and appropriate s. We may also expect that the best methods for the discussion of the formula will be those with which Carlson and others have familiarised us and whose original source is to be found in Mellin.

Proof of (A)

11.4. In what follows I write

$$u = v + iw.$$

I denote the half-plane $v \geqq -\delta$, where $\delta > 0$, by $H(\delta)$. For the present I suppose

$$0 < \delta < 1.$$

If $\phi(u)$ is regular, and

(11.4.1) $$|\phi(u)| < Ce^{Pv + A|w|}$$

throughout $H(\delta)$, then I shall say that $\phi(u)$ belongs to the class $\mathfrak{K}(A, P, \delta)$, or simply $\mathfrak{K}$. We shall be concerned for the most part with functions for which

(11.4.2) $$A < \pi.$$

Let us suppose now that $\phi(u)$ belongs to $\mathfrak{K}(A, P, \delta)$, with $A < \pi$, $0 < \delta < 1$; that $0 < c < \delta$; and that

(11.4.3) $$0 < x < e^{-P}.$$

Then a simple application of Cauchy's theorem gives

(11.4.4)
$$\Phi(x) = \phi(0) - x\phi(1) + x^2\phi(2) - \ldots = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{\pi}{\sin \pi u}\phi(-u)x^{-u}\,du.$$

The series is usually divergent if $x > e^{-P}$, but the integrand is majorised, for all positive x, by a multiple of

$$e^{-(\pi-A)|w|}e^{-Pc}x^{-c},$$

and the integral converges uniformly in any interval $0 < x_0 \leqq x \leqq X$; and therefore $\Phi(x)$ is regular, and represented by the integral, for all positive x.[1]

We can now deduce (A) by the well-known inversion formula of Mellin. This is expressed by the equations

$$f(u) = \int_0^\infty F(x)\,x^{u-1}\,dx, \quad F(x) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} f(u)\,x^{-u}\,du;$$

and (A) will follow if we can take

$$f(u) = \frac{\pi}{\sin u\pi}\phi(-u), \quad F(x) = \Phi(x).$$

We may justify the use of Mellin's formula in various ways, appealing, if we please, to some established general theorem; but it is more interesting to give a direct proof on the lines originally followed by Mellin himself.

[1] If $x = re^{i\theta}$, then $|x^{-c-iw}| = r^{-c}e^{\theta w}$, and $\Phi(x)$ is regular for $|\theta| < \pi - A$; but we do not use this.

Suppose that $s = \sigma + it$ and

$$0 < \sigma < \delta,$$

and choose c_1 and c_2 so that

$$0 < c_1 < \sigma < c_2 < \delta.$$

Then (11.4.4) is true both with $c = c_1$ and with $c = c_2$.

Now

$$\int_0^1 \Phi(x)\, x^{s-1} dx = \frac{1}{2\pi i}\int_0^1 x^{s-1} dx \int_{c_1-i\infty}^{c_1+i\infty} \frac{\pi}{\sin \pi u}\phi(-u)\, x^{-u} du$$

$$= \frac{1}{2\pi i}\int_{c_1-i\infty}^{c_1+i\infty} \frac{\pi}{\sin \pi u}\phi(-u)\, du \int_0^1 x^{s-u-1} dx = \frac{1}{2\pi i}\int_{c_1-i\infty}^{c_1+i\infty} \frac{\pi}{\sin \pi u}\frac{\phi(-u)}{s-u} du,$$

since $\Re(s-u) = \sigma - c_1 > 0$. There is no difficulty in the inversion, the double integral being absolutely convergent. Similarly

$$\int_1^\infty \Phi(x)\, x^{s-1} dx = \frac{1}{2\pi i}\int_{c_2-i\infty}^{c_2+i\infty} \frac{\pi}{\sin \pi u}\phi(-u)\, du \int_1^\infty x^{s-u-1} dx$$

$$= -\frac{1}{2\pi i}\int_{c_2-i\infty}^{c_2+i\infty} \frac{\pi}{\sin \pi u}\frac{\phi(-u)}{s-u} du,$$

since now $\Re(s-u) = \sigma - c_2 < 0$. Combining these equations we obtain

$$\int_0^\infty \Phi(x)\, x^{s-1} dx = \frac{1}{2\pi i}\left(\int_{c_1-i\infty}^{c_1+i\infty} - \int_{c_2-i\infty}^{c_2+i\infty}\right) \frac{\pi}{\sin \pi u}\frac{\phi(-u)}{s-u} du = \frac{\pi}{\sin s\pi}\phi(-s),$$

by Cauchy's theorem.

11.5. We have thus proved (A) for a $\phi(u)$ of $\mathfrak{K}$ and for $0 < \sigma < \delta$, in particular for $0 < s < \delta$. It is plain that, if $\phi(u) = O(e^{A|u|})$, then $\phi(u)$ belongs to $\mathfrak{K}$, with $P = A$, so that our class of $\phi(u)$ includes the Carlson class. Hence (A) gives incidentally a proof of Carlson's theorem.

We may also reduce the second part of the proof to an application of Fourier's theorem. If we put $x = e^{-\xi}$, $u = c + iw$, in (11.4.4), it becomes

$$\Phi(e^{-\xi}) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{\pi}{\sin \pi(c+iw)}\phi(-c-iw)\, e^{\xi(c+iw)} dw;$$

and Fourier's theorem then gives

$$\frac{\pi\phi(-c-iw)}{\sin \pi(c+iw)} = \int_{-\infty}^{\infty} e^{-\xi(c+iw)}\Phi(e^{-\xi})\, d\xi = \int_0^\infty x^{c+iw-1}\Phi(x)\, dx.$$

Such a reduction of a "Mellin" to a "Fourier" transformation is of course always possible, the transformations being formal variants of one another.

I add one remark which applies not only here but often later in the lecture. I have supposed $A < \pi$, in which case all the integrals occurring are absolutely convergent, and with a good deal to spare. In many of the most interesting examples A will be equal to π; and the final integral may not be absolutely convergent, in which case a more delicate argument, not depending simply on majorisation, will be needed. The hypothesis $A < \pi$ is, here and elsewhere, a crude hypothesis; but it is at any rate a "natural" one, the results being usually false when $A > \pi$; and I shall have no time for any more subtle analysis.

Heuristic deductions of (A) *and* (B)

11.6. The formulae (A) and (B) are connected in a very interesting way with "Newton's interpolation formula".

It is well known that

$$e^x \sum_0^\infty (-1)^n \frac{a_n}{n!} x^n = \sum_0^\infty \frac{\Delta^n a_0}{n!} x^n,$$

where $\Delta a_0 = a_0 - a_1, \quad \Delta^2 a_0 = a_0 - 2a_1 + a_2, \ldots.$

Hence, if $s > 0$ and the series

$$\Lambda(x) = \lambda(0) - \frac{\lambda(1)}{1!} x + \frac{\lambda(2)}{2!} x^2 - \ldots$$

is convergent for all x, we have

$$\int_0^\infty \Lambda(x) x^{s-1} dx = \int_0^\infty e^{-x} x^{s-1} \sum_0^\infty \frac{\Delta^n \lambda(0)}{n!} x^n dx;$$

and term-by-term integration gives

$$\sum_0^\infty \frac{\Delta^n \lambda(0)}{n!} \int_0^\infty e^{-x} x^{s+n-1} dx = \Gamma(s) \left\{ \lambda(0) + \frac{s}{1!} \Delta\lambda(0) + \frac{s(s+1)}{2!} \Delta^2 \lambda(0) + \ldots \right\}.$$

Finally

$$\lambda(-s) = \lambda(0) + \frac{s}{1!} \Delta\lambda(0) + \frac{s(s+1)}{1.2} \Delta^2 \lambda(0) + \ldots \tag{11.6.1}$$

is Newton's formula.

The term-by-term integration can always be justified when the resulting series is convergent. We can therefore prove (B) on the assumptions that (i) s is positive, (ii) $\Lambda(x)$ is an integral function, and (iii) Newton's formula holds for $\lambda(-s)$. If (i) and (ii) are satisfied, then (B) is equivalent to Newton's formula.

There is a corresponding reduction of (A), based on Euler's transformation

$$x = \frac{y}{1-y}, \quad y = \frac{x}{1+x}.$$

It is convenient to suppose that $\phi(0) = 0$. Then, writing b_n for $\phi(n)$, we have formally

$$b_1 x - b_2 x^2 + b_3 x^3 - \ldots = b_1 . y + \Delta b_1 . y^2 + \Delta^2 b_1 . y^3 + \ldots,$$

$$\int_0^\infty \Phi(x) x^{s-1} dx = -\int_0^\infty x^{s-1} \sum_0^\infty \Delta^n b_1 \left(\frac{x}{1+x}\right)^{n+1} dx$$

$$= -\sum_0^\infty \Delta^n b_1 \int_0^\infty \frac{x^{s+n}}{(1+x)^{n+1}} dx = -\Gamma(-s) \sum_0^\infty \Delta^n b_1 \frac{\Gamma(s+n+1)}{\Gamma(n+1)}$$

$$= \frac{\pi}{\sin s\pi}\left\{b_1 + (s+1)\Delta b_1 + \frac{(s+1)(s+2)}{2!}\Delta^2 b_1 + \ldots\right\}.$$

In this way (A) also may be reduced to Newton's formula.[1]

Ramanujan's argument

11.7. Ramanujan supposes, in effect, that

(11.7.1) $$\lambda(u) = \chi(e^{-au}),$$

where a is positive and $\chi(z)$ is regular at the origin. Then

$$\Lambda(x) = \sum_{n=0}^\infty \frac{(-1)^n}{n!} x^n \chi(e^{-an})$$

$$= \sum_{n=0}^\infty \frac{(-1)^n}{n!} x^n \sum_{r=0}^\infty \frac{\chi^r(0)}{r!} e^{-arn}$$

$$= \sum_{r=0}^\infty \frac{\chi^r(0)}{r!} \sum_{n=0}^\infty \frac{(-1)^n}{n!} x^n e^{-arn} = \sum_{r=0}^\infty \frac{\chi^r(0)}{r!} e^{-xe^{-ar}},$$

and $$\int_0^\infty \Lambda(x) x^{s-1} dx = \sum_{r=0}^\infty \frac{\chi^r(0)}{r!} \int_0^\infty e^{-xe^{-ar}} x^{s-1} dx$$

$$= \Gamma(s) \sum_{r=0}^\infty \frac{\chi^r(0)}{r!} e^{ars} = \Gamma(s)\chi(e^{as}) = \Gamma(s)\lambda(-s)$$

for $s > 0$. The inversions can be justified if (i) $\chi(z)$ is regular for $|z| \leqq 1$, and (ii) s is sufficiently small. For then

$$\Sigma \frac{|\chi^r(0)|}{r!} Z^r$$

is convergent for some $Z > 1$,

$$\Sigma \frac{|\chi^r(0)|}{r!} \Sigma \frac{1}{n!} x^n e^{-arn} = \Sigma \frac{|\chi^r(0)|}{r!} e^{xe^{-ar}}$$

is convergent, and

$$\Sigma \frac{|\chi^r(0)|}{r!} \int_0^\infty e^{-xe^{-ar}} x^{s-1} dx = \Gamma(s) \Sigma \frac{|\chi^r(0)|}{r!} e^{ars}$$

[1] (11.6.1) with $\phi(u+1)$ for $\lambda(u)$ and $s+1$ for s.

is convergent for sufficiently small positive s. In these circumstances Ramanujan's analysis is valid. But the condition on $\lambda(u)$, which demands that $\lambda(n)$ should be of the form

$$c_0+c_1e^{-an}+c_2e^{-2an}+\dots$$

for large n, is extremely stringent and, though it may be extended a little, excludes practically all of Ramanujan's examples.

It is instructive to consider a case in which an argument like Ramanujan's leads to a demonstrably false result. Suppose that, in (11.7.1), $a=-b$ is negative. Then, if we repeat Ramanujan's calculations, we obtain

$$\text{(11.7.2)} \qquad \Lambda(x)=\sum_{r=0}^{\infty}\frac{\chi^r(0)}{r!}e^{-xe^{br}},$$

and can apparently complete the proof as before. But (11.7.2) is usually false, since $\Lambda(x)$ is regular at the origin, while the series on the right represents a function of x which is regular in the right-hand half-plane and has the imaginary axis as a singular line. Thus the assumptions

$$\chi(u)=e^{-cu} \quad (c>0), \qquad b=\log 2, \qquad \lambda(u)=e^{-c2^u}$$

would lead to $$\sum_0^{\infty}(-1)^n\frac{x^n}{n!}e^{-c2^n}=\sum_0^{\infty}(-1)^r\frac{c^r}{r!}e^{-x2^r},$$

which is obviously false.

Examples and applications

11.8. I quote a few of Ramanujan's special cases.

(i) If $0<s<\operatorname{Min}(\alpha,\beta)$, then

$$\int_0^{\infty}x^{s-1}F(\alpha,\beta,\gamma,-x)\,dx=\frac{\Gamma(\gamma)}{\Gamma(\alpha)\Gamma(\beta)}\frac{\Gamma(s)\Gamma(\alpha-s)\Gamma(\beta-s)}{\Gamma(\gamma-s)}.$$

Here $F(\alpha,\beta,\gamma,x)$ is the hypergeometric function, defined for $0<x<1$ by the usual power-series and for $x>1$ by analytic continuation.

(ii) If $0<s<1$, then

$$\int_0^{\infty}x^{s-1}(1^{-a}-2^{-a}x+3^{-a}x^2-\dots)\,dx=\frac{\pi}{\sin s\pi}(1-s)^{-a}.$$

(iii) If $0<q<1$, $s>0$, and $0<a<q^{s-1}$, then

$$\int_0^{\infty}x^{s-1}\frac{(1+aqx)(1+aq^2x)\dots}{(1+x)(1+qx)(1+q^2x)\dots}\,dx=\frac{\pi}{\sin s\pi}\prod_1^{\infty}\frac{(1-q^{m-s})(1-aq^m)}{(1-q^m)(1-aq^{m-s})}.$$

(iv) If $0<s<\frac{1}{2}$, then

$$\int_0^{\infty}x^{s-1}\sum_0^{\infty}\frac{(-1)^nx^n}{n!\,\zeta(2n+2)}\,dx=\frac{\Gamma(s)}{\zeta(2-2s)}.$$

Of these, (i) and (ii) are straightforward corollaries of what we have proved. To prove (iii), we use the expansion

$$\Phi(x) = \frac{(1+aqx)(1+aq^2x)\dots}{(1+x)(1+qx)(1+q^2x)\dots} = \sum_0^\infty (-1)^n \frac{(1-aq)(1-aq^2)\dots(1-aq^n)}{(1-q)(1-q^2)\dots(1-q^n)} x^n,$$

which is easily deduced from the functional equation

$$(1+aqx)\Phi(qx) = (1+x)\Phi(x)$$

satisfied by $\Phi(x)$. Here

$$\phi(u) = \prod_{m=1}^\infty \frac{(1-aq^m)(1-q^{m+u})}{(1-q^m)(1-aq^{m+u})}.$$

Finally (iv) is a formula found independently by M. Riesz, and used by him to determine a very curious necessary and sufficient condition for the truth of the Riemann hypothesis. On this hypothesis, the formula is true for $0 < s < \frac{3}{4}$. An alternative form of $\Phi(x)$ is

$$\Phi(x) = \sum_1^\infty \frac{\mu(m)}{m} e^{-x/m^2},$$

where $\mu(m)$ is the Möbius function.

11.9. We can use Ramanujan's formulae to obtain many expansions usually derived from Lagrange's or Burmann's series. I take two examples given by Ramanujan himself.

(i) A problem familiar in text-books is that of expanding e^{-ax} in powers of xe^{bx}. Ramanujan argues as follows. If $y = xe^{bx}$ and

$$e^{-ax} = \sum_0^\infty (-1)^n \frac{\lambda(n)}{n!} y^n,$$

then, by (B),

$$\Gamma(s)\lambda(-s) = \int_0^\infty y^{s-1}e^{-ax}\,dy = \int_0^\infty x^{s-1}(1+bx)\,e^{-(a-sb)x}\,dx$$

$$= \Gamma(s)\,a(a-sb)^{-s-1};$$

so that

$$\lambda(n) = a(a+nb)^{n-1}.$$

The argument obviously requires that a and b shall be positive.[1]

(ii) It is well known that the roots of trinomial equations can be expanded as hypergeometric series. Ramanujan finds an expansion for any power x^r of a root of

$$aqx^p + x^q = 1$$

as follows. If

$$x^r = \sum_0^\infty \frac{(-1)^n\lambda(n)}{n!} a^n,$$

[1] When the function

$$\frac{a(a+bu)^{u-1}}{\Gamma(1+u)}$$

does not quite satisfy the conditions of § 11.4 (the "A" being π).

then, by (B),

$$\Gamma(s)\lambda(-s) = \int_0^\infty a^{s-1}x^r da = \int_0^1 x^r\left(\frac{1-x^q}{qx^p}\right)^{s-1} d\left(\frac{1-x^q}{qx^p}\right).$$

Calculating the integral, he finds the expansion

$$x^r = 1 - \frac{r}{1!}a + \frac{r(r+2p-q)}{2!}a^2 - \frac{r(r+3p-q)(r+3p-2q)}{3!}a^3 + \dots.$$

I do not know whether this formula is new, and I have not attempted to find conditions under which the analysis can be justified.

Formula (C)

11.10. The formula (C) may be regarded as one for the remainder in the "Euler-Maclaurin sum formula". Since

$$\int_0^\infty \frac{dx}{x\{\pi^2+(\log x)^2\}} = \int_{-\infty}^\infty \frac{dy}{\pi^2+y^2} = 1,$$

we may multiply $\phi(0)$, in the two places where it occurs, by any constant factor. It is convenient to write the formula as

$$\tfrac{1}{2}\phi(0)+\phi(1)+\phi(2)+\dots-\int_0^\infty \phi(x)\,dx = \int_0^\infty \frac{\frac{1}{2}\phi(0)-x\phi(1)+x^2\phi(2)-\dots}{x\{\pi^2+(\log x)^2\}}\,dx$$

or, replacing $\phi(u)$ by $y^u\phi(u)$, as

(11.10.1) $$R(y) = \int_0^\infty \frac{\Phi_1(-yx)}{x\{\pi^2+(\log x)^2\}}\,dx,$$

where now

(11.10.2) $$\Phi_1(x) = \tfrac{1}{2}\phi(0)+\sum_1^\infty \phi(n)\,x^n,$$ [1]

and

(11.10.3) $$R(y) = \Phi_1(y) - \int_0^\infty y^x\phi(x)\,dx.$$

The most familiar formulae for $R(y)$ are Poisson's, viz.

(11.10.4) $$R(y) = 2\sum_1^\infty \int_0^\infty y^x\phi(x)\cos 2n\pi x\,dx,$$

and Plana's, viz.

(11.10.5) $$R(y) = i\int_0^\infty \frac{y^{iw}\phi(iw)-y^{-iw}\phi(-iw)}{e^{2\pi w}-1}\,dw.$$

Ramanujan's is apparently new.

[1] $\Phi_1(x) = \Phi(-x) - \frac{1}{2}\phi(0)$, in the notation of § 11.4.

It is convenient to begin by proving a special case of the formula. Suppose that

$$\phi(u) = \frac{1}{\Gamma(1+u)}$$

and

$$J(y) = \int_0^\infty y^x \phi(x)\,dx = \int_0^\infty \frac{y^x}{\Gamma(1+x)}\,dx.$$

Then

$$\int_0^\infty e^{-sy} J(y)\,dy = \int_0^\infty \frac{dx}{\Gamma(1+x)} \int_0^\infty e^{-sy} y^x\,dy$$

$$= \int_0^\infty s^{-x-1}\,dx = \frac{1}{s \log s},$$

if $s > 1$. It follows from the inversion formula of Laplace that

$$J(y) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{e^{ys}}{s \log s}\,ds,$$

where $c > 1$. If we deform the contour into a lacet round the negative real axis, and allow for the residue at the pole $s = 1$, we obtain

$$J(y) = e^y - \int_0^\infty \frac{e^{-yx}}{x\{\pi^2 + (\log x)^2\}}\,dx.$$

11.11. Ramanujan gives a very ingenious proof of this special formula. He proves, more generally, that

(11.11.1)

$$\int_{-\xi}^\infty \frac{y^x}{\Gamma(1+x)}\,dx + \int_0^\infty x^{\xi-1} e^{-yx} \left(\cos \pi\xi - \frac{\sin \pi\xi}{\pi} \log x\right) \frac{dx}{\pi^2 + (\log x)^2} = e^y$$

if $y \geqq 0$ and $\xi \geqq 0$. If we denote the left-hand side of (11.11.1) by $p(y, \xi)$, and differentiate with respect to ξ, we find that

$$\frac{\partial p}{\partial \xi} = \frac{y^{-\xi}}{\Gamma(1-\xi)} - \frac{\sin \pi\xi}{\pi} \int_0^\infty x^{\xi-1} e^{-yx}\,dx$$

$$= y^{-\xi}\left\{\frac{1}{\Gamma(1-\xi)} - \frac{\Gamma(\xi) \sin \pi\xi}{\pi}\right\} = 0.$$

Hence $p(y, \xi)$ is a function $P(y)$ of y only. But if we differentiate with respect to y, we obtain

$$\frac{dP}{dy} = \int_{-\xi}^\infty \frac{y^{x-1}}{\Gamma(x)}\,dx - \int_0^\infty x^{\xi} e^{-yx} \left(\cos \pi\xi - \frac{\sin \pi\xi}{\pi} \log x\right) \frac{dx}{\pi^2 + (\log x)^2}$$

$$= \int_{-\xi-1}^\infty \frac{y^x}{\Gamma(x+1)}\,dx - \int_0^\infty x^{\xi} e^{-yx} \left(\cos \pi\xi - \frac{\sin \pi\xi}{\pi} \log x\right) \frac{dx}{\pi^2 + (\log x)^2}$$

$$= p(y, \xi+1) = P(y).$$

Hence $P(y) = Ce^y$, and we can find C by supposing that $y = 0$ and $\xi = 0$.[1]

[1] We can also prove (11.11.1) by the "Laplace transform" method.

We may now replace $\phi(u)$ by

$$\phi(u)-\frac{\phi(0)}{\Gamma(1+u)}$$

in the general formula, so that we may suppose $\phi(0)=0$. We may also suppose that $\phi(u)$ is real for real u. These reductions are not essential, but they simplify our formal analysis.[1]

11.12. We can deduce Ramanujan's formula from Plana's as follows. We suppose again that $\phi(u)$ belongs to $\mathfrak{K}(A,P,\delta)$. Then

$$(11.12.1)\qquad \Phi_1(-x)=\sum_1^\infty(-1)^n\phi(n)x^n=\frac{1}{2\pi i}\int_{-i\infty}^{i\infty}\frac{\pi}{\sin\pi u}x^{-u}\phi(-u)\,du$$

$$=\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{\pi}{\sin\pi iw}x^{-iw}\phi(-iw)\,dw;$$

we can now take $c=0$ because $\phi(0)=0$. Hence

$$\int_0^\infty\frac{\Phi_1(-yx)}{x\{\pi^2+(\log x)^2\}}dx=\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{\pi}{\sin\pi iw}y^{-iw}\phi(-iw)\,dw\int_0^\infty\frac{x^{-iw}}{x\{\pi^2+(\log x)^2\}}dx.$$

But $$\int_0^\infty\frac{x^{-iw}}{x\{\pi^2+(\log x)^2\}}dx=\int_{-\infty}^{\infty}\frac{e^{-iwt}}{\pi^2+t^2}dt=e^{-\pi|w|},$$

and so $$\int_0^\infty\frac{\Phi_1(-yx)}{x\{\pi^2+(\log x)^2\}}dx=\frac{1}{2}\int_{-\infty}^{\infty}\frac{e^{-\pi|w|}}{\sin\pi iw}y^{-iw}\phi(-iw)\,dw,$$

which reduces to Plana's integral (11.10.5) by trivial transformations.

There is a full discussion of Plana's formula in Lindelöf's *Calcul des résidus*. Lindelöf proves (11.10.5) under the conditions that

$$\text{(i)}\qquad e^{-2\pi|w|}y^{v+iw}\phi(v+iw)\to 0$$

when $|w|\to\infty$, uniformly in any finite strip $-\delta\leqq v\leqq V$, and

$$\text{(ii)}\qquad \int_{-\infty}^{\infty}e^{-2\pi|w|}|y^{v+iw}||\phi(v+iw)|\,dw\to 0$$

when $v\to\infty$. It is plain that these conditions are satisfied if $\phi(u)$ belongs to $\mathfrak{K}(A,P,\delta)$, with $A<2\pi$, and

$$0<y<e^{-P};$$

and in particular, for such y, by the $\phi(u)$ considered here. Thus Ramanujan's formula (C) holds, in the form (11.10.1), and for these y, for the same $\phi(u)$ for which we proved (A).

[1] We could begin by verifying the formula for any special $\phi(u)$ for which $\phi(0)\neq 0$, but I know of none for which (C) becomes quite trivial.

11.13. I add a remark about Plana's formula. It may be proved directly as a corollary of Cauchy's theorem: but it is worth while noticing how it can be deduced from Poisson's formula, which, being a "real variable" formula, is probably now more familiar. Let us write Poisson's formula (11.10.4) as

$$R(y) = 2\sum_1^\infty J_n;$$

write $e^{-\beta}$, where $\beta > P$, for y; and v for x. Then

$$J_n = \Re\left\{\int_0^\infty \phi(v)\, e^{-(\beta-2n\pi i)v}\, dv\right\} = \Re\left\{i\int_0^\infty \phi(iw)\, e^{-\beta iw-2n\pi w}\, dw\right\},$$

again by Cauchy's theorem; the argument here demands only that $A < 2\pi$.[1] When we sum, we obtain Plana's formula.

11.14. Suppose, for example, that

$$\phi(u) = \frac{\Gamma(r+u)}{\Gamma(r)\,\Gamma(1+u)},$$

where $r > 0$. Then $\phi(u)$ belongs to $\mathfrak{K}$ for $0 < \delta < r$, any positive A, and any P; and (11.10.1) gives

$$(1-y)^{-r} - \int_0^\infty \frac{\Gamma(r+x)}{\Gamma(r)\,\Gamma(1+x)}\, y^x dx = \int_0^\infty \frac{(1+yx)^{-r}}{x\{\pi^2+(\log x)^2\}}\, dx.$$

The integral on the left is elementary when r is an integer, and then that on the right can be evaluated in terms of elementary functions.

Fourier transformations

11.15. The formulae (D) embody a heuristic theory of Fourier transforms, a theory which is naturally valid only under very restrictive conditions.

Suppose, for example, that $\lambda(u)$ is an integral function, and that

(11.15.1) $$\lambda(1) = \lambda(3) = \lambda(5) = \ldots = 0,$$

so that

(11.15.2) $$A(x) = \sum_0^\infty \frac{(-1)^n \lambda(n)}{n!}\, x^n = \sum_0^\infty \frac{\lambda(2m)}{2m!}\, x^{2m}$$

is even; and let

(11.15.3) $$\lambda_1(u) = \sqrt{\left(\frac{2}{\pi}\right)}\, \Gamma(1+u) \cos \tfrac{1}{2}u\pi\, \lambda(-u-1).$$

Then $\lambda_1(u)$ is also integral, since the poles of $\Gamma(1+u)$ at $-1, -3, \ldots$ are cancelled by zeros of $\cos\frac{1}{2}u\pi$, and those at $-2, -4, \ldots$ by zeros of $\lambda(-u-1)$; and

(11.15.4) $$\lambda_1(1) = \lambda_1(3) = \lambda_1(5) = \ldots = 0.$$

[1] Instead of $A < \pi$.

Also
$$\lambda_1(2m) = (-1)^m \sqrt{\left(\frac{2}{\pi}\right)}\, 2m!\,\lambda(-2m-1)$$
and
$$\lambda_1(-2m-1) = \sqrt{\left(\frac{2}{\pi}\right)}\,\lambda(2m) \lim_{\epsilon\to 0} \{\Gamma(-2m+\epsilon)\cos\tfrac{1}{2}(2m+1-\epsilon)\pi\}$$
$$= (-1)^m \sqrt{\left(\frac{\pi}{2}\right)}\,\frac{\lambda(2m)}{2m!}.$$
Hence, if we write
$$\mathrm{M}(x) = \sqrt{\left(\frac{2}{\pi}\right)} \sum_0^\infty (-1)^m \lambda(-2m-1)\,x^{2m}$$
and denote by Λ_1, M_1 the functions formed from λ_1 as Λ, M are formed from λ, then
$$\Lambda_1(x) = \mathrm{M}(x), \quad \mathrm{M}_1(x) = \Lambda(x);$$
and (D 1) and the corresponding formula with λ_1 become
$$\sqrt{\left(\frac{2}{\pi}\right)} \int_0^\infty \Lambda(x)\cos yx\,dx = \mathrm{M}(y), \quad \sqrt{\left(\frac{2}{\pi}\right)} \int_0^\infty \mathrm{M}(x)\cos yx\,dx = \Lambda(y).$$

These are the ordinary formulae of the Fourier cosine transformation. If we suppose, instead of (11.15.1), that
$$\lambda(0) = \lambda(2) = \lambda(4) = \ldots = 0,$$
and define $\lambda_1(u)$ by
$$\lambda_1(u) = \sqrt{\left(\frac{2}{\pi}\right)}\,\Gamma(1+u)\sin\tfrac{1}{2}u\pi\,\lambda(-u-1),$$
we are led similarly to the formulae of the sine transformation.

Thus
$$\lambda(u) = \frac{2^{\frac{1}{2}u}\sqrt{\pi}}{\Gamma(\frac{1}{2}-\frac{1}{2}u)} = 2^{-1-\frac{1}{2}u}\frac{\Gamma(-\frac{1}{2}u)}{\Gamma(-u)}$$
satisfies (11.15.1). In this case
$$\lambda_1(u) = \sqrt{\left(\frac{2}{\pi}\right)}\,\Gamma(1+u)\cos\tfrac{1}{2}u\pi\,.\,\frac{2^{-\frac{1}{2}+\frac{1}{2}u}\Gamma(\frac{1}{2}+\frac{1}{2}u)}{\Gamma(1+u)} = \frac{2^{\frac{1}{2}u}\sqrt{\pi}}{\Gamma(\frac{1}{2}-\frac{1}{2}u)} = \lambda(u)$$
and
$$\Lambda(x) = \mathrm{M}(x) = \sum_0^\infty \frac{(-1)^m}{m!}(\tfrac{1}{2}x^2)^m = e^{-\frac{1}{2}x^2}.$$
The formulae express the "self-reciprocal" property of $e^{-\frac{1}{2}x^2}$.

On the other hand there are many familiar reciprocities which cannot be expressed in this way. Thus

$$(11.15.5) \quad \int_0^\infty e^{-x}\cos yx\,dx = \frac{1}{1+y^2}, \quad \int_0^\infty \frac{\cos yx}{1+x^2}\,dx = \tfrac{1}{2}\pi e^{-|y|}.$$

The first of these is (when $|y| < 1$) a case of (D 1), with $\lambda(u) = 1$; but then
$$\lambda_1(u) = \sqrt{\left(\frac{2}{\pi}\right)}\,\Gamma(1+u)\cos\tfrac{1}{2}u\pi$$

is not an integral function, so that we cannot account for the second formula in this way (as is obvious because $e^{-|y|}$ is not expansible as a power series in y).

11.16. I worked out a theory of the formulae (D) in my paper **11**. I shall state it here in a rather more general form due to Mr F. M. Goodspeed; his theory accounts for the formulae (11.15.5).

Suppose that $\chi(u)$ is an integral function and that

$$(11.16.1) \qquad \left|\frac{\chi(u)}{2^{\frac{1}{2}|v|}\Gamma(\frac{1}{2}|v|+\frac{1}{2}iw)}\right| < Ce^{P|v|+A|w|},$$

where $A<\pi$, for all u. Then it follows from Cauchy's theorem that

$$(11.16.2) \qquad \Lambda(x) = \sum_0^\infty \frac{(-1)^n\chi(n)}{2^{\frac{1}{2}n}\Gamma(\frac{1}{2}n+\frac{1}{2})}x^n = -\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{\pi}{\sin\pi u}\frac{x^u\chi(u)}{2^{\frac{1}{2}u}\Gamma(\frac{1}{2}u+\frac{1}{2})}du$$

when $\qquad -1<c<0, \quad 0<x<e^{-P},$

the series being then convergent. The integral is uniformly convergent in any interval $0<\delta\leqq x\leqq\Delta<\infty$, so that $\Lambda(x)$ is regular for all positive x.

If now $y>0$ then

$$(11.16.3) \qquad \sqrt{\left(\frac{2}{\pi}\right)}\int_0^\infty \Lambda(x)\cos yx\,dx$$

$$= -\sqrt{\left(\frac{2}{\pi}\right)}\int_0^\infty \cos yx\,dx\,\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{\pi}{\sin\pi u}\frac{x^u\chi(u)}{2^{\frac{1}{2}u}\Gamma(\frac{1}{2}u+\frac{1}{2})}du$$

$$= -\sqrt{\left(\frac{2}{\pi}\right)}\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{\pi}{\sin\pi u}\frac{\chi(u)}{2^{\frac{1}{2}u}\Gamma(\frac{1}{2}u+\frac{1}{2})}du\int_0^\infty x^u\cos yx\,dx.$$

The justification of the inversion is not quite trivial but proceeds on the same lines as the discussion of the same point in **11**.

Now $\qquad \int_0^\infty x^u\cos yx\,dx = \Gamma(u+1)\cos\frac{1}{2}(u+1)\pi\, y^{-u-1}.$

If we substitute this value in (11.16.3), and simplify by means of the duplication formula for the Gamma-function, we obtain

$$\sqrt{\left(\frac{2}{\pi}\right)}\int_0^\infty \Lambda(x)\cos yx\,dx = -\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{\pi}{\sin\pi u}2^{\frac{1}{2}u+\frac{1}{2}}\frac{y^{-u-1}\chi(u)}{\Gamma(-\frac{1}{2}u)}du$$

$$= -\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{\pi}{\sin\pi u}\frac{y^u\chi(-u-1)}{2^{\frac{1}{2}u}\Gamma(\frac{1}{2}+\frac{1}{2}u)}du$$

(with a different c, but still with $-1<c<0$). This integral may again be calculated by Cauchy's theorem, with the result

$$(11.16.4) \qquad \sqrt{\left(\frac{2}{\pi}\right)}\int_0^\infty \Lambda(x)\cos yx\,dx = \mathrm{M}(y),$$

where

(11.16.5) $$\mathrm{M}(x) = \sum_0^\infty \frac{(-1)^n \chi(-n-1)}{2^{\frac{1}{2}n}\Gamma(\frac{1}{2}n+\frac{1}{2})} x^n.$$

This series also is convergent for $0 < x < e^{-P}$, and $\mathrm{M}(x)$ is regular for all positive x. And it is plain from the symmetry of the argument that

(11.16.6) $$\sqrt{\left(\frac{2}{\pi}\right)} \int_0^\infty \mathrm{M}(x) \cos yx\, dx = \Lambda(y).$$

11.17. Ramanujan's own formulae for cosine transforms are slightly less general. If we write

$$\frac{\chi(u)}{2^{\frac{1}{2}u}\Gamma(\frac{1}{2}u+\frac{1}{2})} = \frac{\lambda(u)}{\Gamma(u+1)}$$

or $$\chi(u) = \frac{2^{-\frac{1}{2}u}\sqrt{\pi}}{\Gamma(\frac{1}{2}u+1)} \lambda(u),$$

and suppose that $\lambda(u)$ satisfies the conditions (11.15.1), then it will be found that the formulae of § 11.16 reduce to those of § 11.15. The "order" conditions are equivalent. The analysis of § 11.16 has the advantage of symmetry, and is more general in two respects. In the first place, the assumption that $\chi(u)$ is integral is less exacting than the assumption that $\lambda(u)$ is integral, since it allows poles of $\lambda(u)$ for $u = -2, -4, \ldots$. A more important point is that, in § 11.15, both $\Lambda(x)$ and $\mathrm{M}(x)$ are even analytic functions regular at the origin, while in § 11.16 they are defined by the power series for positive x only, and then for negative x by evenness.[1]

Thus, if we take $$\chi(u) = \frac{2^{\frac{1}{2}u}\Gamma(\frac{1}{2}u+\frac{1}{2})}{\Gamma(u+1)},$$

then $$\Lambda(x) = e^{-x}, \quad \mathrm{M}(x) = \sqrt{\left(\frac{2}{\pi}\right)} \frac{1}{1+x^2}$$

for $x > 0$. The conditions of § 11.16 are satisfied, and, if we define $\Lambda(x)$ and $\mathrm{M}(x)$ by evenness for $x < 0$, we obtain the formulae (11.15.5).

If we take $$\chi(u) = 1$$

we find that $$\Lambda(x) = \mathrm{M}(x) = \sum_0^\infty \frac{(-1)^n}{2^{\frac{1}{2}n}\Gamma(\frac{1}{2}n+\frac{1}{2})} x^n$$

[1] In (D 1) we are in an intermediate position. If we write the formula as

$$\sqrt{\left(\frac{2}{\pi}\right)} \int_0^\infty \Lambda(x) \cos yx\, dx = \mathrm{M}(y),$$

then $\Lambda(x)$ is defined like the $\Lambda(x)$ of § 11.16, but $\mathrm{M}(y)$ is regular at the origin. In § 11.15 we make the relation symmetrical by specialisation.

for $x > 0$, and the same remarks apply. This $\Lambda(x)$, which may also be expressed by

$$1 - xe^{\frac{1}{2}x^2}\int_x^\infty e^{-\frac{1}{2}t^2}\,dt = \int_0^\infty e^{-\frac{1}{2}w^2 - xw}\,w\,dw,$$

is one of the examples of "self-reciprocal" functions given by Hardy and Titchmarsh.

Finally, the choice

$$\chi(u) = \frac{\pi 2^{-u}}{\Gamma(\frac{1}{2} - \frac{1}{2}u)\,\Gamma(1 + \frac{1}{2}u)}$$

leads to

$$\Lambda(x) = \mathrm{M}(x) = e^{-\frac{1}{2}x^2}.$$

This case is covered by § 11.15.

11.18. I said in my introductory lecture that Ramanujan was familiar with most of the formal ideas which underlie the recent work of Watson, and of Titchmarsh and myself, on "Fourier kernels" and "self-reciprocal functions". Here we have a case in point.

It is plain that a necessary and sufficient condition for the $\Lambda(x)$ of § 11.16 to be self-reciprocal is that

(11.18.1) $$\chi(u) = \chi(-u-1).$$

If we write

(11.18.2) $$\psi(u) = \pi^{-1}\Gamma(\tfrac{1}{2} + \tfrac{1}{2}u)\,\Gamma(1 - \tfrac{1}{2}u)\,\chi(-u),$$

then (11.18.1) becomes

(11.18.3) $$\psi(u) = \psi(1-u).$$

The integral expression for $\Lambda(x)$ is

$$\Lambda(x) = -\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{\pi}{\sin \pi u}\,\frac{x^u \chi(u)}{2^{\frac{1}{2}u}\Gamma(\frac{1}{2}u + \frac{1}{2})}\,du.$$

If we change u into $-u$, substitute for $\chi(-u)$ from (11.18.2), and perform some simple reductions, we obtain

(11.18.4) $$\Lambda(x) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} 2^{\frac{1}{2}u-1}\Gamma(\tfrac{1}{2}u)\,x^{-u}\psi(u)\,du,$$

which is (apart from a numerical factor) the formula for self-reciprocal functions given by Titchmarsh and myself.

All this has its analogue in the theory of "general transforms", but I postpone this to the end of the lecture.

Formula (E)

11.19. We may write (E) as

(11.19.1) $$\int_0^\infty \Lambda(x)\,\mathrm{M}(x)\,dx = \sum_0^\infty (-1)^n \lambda(-n-1)\,\mu(n),$$

$\mathit{\Lambda}(x)$ and $\mathrm{M}(x)$ being the two infinite series which occur in the integrand of (E). We may as well consider a more general formula, viz.

$$(11.19.2)\qquad \int_0^\infty \mathit{\Lambda}(x)\,\mathrm{M}(x)\,x^{s-1}\,dx = \sum_0^\infty (-1)^n \frac{\Gamma(s+n)}{n!}\lambda(-n-s)\,\mu(n)$$

$$= \Gamma(s)\left\{\lambda(-s)\,\mu(0) - \frac{s}{1}\lambda(-s-1)\,\mu(1) + \frac{s(s+1)}{1.2}\lambda(-s-2)\,\mu(2) - \ldots\right\}.$$

We can deduce (11.19.2) formally from (B) by writing

$$\int_0^\infty \mathit{\Lambda}(x)\,\mathrm{M}(x)\,x^{s-1}\,dx = \sum_0^\infty \frac{(-1)^n}{n!}\mu(n)\int_0^\infty \mathit{\Lambda}(x)\,x^{n+s-1}\,dx$$

$$= \sum_0^\infty (-1)^n \frac{\Gamma(s+n)}{n!}\lambda(-s-n)\,\mu(n),$$

and there are some few cases in which this procedure can be justified. They are however extremely special; and Ramanujan seems to have had no other argument, even when $\mathrm{M}(x)$ is $\cos yx$, when proof on these lines is quite impossible. In this case

$$\mu(2m) = (-1)^m y^{2m}, \quad \mu(2m+1) = 0,$$

and (11.19.1) reduces to (D1).

The formula becomes simpler when expressed in the notation of § 11.4. Then

$$\phi(u) = \frac{\lambda(u)}{\Gamma(1+u)}, \quad \psi(u) = \frac{\mu(u)}{\Gamma(1+u)},$$

$\Phi(x)$ is defined as in § 11.4 and $\Psi(x)$ similarly, and (11.19.2) takes the form

$$(11.19.3)\qquad \int_0^\infty \Phi(x)\,\Psi(x)\,x^{s-1}\,dx = \frac{\pi}{\sin s\pi}\sum_0^\infty \phi(-n-s)\,\psi(n).$$

This formula is simpler, but rather misleading, and (11.19.2) is really the better standard form. We can see this by considering the most obvious special cases.

If we take $\qquad \lambda(u) = \alpha^u, \quad \mu(u) = \beta^u,$

where $0<\beta<\alpha$, in (11.19.2), we obtain

$$\int_0^\infty e^{-(\alpha+\beta)x}x^{s-1}\,dx = \Gamma(s)\left\{\alpha^{-s} - \frac{s}{1}\alpha^{-s-1}\beta + \frac{s(s+1)}{1.2}\alpha^{-s-2}\beta^2 + \ldots\right\} = \frac{\Gamma(s)}{(\alpha+\beta)^s},$$

which is true for all positive s. But if we take $\phi(u) = \alpha^u$, $\psi(u) = \beta^u$ in (11.19.3), we obtain

$$\int_0^\infty \frac{x^{s-1}\,dx}{(1+\alpha x)(1+\beta x)} = \frac{\pi}{\sin s\pi}(\alpha^{-s} + \alpha^{-s-1}\beta + \alpha^{-s-2}\beta^2 + \ldots) = \frac{\pi}{\sin s\pi}\frac{\alpha^{1-s}}{\alpha-\beta},$$

which is always false.

The correct formula is

$$\int_0^\infty \frac{x^{s-1}dx}{(1+\alpha x)(1+\beta x)} = \frac{\pi}{\sin s\pi}\frac{\alpha^{1-s}-\beta^{1-s}}{\alpha-\beta},$$

which is true for $0 < s < 2$. We can write the right-hand side as

$$\frac{\pi}{\sin s\pi}(\alpha^{-s}+\alpha^{-s-1}\beta+\alpha^{-s-2}\beta^2+\ldots-\alpha^{-1}\beta^{1-s}-\alpha^{-2}\beta^{2-s}-\ldots)$$

$$=\frac{\pi}{\sin s\pi}\{\phi(-s)\psi(0)+\phi(-s-1)\psi(1)+\ldots-\phi(-1)\psi(1-s)-\phi(-2)\psi(2-s)-\ldots\}.$$

Here there are two series instead of one, and in fact, as we shall see in a moment, the "right" formula for the general integral is not (11.19.3) but

(11.19.4)

$$\int_0^\infty \Phi(x)\Psi(x)x^{s-1}dx = \frac{\pi}{\sin s\pi}\left\{\sum_0^\infty \phi(-n-s)\psi(n) - \sum_0^\infty \phi(-n-1)\psi(n+1-s)\right\}.$$

This reduces to (11.19.3) when $\phi(u)$ vanishes for $u = -1, -2, \ldots$, as when

$$\phi(u) = \frac{\alpha^u}{\Gamma(1+u)}.$$

It is therefore better to work with (11.19.2). I shall not do more than indicate the connection of this formula with the standard formulae of the theory of Mellin transforms. There is a rigorous investigation in an unpublished paper of Goodspeed.

11.20. We may regard (11.19.2) as a form of the "Parseval" theorem for Mellin transforms. This theorem asserts that, if

(11.20.1) $$f(x) = \int_0^\infty F(x)x^{s-1}dx, \quad g(s) = \int_0^\infty G(x)x^{s-1}dx,$$

then, under certain conditions,

(11.20.2) $$\int_0^\infty F(x)G(x)x^{s-1}dx = \frac{1}{2\pi i}\int_{k-i\infty}^{k+i\infty} f(u)g(s-u)\,du.$$

Now in certain circumstances, which we considered in detail in § 11.4, we have

(11.20.3)

$$\int_0^\infty \Lambda(x)x^{s-1}dx = \Gamma(s)\lambda(-s), \quad \int_0^\infty \mathrm{M}(x)x^{s-1}dx = \Gamma(s)\mu(-s),$$

(11.20.4)

$$\Lambda(x) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\Gamma(u)\lambda(-u)x^{-u}du, \quad \mathrm{M}(x) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\Gamma(u)\mu(-u)x^{-u}du.$$

In fact the formulae (11.20.3) are cases of (B), and if we write

$$\phi(u) = \frac{\lambda(u)}{\Gamma(1+u)}, \quad \psi(u) = \frac{\mu(u)}{\Gamma(1+u)},$$

then the formulae (11.20.4) become cases of (11.4.4).

Taking $F(x) = \Lambda(x)$, $G(x) = \mathrm{M}(x)$ in (11.20.1), (11.20.2) gives

$$\int_0^\infty F(x)\, G(x)\, x^{s-1}\, dx = \frac{1}{2\pi i}\int_{k-i\infty}^{k+i\infty} \Gamma(u)\, \Gamma(s-u)\, \lambda(-u)\, \mu(u-s)\, du.$$

If this is true for a k between 0 and σ, then the last integral, when calculated by residues on the right, is

$$\sum_0^\infty \frac{(-1)^n}{n!}\Gamma(s+n)\, \lambda(-s-n)\, \mu(n),$$

and we obtain (11.19.2). The whole argument, of course, requires careful consideration.

If we argue with ϕ and ψ instead of with λ and μ, we have

$$\int_0^\infty \Phi(x)\, x^{s-1}\, dx = \frac{\pi}{\sin s\pi}\phi(-s), \quad \int_0^\infty \Psi(x)\, x^{s-1}\, dx = \frac{\pi}{\sin s\pi}\psi(-s),$$

$$\int_0^\infty \Phi(x)\, \Psi(x)\, x^{s-1}\, dx = \frac{1}{2\pi i}\int_{k-i\infty}^{k+i\infty} \frac{\pi}{\sin u\pi}\, \frac{\pi}{\sin(s-u)\pi}\, \phi(-u)\, \psi(u-s)\, du,$$

and when we calculate this integral by residues we are led to (11.19.4). It is only when $\phi(u)$ vanishes for negative integral values of u that our earlier formula (11.19.3) is correct.

11.21. Formula (11.19.2) is a very powerful tool for the evaluation of definite integrals. If for example

$$\lambda(u) = \frac{1}{\Gamma(a+1+u)}(\tfrac{1}{2}\alpha)^{2u}, \quad \mu(u) = \frac{1}{\Gamma(b+1+u)}(\tfrac{1}{2}\beta)^{2u},$$

then $$\Lambda(x) = 2^a \alpha^{-a} x^{-\frac{1}{2}a} J_a(\alpha\sqrt{x}), \quad \mathrm{M}(x) = 2^b \beta^{-b} x^{-\frac{1}{2}b} J_b(\beta\sqrt{x}).$$

The formula reduces, after some trivial transformations,[1] to

$$\int_0^\infty J_a(\alpha x)\, J_b(\beta x)\, x^{s-1}\, dx = 2^{s-1}\alpha^{-s-b}\beta^b \frac{\Gamma(\frac{1}{2}s+\frac{1}{2}a+\frac{1}{2}b)}{\Gamma(b+1)\, \Gamma(1-\frac{1}{2}s-\frac{1}{2}a-\frac{1}{2}b)}$$
$$\times F\left(\tfrac{1}{2}s+\tfrac{1}{2}a+\tfrac{1}{2}b, \tfrac{1}{2}s-\tfrac{1}{2}a+\tfrac{1}{2}b, b+1, \frac{\beta^2}{\alpha^2}\right),$$

which is true whenever $0 < \beta < \alpha$ and the integral is convergent.[2]

[1] x^2 for x, $\frac{1}{2}(s+a+b)$ for s.

[2] I.e. if $-a-b < s < 2$.

Formulae (F)

11.22. The last group of formulae contain a further justification of the assertion of my own which I quoted at the beginning of § 11.18. I shall regard them, as did Ramanujan, simply as formulae, and say nothing at all here about conditions for their validity. Suppose that, in our previous notation,

$$\lambda(u) = \alpha^u p(u), \quad \mu(u) = \beta^u q(u),$$

where $0 < \beta < \alpha$ and

$$p(u)\, q(-u-1) = 1.$$

Then

$$A(x) = F(\alpha x), \quad \mathrm{M}(x) = G(\beta x),$$

where

$$F(x) = \sum_0^\infty \frac{(-1)^n}{n!} p(n)\, x^n, \quad G(x) = \sum_0^\infty \frac{(-1)^n}{n!} q(n)\, x^n,$$

and (11.19.2) gives

$$(11.22.1) \qquad \int_0^\infty F(\alpha x)\, G(\beta x)\, dx = \sum_0^\infty (-1)^n \frac{\beta^n}{\alpha^{n+1}} = \frac{1}{\alpha+\beta}.$$

For example, a solution of this integral equation is

$$F(x) = \Sigma \frac{(-1)^n}{n!} \cos(c\sqrt{n})\, x^n, \quad G(x) = \Sigma \frac{(-1)^n}{n!} \frac{x^n}{\cosh\{c\sqrt{(n+1)}\}}.$$

Now

$$\int_0^\infty \alpha^{s-1} d\alpha \int_0^\infty F(\alpha x)\, G(\beta x)\, dx = \int_0^\infty G(\beta x)\, dx \int_0^\infty F(\alpha x)\, \alpha^{s-1} d\alpha$$

$$= f(s) \int_0^\infty x^{-s} G(\beta x)\, dx = \beta^{s-1} f(s)\, g(1-s),$$

where $f(s)$ and $g(s)$ are defined by (F 2). Since the first integral here is

$$\int_0^\infty \frac{\alpha^{s-1}}{\alpha+\beta} d\alpha = \frac{\pi}{\sin s\pi} \beta^{s-1},$$

we obtain (F 3).

I now recall some fundamental formulae in the theory of "Fourier kernels". If $K(x)$ is a Fourier kernel, that is to say if

$$(11.22.2) \qquad \int_0^\infty A(x)\, K(xy)\, dx = B(y)$$

implies

$$(11.22.3) \qquad \int_0^\infty B(x)\, K(xy)\, dx = A(y)$$

for "arbitrary" $A(x)$, and

$$k(s) = \int_0^\infty K(x)\, x^{s-1} dx,$$

then

$$(11.22.4) \qquad k(s)\, k(1-s) = 1.$$

More generally, if (11.22.2) implies

(11.22.5) $$\int_0^\infty B(x)\,H(xy)\,dx = A(y),$$

and $h(s)$ is defined like $k(s)$, then

(11.22.6) $$k(s)\,h(1-s) = 1.$$

Ramanujan never, I think, writes down quite these equations, but (F4) and (F5) are formally much the same. For if

$$K(x) = \frac{1}{\sqrt{(2\pi)}}\{F(xi)+F(-xi)\},\quad H(x) = \frac{1}{\sqrt{(2\pi)}}\{G(xi)+G(-xi)\},$$

then $$k(s) = \frac{1}{\sqrt{(2\pi)}}\left\{\int_0^\infty F(xi)\,x^{s-1}dx+\int_0^\infty F(-xi)\,x^{s-1}dx\right\}$$

$$= \frac{1}{\sqrt{(2\pi)}}\{i^{-s}+(-i)^{-s}\}f(s) = \sqrt{\left(\frac{2}{\pi}\right)}\cos\tfrac{1}{2}s\pi\, f(s);$$

and similarly $$h(s) = \sqrt{\left(\frac{2}{\pi}\right)}\cos\tfrac{1}{2}s\pi\, g(s),$$

so that (11.22.6) becomes (F3).

All this was naturally simple formalism in Ramanujan's hands. Its translation into substantial analysis demands ideas which were quite beyond his ken, Plancherel's theory and Watson's generalisation. But a formal basis is essential as a foundation for all such theories, and that Ramanujan possessed.

11.23. I conclude with a few remarks about the topic which I postponed in § 11.18, the extension to "general transforms" of formulae (D).

Suppose that

(11.23.1) $$k(s) = \frac{\kappa(s)}{\kappa(1-s)},$$

so that (11.22.4) is satisfied identically. Then

$$K(x) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} k(u)\,x^{-u}du$$

is a Fourier kernel whose Mellin transform is $k(s)$.

If we start from the formula

$$R(x) = \sum_0^\infty (-1)^n \frac{r(n)}{\kappa(n+1)}x^n = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{\pi}{\sin u\pi}\frac{r(-u)}{\kappa(1-u)}x^{-u}du,$$

and argue on the formal lines of § 11.16, we obtain

$$\int_0^\infty R(x)\,K(xy)\,dx = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{\pi}{\sin u\pi}\,\frac{r(-u)}{\kappa(1-u)}\,du \int_0^\infty K(xy)\,x^{-u}\,dx$$

$$= \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{\pi}{\sin u\pi}\,\frac{r(-u)}{\kappa(1-u)}\,k(1-u)\,y^{u-1}\,du$$

$$= \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{\pi}{\sin u\pi}\,\frac{r(-u)}{\kappa(u)}\,y^{u-1}\,du$$

$$= \sum_0^\infty (-1)^n \frac{r(-n-1)}{\kappa(n+1)}\,y^n;$$

so that

(11.23.2)

$$R(x) = \sum_0^\infty (-1)^n \frac{r(n)}{\kappa(n+1)}\,x^n, \quad S(x) = \sum_0^\infty (-1)^n \frac{r(-n-1)}{\kappa(n+1)}\,x^n$$

are a pair of "K-transforms" satisfying

(11.23.3)

$$\int_0^\infty R(x)\,K(xy)\,dx = S(y), \quad \int_0^\infty S(x)\,K(xy)\,dx = R(y).$$

The condition for a "self-reciprocal" $R(x)$ is that

$$r(u) = r(-u-1).$$

Thus if $$K(x) = \sqrt{x}\,J_\nu(x),$$

then $$k(s) = 2^{s-\frac{1}{2}}\frac{\Gamma(\frac{1}{2}s+\frac{1}{2}\nu+\frac{1}{4})}{\Gamma(\frac{1}{2}\nu+\frac{3}{4}-\frac{1}{2}s)},$$

and we may take $$\kappa(s) = 2^{\frac{1}{2}s}\Gamma(\tfrac{1}{2}s+\tfrac{1}{2}\nu+\tfrac{1}{4}).$$

Then $$R(x) = \sum_0^\infty (-1)^n \frac{r(n)}{\Gamma(\frac{1}{2}n+\frac{1}{2}\nu+\frac{3}{4})}\,2^{-\frac{1}{2}n}x^n,$$

and $$S(x) = \sum_0^\infty (-1)^n \frac{r(-n-1)}{\Gamma(\frac{1}{2}n+\frac{1}{2}\nu+\frac{3}{4})}\,2^{-\frac{1}{2}n}x^n$$

are a pair of Hankel transforms.

If $$K(x) = \frac{2}{\pi}\,\frac{1}{1-x^2},$$

then $$k(s) = \cot\tfrac{1}{2}s\pi,$$

and we may take $$\kappa(s) = \operatorname{cosec}\tfrac{1}{2}s\pi.$$

In this case

$$R(x) = \sum_0^\infty (-1)^n\,r(2n)\,x^{2n}, \quad S(x) = \sum_0^\infty (-1)^n\,r(-2n-1)\,x^{2n};$$

the integrals (11.23.3) are principal values. If $r(u) = 1$, then

$$R(x) = S(x) = \frac{1}{1+x^2},$$

and other simple rational functions self-reciprocal for $K(x)$ may be obtained by making $r(u)$ an appropriate polynomial. Here also I must restrict myself to a mere sketch of the formal lines of the theory, adding only that all the analysis of §§ 11.22–23 has also been made exact by Goodspeed.

NOTES ON LECTURE XI

§ 11.2. The quotation from Littlewood occurs in his review of the *Papers*.

§ 11.3. Carlson's theorem is proved in Titchmarsh, *Theory of functions*, 185–186.

The theorem has been generalised in various ways. In the first place, we can weaken the order conditions on $\phi(u)$; and secondly we may replace the conditions

$$\phi(0) = \phi(1) = \ldots = 0$$

by more general conditions of the type $\phi(\lambda_n) = 0$ $(n = 0, 1, 2, \ldots)$. The most general theorem of this kind which I have seen was communicated to me by Mr N. Levinson, and runs: *if* (i) $\phi(u)$ *is regular and* $O(e^{A|u|})$, *for some* A, *in the right-hand half-plane;*

(ii) $$\int_{-\infty}^{\infty} \frac{\log^+ |\phi(iw)| - \pi |w|}{1+w^2}\, dw = -\infty;$$

(iii) $$\phi(\lambda_n) = 0,$$

where λ_n *is an increasing sequence of positive numbers such that*

$$\lambda_n < Bn, \quad \sum_{n \leq R} \frac{1}{\lambda_n} > \log R - C,$$

for some B *and* C; *then* $\phi(u) = 0$.

§ 11.4. For Mellin's formula see Titchmarsh, *Fourier integrals*, 7–9 and 46–48.

The proof of (A) is that given by Hardy, *Acta Math.* 42 (1920), 327–339. The conditions on $\phi(u)$ may be relaxed here also, for example to

$$\phi(u) = O(e^{A|u|}) \text{ (some } A\text{)}, \quad \phi(iw) = O(|w|^{-2} e^{\pi |w|}).$$

Goodspeed, in an unpublished manuscript, has proved a 'λ_n' generalisation of (A). *If* $u = v + iw$,

$$0 < \lambda_n < \lambda_{n+1}, \quad 0 < An \leqq \lambda_n \leqq Bn,$$

$$g(u) = \prod_1^\infty \left(1 - \frac{u^2}{\lambda_n^2}\right),$$

and $f(u)$ *belongs to* $\mathfrak{K}(C, D, \delta)$, *where* $C < \pi/B$ *and* $\delta > 0$, *for some* D, *then the series*

$$\sum_{k=1}^{\infty} \frac{f(\lambda_n)}{g'(\lambda_n)} x^{\lambda_n}$$

converges, when the terms are grouped appropriately, for sufficiently small x, *and represents an analytic function* $H(x)$ *regular for all positive* x; *and*

$$\int_0^\infty x^{s-1} H(x)\, dx = \frac{f(-s)}{g(s)}$$

for $0 < s < \delta$. *If also* $\lambda_{n+1} - \lambda_n \geqq h > 0$, *then the series converges in the ordinary sense.*

§ 11.6. The connection of (B) with Newton's interpolation formula was pointed out by Narayana Aiyar (**2**).

For the theory of Newton's formula see Nörlund, *Vorlesungen über Differenzenrechnung*, 222 *et seq*. There is much useful information, but no function-theory, in Whittaker and Robinson's *Calculus of observations*.

For the justification of the term-by-term integration see Hardy, *Trans. Camb. Phil. Soc.* 21 (1912), 1–48 (5–6), and *Messenger of Math.* 39 (1910), 136–139. There is a similar theorem relevant to the reduction of (A) at the end of the section.

§ 11.7. Ramanujan's analysis may be extended, for example, to cases in which

$$\chi(z) = \Sigma c_r z^{p_r},$$

where $p_r \to \infty$, the p_r need not be integers, and a finite number of them may be negative. This extension covers the case $\lambda(u) = e^{\alpha u}$, for any real α. A case actually covered by his analysis as it stands is

$$\lambda(u) = \frac{1}{a+e^u} \quad (0 < a < 1).$$

§ 11.8. There is a direct proof of formula (iii) by Cauchy's theorem in Hardy (**5**).

For formula (iv) see M. Riesz, *Acta Math.* 40 (1916), 185–190, and Hardy and Littlewood, *ibid.* 41 (1917), 119–196 (156–162).

§ 11.9. For Lagrange's and Burmann's series see, for example, Whittaker and Watson, *Modern analysis*, ed. 4 (1927), 128–133.

The most complete results concerning the expansion of roots of trinomial equations are those of Birkeland, *Math. Zeitschrift*, 26 (1927), 566–578. Birkeland gives full references to earlier work of Mellin and other writers.

§§ 11.10–13. The substance of the proof is taken from Hardy (**3**), but Ramanujan's argument in § 11.11 has not been printed before.

Laplace's inversion formula, used in § 11.10, is another formal variant of Fourier's. See Titchmarsh, *Fourier integrals*, 6–7 and 48–49.

For a direct proof of Plana's formula by means of Cauchy's theorem see Lindelöf, *Le calcul des résidus*, 55–62.

§§ 11.15–18. See Hardy (**11**) and Goodspeed (**1**). The penultimate example of § 11.17 occurs in Hardy and Titchmarsh, *Quarterly Journal of Math.* (Oxford), 1 (1930), 196–231 (210); and (11.18.4) on p. 198 of the same paper.

§§ 11.19–20. It is curious that Ramanujan seems never to have written down (11.19.2), or even (11.19.1) in its general form. In the notebooks and reports one of the functions is always even.

For the Parseval formula for Mellin transforms see Titchmarsh, *Fourier integrals*, 94–95. There $s = \sigma + it$, $0 < k < \sigma$, and $x^k F$ and $x^{\sigma-k} G$ are $L^2(0, \infty)$. The integrals (11.20.1) are 'mean square' integrals, and (11.20.2) is true, in the ordinary sense, for all s whose real part is σ.

Goodspeed, using a method more like that of § 11.4, has proved that (11.19.4) *is true whenever*

(1) $\phi(u)$ *is an integral function, and belongs to* $\mathfrak{K}(A, B, \eta)$, *where* $A < \pi$, *for some* B *and any positive* η;

(2) $\psi(u)$ *belongs to* $\mathfrak{K}(C, D, \delta)$, *where* $C < \pi$, *for some* D *and a* δ *not less than* 1;

(3) $$0 < s < 2;$$

(4) $$\phi(-u-s)\,\psi(u) = o(e^{E|w|}),$$

where $E < 2\pi$, *uniformly in* w, *when* $v \to \infty$.

It should be observed that $\phi(u)$ occurs in the theorem with arguments n, $-n-s$, and $-n-1$ and $\psi(u)$ with arguments n and $n+1-s$. The conditions that $\phi(u)$ should be integral, and $\psi(u)$ regular for $\sigma > -1$, are therefore natural.

§ 11.21. See the remarks at the end of § 11.5. In this case each of $\lambda(u)$ and $\mu(u)$ behaves roughly like $e^{\frac{1}{2}\pi|w|}$, and each of $\phi(u)$ and $\psi(u)$ roughly like $e^{\pi|w|}$, in the direction of the imaginary axis, and the final integral is not necessarily absolutely convergent. It is not difficult to justify the transformations in this or any other particular case.

The integral was first evaluated by Weber and Schafheitlin: see Watson, *Bessel functions*, 398–410.

§ 11.22. For the theory of 'Fourier kernels' and 'general transforms' see Hardy and Titchmarsh, *Proc. London Math. Soc.* (2), 35 (1933), 116–155; Watson, *ibid.* 156–199; or Titchmarsh, *Fourier integrals*, ch. VIII.

The subject has attracted a good deal of attention since the appearance of Watson's paper. Full references, up to 1937, will be found in Titchmarsh's book.

§ 11.23. Goodspeed has proved that *if $\kappa(u)$ is real on the real axis, and is the reciprocal of a function regular for $v \geqq -\delta$, where δ is positive; $r(u)$ is an integral function; and*

$$\frac{r(u)}{\kappa(1+u)}, \quad \frac{r(-1-u)}{\kappa(1+u)}$$

belong to the classes $\mathfrak{K}(A, B, \delta)$ and $\mathfrak{K}(C, D, \delta)$, with $A < \pi$ and $C < \pi$, then $k(s)$ defines a Watson transformation in which $R(x)$ and $S(x)$ are reciprocal.

XII

ELLIPTIC AND MODULAR FUNCTIONS

12.1. I must end by saying something about Ramanujan's work on elliptic and modular functions, and it is this which I find my most difficult task. It is here that both the profundity and the limitations of Ramanujan's knowledge stand out most sharply, and that it is least possible to decide how much he may have learnt from others. Besides, I do not know the subject very well.

Ramanujan never professed to have made any major advance in the general theory of elliptic functions, and it seems that he must have learnt the fundamentals of the theory, so far as he was interested in them, from books. There is a sharp contrast between his attitude here and his attitude about the theory of primes, where he certainly regarded all his results as his own. He never writes as if he had *invented* theta-functions or modular equations, though he sets out a whole theory of them in a language of his own. Cayley's and Greenhill's books were in the Madras University Library, and he could have found a good deal about them there. Indeed Littlewood says that "Ramanujan somehow acquired an effectively complete knowledge of the formal side of the theory of elliptic functions", and that his ignorance of complex function theory and of Cauchy's theorem may seem difficult to reconcile with this; and adds that "a sufficient, and I think necessary, explanation would be that Greenhill's very odd and individual *Elliptic functions* was his text-book". In Greenhill's book the complex variable and double periodicity are not mentioned until p. 254, and the double periodicity is deduced somehow from properties of Cartesian ovals. In fact Greenhill knew very little more "function theory" than Ramanujan.

I said that I did not know the subject well, and I shall rely very largely, in all that follows, upon Professor Watson. In the first place, his lecture to the London Mathematical Society contains a short account of the relevant chapters of the notebooks. Secondly, he has very kindly lent me a manuscript containing proofs of nearly all Ramanujan's formulae, without which I should have been very much lost.

12.2. The relevant chapters are Chs. XVI–XXI; these, in Watson's opinion, "show Ramanujan at his best". In Ch. XVI he starts with the function

$$\Pi(a,b) = (1+a)(1+ab)(1+ab^2)\ldots,$$

and all his later work is based on it. He develops a whole series of theorems, mostly to be found in Euler, Gauss, Jacobi, Heine, or other writers, though there are some which seem to be new.[1] I shall not refer to these formulae except when it is necessary for my immediate purpose, which is to track down in the notebooks the proof of the modular equation of degree 3.

Ramanujan also writes

$$f(a,b) = 1+(a+b)+ab(a^2+b^2)+a^3b^3(a^3+b^3)+\dots$$

(the indices of powers of ab being the triangular numbers). Thus, in the orthodox notation[2]

$$f(a,b) = \vartheta_3(v,\tau),$$

where

$$v = \frac{1}{4\pi i}\log\frac{a}{b}, \quad \tau = \frac{1}{2\pi i}\log ab;$$

or again

$$f(a,b) = \rho(z,q),$$

where

$$\rho(z,q) = \sum_{-\infty}^{\infty} q^{n^2}z^n, \quad z = e^{2\pi iv} = \left(\frac{a}{b}\right)^{\frac{1}{2}}, \quad q = e^{\pi i\tau} = (ab)^{\frac{1}{2}}.$$

Ramanujan gives Jacobi's fundamental factorisation formula in the form

$$f(a,b) = \Pi(a,ab)\,\Pi(b,ab)\,\Pi(-ab,ab).$$

Finally, he writes[3]

$$\begin{aligned}
\phi(q) &= f(q,q) = 1+2q+2q^4+\dots,\\
\psi(q) &= f(q,q^3) = 1+q+q^3+q^6+\dots,\\
f(-q) &= f(-q,-q^2) = 1-q-q^2+q^5+\dots,\\
\chi(q) &= \Pi(q,q^2) = (1+q)(1+q^3)(1+q^5)\dots.
\end{aligned}$$

Thus

$$\phi(q) = \vartheta_3(0,\tau).$$

Further, if, with Tannery and Molk, we define q_0, q_1, q_2 and q_3 by

$$q_0 = \prod_1^\infty(1-q^{2n}), \quad q_1 = \prod_1^\infty(1+q^{2n}), \quad q_2 = \prod_1^\infty(1+q^{2n-1}), \quad q_3 = \prod_1^\infty(1-q^{2n-1}),$$

then

$$\begin{aligned}
\chi(q) &= q_2,\\
f(-q) &= (1-q)(1-q^2)(1-q^3)\dots = q_0q_3,\\
\psi(q) &= \frac{(1-q^2)(1-q^4)(1-q^6)\dots}{(1-q)(1-q^3)(1-q^5)\dots} = \frac{q_0}{q_3}.
\end{aligned}$$

The last pair of formulae embody famous identities of Euler and Gauss. We have also

$$\psi(q^2) = \tfrac{1}{2}q^{-\frac{1}{4}}\vartheta_2(0,\tau)$$

(a formula which we shall want later).

[1] Especially 16.17 (in the numeration of the notebooks), which I shall refer to again in § 12.12.

[2] That of Tannery and Molk.

[3] I replace his x by the customary q.

12.3. In this notation many fundamental formulae appear in odd disguises. Thus the "inversion theorem" (there is a τ corresponding to a given k^2) appears as follows. Writing c for k^2, defining K and K' as definite integrals or hypergeometric series, and putting

$$e^{-\pi K'/K} = H(c),$$

then the equation $$q = H(c)$$

has the solution $$c = 1 - \frac{\phi^4(-q)}{\phi^4(q)}.$$

Again the theorem $$\operatorname{sn}^2 u + \operatorname{cn}^2 u = 1$$

appears in the form: if $q = e^{-y}$ and

$$S(\theta, y) = \frac{\sin\frac{1}{2}\theta}{\sinh\frac{1}{2}y} + \frac{\sin\frac{3}{2}\theta}{\sinh\frac{3}{2}y} + \dots, \quad C(\theta, y) = \frac{\cos\frac{1}{2}\theta}{\cosh\frac{1}{2}y} + \frac{\cos\frac{3}{2}\theta}{\cosh\frac{3}{2}y} + \dots,$$

then $$S^2 + C^2 = \tfrac{1}{4}k^2\phi^4(y).$$

In fact $$S(\theta, y) = \frac{kK}{\pi}\operatorname{sn}\frac{K\theta}{\pi}, \quad C(\theta, y) = \frac{kK}{\pi}\operatorname{cn}\frac{K\theta}{\pi}$$

and $$K = \tfrac{1}{2}\pi\vartheta_3^2(0, \tau) = \tfrac{1}{2}\pi\phi^2(y).$$

There are proofs of these formulae, by direct squaring of series, in Jacobi's *Fundamenta nova* and Enneper's *Elliptische Funktionen*. It is very unlikely that Ramanujan had seen either book, but this type of argument was familiar to him.

The modular equation of degree 3

12.4. The modular equation of degree n is the relation between the moduli k and l which corresponds to the change of q into $q^{1/n}$,[1] i.e. to

(12.4.1) $$n\frac{L'}{L} = \frac{K'}{K},$$

where K, K', L, L' are the complete elliptic integrals with moduli k and l.

Ramanujan deduced his modular equations from relations between his functions f, ϕ, ψ, that is to say from relations between ϑ-functions. Thus his forms of the equation of degree 5 are derived from the identities

$$\phi^2(q) - \phi^2(q^5) = 4qf(q, q^9)f(q^3, q^7)^{2}$$

and $$\psi^2(q) - q\psi^2(q^5) = f(q, q^4)f(q^2, q^3).^{3}$$

The first of these, in the orthodox notation, is

$$\vartheta_3^2(0, \tau) - \vartheta_3^2(0, 5\tau) = 4e^{\pi i\tau}\vartheta_3(-2\tau, 5\tau)\vartheta_3(-\tau, 5\tau).$$

[1] Or into q^n, when $L'/L = nK'/K$. See § 12.7.

[2] 19.10 (iv).

[3] 19.10 (v).

The identities are proved by "elementary" arguments, the whole process being "algebraical".

I shall confine myself to the equation of order 3, and I shall begin by explaining two of the standard methods of proof. The first rests on direct transformations of integrals, and is Jacobi's version of Legendre's original proof. The second, found much later by Schröter, is much more like Ramanujan's.

12.5. Jacobi's argument is expounded in many text-books. We try to satisfy the equation

(12.5.1) $$\frac{m\,dy}{\sqrt{Y}} = \frac{dx}{\sqrt{X}},$$

where $$X = (1-x^2)(1-k^2x^2), \quad Y = (1-y^2)(1-l^2y^2),$$
$$0<k<1, \quad 0<l<1,$$

by taking

(12.5.2) $$y = \frac{U}{V},$$

where U and V are coprime polynomials in x of degree not exceeding 3, and y vanishes with x and is positive for positive x. It is plain (since y is an odd function of x) that

$$U = x(\alpha+\beta x^2), \quad V = \gamma+\delta x^2,$$

where α, β, γ, δ are constants.

Next $$m\frac{dy}{dx} = \sqrt{\left\{\frac{(1-y^2)(1-l^2y^2)}{(1-x^2)(1-k^2x^2)}\right\}} > 0$$

for small x, so that m is positive; and $y = 1$ when $x = 1$, since y would otherwise have a branch point for $x = 1$. Further (12.5.1), and so (12.5.2), is unaltered when we substitute $1/kx$ and $1/ly$ for x and y, so that the values

$$y = 1,\ -1,\ \frac{1}{l},\ -\frac{1}{l}$$

correspond to $$x = 1,\ -1,\ \frac{1}{k},\ -\frac{1}{k}.$$

Since $$\frac{dy}{\sqrt{Y}} = \frac{U'V-UV'}{\sqrt{\{(V^2-U^2)(V^2-l^2U^2)\}}}\,dx,$$

we must have $$(V^2-U^2)(V^2-l^2U^2) = T^2(1-x^2)(1-k^2x^2),$$

where T is a quartic; so that

$$\frac{dy}{\sqrt{Y}} = \frac{U'V-UV'}{T\sqrt{X}}\,dx,$$

$$U'V-UV' = \frac{T}{m}.$$

Next, since no two of $V \pm U$, $V \pm lU$ can have a common factor, we must have

$$V+U = (1+x)A^2, \quad V+lU = (1+kx)C^2,$$
$$V-U = (1-x)B^2, \quad V-lU = (1-kx)D^2,$$

where A, B, C, D are linear. It follows (since y is odd) that

$$\frac{1-y}{1+y} = \frac{1-x}{1+x}\left(\frac{1-cx}{1+cx}\right)^2,$$

or

$$y = \frac{x(2c+1+c^2x^2)}{1+c(c+2)x^2},$$

for some c. This equation must be unchanged if we substitute $1/kx$, $1/ly$ for x, y, when it becomes

$$y = \frac{kx}{l}\frac{c(c+2)+k^2x^2}{c^2+(2c+1)k^2x^2};$$

and therefore

(12.5.3) $$k^2 = \frac{c^3(c+2)}{2c+1}, \quad l^2 = c\left(\frac{c+2}{2c+1}\right)^3.$$

We obtain a relation between k and l by eliminating c. In fact

(12.5.4) $$k'^2 = 1-k^2 = \frac{(1+c)^3(1-c)}{2c+1}, \quad l'^2 = 1-l^2 = (1+c)\left(\frac{1-c}{2c+1}\right)^3,$$

and so

(12.5.5) $$\sqrt{(kl)}+\sqrt{(k'l')} = \frac{c(c+2)}{2c+1}+\frac{(1+c)(1-c)}{2c+1} = 1$$

when appropriate values of the radicals are taken. This is Legendre's form of the relation. It is easy to deduce Jacobi's form

(12.5.6) $$u^4 - v^4 + 2uv(1-u^2v^2) = 0,$$

where $k = u^4$ and $l = v^4$.

Comparing the values of x and y for small x, we see that

(12.5.7) $$m = \frac{1}{2c+1}.$$

12.6. The equations (12.5.3), (12.5.5), and (12.5.6) are in fact different forms of the modular equation of degree 3; but in order to see this, we must show that they correspond to (12.4.1), with $n = 3$, as well as to (12.5.1).

It is easy to see[1] that, if $0 < k^2 < 1$, then the equation

(12.6.1) $$k^2 = c^3\frac{c+2}{2c+1}$$

[1] By means of "graphical" considerations and the equation

$$\frac{k^2}{l^2} = \left(c\frac{2c+1}{c+2}\right)^2.$$

in c has just two real roots, one between 0 and 1 and one between -2 and -1, and that these roots make $l^2 > k^2$ and $l^2 < k^2$ respectively. If, in the equations (12.5.4), we put

(12.6.2) $$\frac{1-c}{2c+1} = \gamma, \quad \frac{1-\gamma}{2\gamma+1} = c,$$

then

(12.6.3) $$k'^2 = \gamma\left(\frac{\gamma+2}{2\gamma+1}\right)^3, \quad l'^2 = \gamma^3\frac{\gamma+2}{2\gamma+1}.$$

It now follows from the analysis of § 12.5 that

$$\frac{1}{m'}\frac{dx}{\sqrt{\{(1-x^2)(1-l'^2x^2)\}}} = \frac{dy}{\sqrt{\{(1-y^2)(1-k'^2y^2)\}}},$$

with $$m' = \frac{1}{2\gamma+1}.$$

Hence, integrating between 0 and 1,

$$\frac{L'}{m'} = K',$$

while (12.5.1) gives $$mL = K.$$

Thus $$\frac{L'}{L} = mm'\frac{K'}{K}$$

and it is only necessary to show that

$$mm' = \tfrac{1}{3}.$$

But $$mm' = \frac{1}{(2c+1)(2\gamma+1)}$$

and $$2\gamma+1 = \frac{3}{2c+1}, \quad mm' = \tfrac{1}{3},$$

by (12.6.2).

12.7. The equation (12.5.5) is symmetrical in k and l, and corresponds to

$$\frac{L'}{L} = n\frac{K'}{K}$$

(i.e. to the change of q into q^n) as well as to (12.4.1). We can see this more directly. If we put

$$c = -\frac{c'+2}{2c'+1}, \quad c' = -\frac{c+2}{2c+1}$$

in (12.5.3), we obtain

$$k^2 = c'\left(\frac{c'+2}{2c'+1}\right)^3, \quad l^2 = \frac{c'^3(c'+2)}{2c'+1},$$

i.e. (12.5.3) with c' for c and k and l interchanged. Also c' lies between -2 and -1 when c lies between 0 and 1, and conversely, so that c' is the second root of (12.6.1). If γ' is associated with c' as γ is with c, then

$$(1+2c)(1+2c') = -3, \quad (1+2\gamma)(1+2\gamma') = -3,$$

and so
$$(1+2c')(1+2\gamma') = 3$$

and
$$\frac{L'}{L} = 3\frac{K'}{K}.$$

12.8. The second method of proof depends on direct manipulation of theta-series. The equation (12.5.5) is equivalent to

$$(12.8.1) \quad \vartheta_3(0,\tau)\vartheta_3(0,3\tau) - \vartheta_4(0,\tau)\vartheta_4(0,3\tau) = \vartheta_2(0,\tau)\vartheta_2(0,3\tau)$$

or to

$$(12.8.2) \qquad 2\Sigma' q^{m^2+3n^2} = \Sigma q^{(\mu+\frac{1}{2})^2+3(\nu+\frac{1}{2})^2},$$

where Σ indicates summation over all integers and Σ' summation over all integers of opposite parity. If now we put

$$m+n = u, \quad m-n = v,$$

so that

$$m^2+3n^2 = \left(\frac{u+v}{2}\right)^2 + 3\left(\frac{u-v}{2}\right)^2 = u^2-uv+v^2 = (u-\tfrac{1}{2}v)^2+3(\tfrac{1}{2}v)^2,$$

then the left-hand side of (12.8.2) becomes

$$2\sum_{u,v\ \mathrm{odd}} q^{(u-\frac{1}{2}v)^2+3(\frac{1}{2}v)^2}.$$

Next, if we put
$$u-\tfrac{1}{2}v = \mu+\tfrac{1}{2}, \quad \tfrac{1}{2}v = \nu+\tfrac{1}{2},$$

so that
$$u = \mu+\nu+1, \quad v = 2\nu+1,$$

and let μ and ν run through all values of opposite parity, then u and v run through all odd values, and so

$$\sum_{u,v\ \mathrm{odd}} q^{(u-\frac{1}{2}v)^2+3(\frac{1}{2}v)^2} = \sum{}' q^{(\mu+\frac{1}{2})^2+3(\nu+\frac{1}{2})^2},$$

where the dash has the same meaning as before. Finally, if

$$\mu' = -\mu-1, \quad \nu' = \nu,$$

then
$$(\mu'+\tfrac{1}{2})^2+3(\nu'+\tfrac{1}{2})^2 = (\mu+\tfrac{1}{2})^2+3(\nu+\tfrac{1}{2})^2,$$

and μ' and ν' run through all values of the same parity. Hence

$$\sum_{-\infty}^{\infty} q^{(\mu+\frac{1}{2})^2+3(\nu+\frac{1}{2})^2} = 2\sum{}' q^{(\mu+\frac{1}{2})^2+3(\nu+\frac{1}{2})^2} = 2\sum{}' q^{m^2+3n^2},$$

which is (12.8.2).

12.9. This proof is much like those given by Schröter of the equations for this and certain higher degrees. Schröter found the general identity

(12.9.1) $$\vartheta_3(x,\alpha\tau)\vartheta_3(y,\beta\tau)$$
$$= \sum_{r=1}^{\alpha+\beta-1} q^{\alpha r^2} e^{2r\pi i x}\vartheta_3\{x+y+r\alpha\tau, (\alpha+\beta)\tau\}\,\vartheta_3\{\beta x-\alpha y+r\alpha\beta\tau, \alpha\beta(\alpha+\beta)\tau\},$$

where α and β are any positive integers. There is a fairly simple proof of this formula in Tannery and Molk, where it is applied to the cases $n = 3$ and $n = 5$. Schröter also worked out the cases 11, 23 and 31; thus his form of the equation of degree 23 is

$$(kl)^{\frac14} + (k'l')^{\frac14} + 2^{\frac23}(klk'l')^{\frac1{12}} = 1,$$

which also appears in the notebook.[1] It would seem that Ramanujan must have known (12.9.1). He has a formula[2] for

$$f(a,b)f(c,d),$$

i.e. for $\vartheta_3(v,\tau)\vartheta_3(v',\tau')$, as the sum of an infinite series, which was also found by Schröter, and in certain cases this series becomes finite. "These compact formulae", says Watson, "were not given in the notebook, but in view of the numerous special cases which he does give, he must have been aware of their existence."

Ramanujan gave the known forms for

3, 5, 7, 11, 23

(with many variants in the simpler cases), and new forms for

13, 17, 19, 31, 47, 71.

Thus the equation of degree 13 appears as[3]

$$\begin{cases} m = \left(\dfrac{l}{k}\right)^{\frac12} + \left(\dfrac{l'}{k'}\right)^{\frac12} - \left(\dfrac{ll'}{kk'}\right)^{\frac12} - 4\left(\dfrac{ll'}{kk'}\right)^{\frac16}, \\ \dfrac{13}{m} = \left(\dfrac{k}{l}\right)^{\frac12} + \left(\dfrac{k'}{l'}\right)^{\frac12} - \left(\dfrac{kk'}{ll'}\right)^{\frac12} - 4\left(\dfrac{kk'}{ll'}\right)^{\frac16}; \end{cases}$$

and that of degree 17 as[4]

$$\begin{cases} m = \left(\dfrac{l}{k}\right)^{\frac12} + \left(\dfrac{l'}{k'}\right)^{\frac12} + \left(\dfrac{ll'}{kk'}\right)^{\frac12} - 2\left(\dfrac{ll'}{kk'}\right)^{\frac14}\left\{1 + \left(\dfrac{l}{k}\right)^{\frac14} + \left(\dfrac{l'}{k'}\right)^{\frac14}\right\}, \\ \dfrac{17}{m} = \left(\dfrac{k}{l}\right)^{\frac12} + \left(\dfrac{k'}{l'}\right)^{\frac12} + \left(\dfrac{kk'}{ll'}\right)^{\frac12} - 2\left(\dfrac{kk'}{ll'}\right)^{\frac14}\left\{1 + \left(\dfrac{k}{l}\right)^{\frac14} + \left(\dfrac{k'}{l'}\right)^{\frac14}\right\}. \end{cases}$$

Watson has now worked out proofs of all of these. He remarks that "although modular equations of degrees 13, 17 and 19 have been obtained

[1] 20.15 (i). [2] 16.36 (i), (ii).
[3] 20.8 (iii), (iv). [4] 20.12 (iii), (iv).

by other authors, they have usually been given in more complicated forms; and in fact, when dealing with Ramanujan's modular equations generally, it has always seemed to me that knowledge of other people's work is a positive disadvantage in that it tends to put one off the shortest track".

There are also many "mixed" modular equations or equations of composite degree. For example, if the moduli k, κ, l, λ correspond to q, q^3, q^5 and q^{15} respectively, then[1]

$$(k\lambda)^{\frac{1}{4}} - (k'\lambda')^{\frac{1}{4}} = (\kappa l)^{\frac{1}{4}} - (\kappa' l')^{\frac{1}{4}}$$

and[2]

$$(k\lambda)^{\frac{1}{8}}\{(1+k)^{\frac{1}{4}}(1+\lambda)^{\frac{1}{4}} - (1-k)^{\frac{1}{4}}(1-\lambda)^{\frac{1}{4}}\} + (k'\lambda')^{\frac{1}{8}}\{(1+k')^{\frac{1}{4}}(1+\lambda')^{\frac{1}{4}} - (1-k')^{\frac{1}{4}}(1-\lambda')^{\frac{1}{4}}\} = \sqrt{2}.$$

I shall say no more about results of this kind. I prefer to disentangle from the notebook Ramanujan's own proof of the equation of degree 3, which appears there as the climax of a very intricate chain of reasoning. It is of course quite likely that he possessed a shorter proof on the lines of § 12.8.

12.10. Ramanujan actually deduces (12.8.1) and (12.5.5) from two identities[3] involving series of the "Lambert" type, viz.

$$(12.10.1)\qquad q\psi(q^2)\,\psi(q^6) = \frac{q}{1-q^2} - \frac{q^5}{1-q^{10}} + \frac{q^7}{1-q^{14}} - \frac{q^{11}}{1-q^{22}} + \dots,$$

$$(12.10.2)\qquad \phi(q)\,\phi(q^3) = 1 + 2\left(\frac{q}{1-q} - \frac{q^2}{1+q^2} + \frac{q^4}{1+q^4} - \frac{q^5}{1-q^5} + \dots\right),$$

the signs in each series recurring to modulus 6. From these it follows that

$$(12.10.3)\qquad 4q\psi(q^2)\,\psi(q^6) = \phi(q)\,\phi(q^3) - \phi(-q)\,\phi(-q^3);$$[4]

and, since

$$\phi(q) = \vartheta_3(0,\tau), \quad \phi(-q) = \vartheta_4(0,\tau), \quad \psi(q^2) = \tfrac{1}{2}q^{-\frac{1}{4}}\vartheta_2(0,\tau),$$

(12.10.3) is equivalent to (12.8.1). It remains to trace the genesis of (12.10.1) and (12.10.2): this I have only been able to do with the help of Watson's solutions.

I call a relation between f, ϕ, ψ "trivial" if it is a direct consequence of their expressions as products. Thus

$$(12.10.4)\qquad \frac{f(q,-q^2)}{f(-q,q^2)} = \frac{\phi(q)}{\phi(q^3)}$$

is "trivial". For if we write

$$1-q^n = (n), \quad 1+q^n = (\bar{n}),$$

[1] 20.11 (iii). [2] 20.11 (vi). [3] 19.3 (i), (ii).

[4] This formula does not occur explicitly in the notebook.

then it is

$$\frac{(\overline{1})(4)(\overline{7})(10)\ldots(2)(\overline{5})(8)(\overline{11})\ldots}{(1)(\overline{4})(7)(\overline{10})\ldots(\overline{2})(5)(\overline{8})(11)\ldots}=\frac{(2)(4)(6)\ldots(\overline{1})^2(\overline{3})^2(\overline{5})^2\ldots}{(6)(12)(18)\ldots(3)^2(\overline{9})^2(\overline{15})^2\ldots}.$$

Such a formula may be verified by observing that

$$(\overline{n})=\frac{(2n)}{(n)}$$

and comparing factors.

Similarly

(12.10.5) $$\frac{qf(-q^4,-q^8)}{f(-q^2,-q^{10})}\phi(q^6)\phi(q^{12})=q\psi(q^2)\psi(q^6)$$

is "trivial".

12.11. The proofs of (12.10.1) and (12.10.2) may be traced back to earlier formulae, viz.

(12.11.1) $$\frac{f(a,b)}{f(-a,-b)}\phi^2(-ab)=1+2\left(\frac{a+b}{1+ab}+\frac{a^2+b^2}{1+a^2b^2}+\ldots\right)^{1}$$

and

(12.11.2) $$f^2(a,b)-f^2(-a,-b)=4af\left(\frac{b}{a},a^3b\right)\psi(a^2b^2).^{2}$$

Let us assume these formulae provisionally. The series in brackets in (12.11.1) is

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{a^n+b^n}{1+a^nb^n}&=\sum_{n=1}^{\infty}\sum_{m=0}^{\infty}(-1)^m\{a^{(m+1)n}b^{mn}+a^{mn}b^{(m+1)n}\}\\&=\sum_{m=0}^{\infty}(-1)^m\left(\frac{a^{m+1}b^m}{1-a^{m+1}b^m}+\frac{a^mb^{m+1}}{1-a^mb^{m+1}}\right).\end{aligned}$$

Hence (12.11.1) is equivalent to

(12.11.3) $$\frac{f(a,b)}{f(-a,-b)}\phi^2(-ab)=1+2\sum_{m=0}^{\infty}(-1)^m\left(\frac{a^{m+1}b^m}{1-a^{m+1}b^m}+\frac{a^mb^{m+1}}{1-a^mb^{m+1}}\right).$$

If we put $a=q$, $b=-q^2$, and use (12.10.4), then (12.11.3) gives (12.10.2).

It also follows from (12.11.3) that

$$\left\{\frac{f(a,-b)}{f(-a,b)}-\frac{f(-a,b)}{f(a,-b)}\right\}\phi^2(ab)=4\sum_{m=0}^{\infty}\left(\frac{a^{m+1}b^m}{1-a^{2m+2}b^{2m}}-\frac{a^mb^{m+1}}{1-a^{2m}b^{2m+2}}\right).$$

If we transform the left-hand side by (12.11.2) and the "trivial" identity

$$f(a,-b)f(-a,b)=f(-a^2,-b^2)\phi(ab),$$

[1] 16.33 (corollary).
[2] 16.30 (vi).

we obtain

$$\frac{f^2(a,-b)-f^2(-a,b)}{f(a,-b)f(-a,b)}\phi^2(ab) = \frac{4af\left(-\frac{b}{a}, -a^3b\right)}{f(a,-b)f(-a,b)}\phi^2(ab)\,\psi(a^2b^2)$$

$$= \frac{4af\left(-\frac{b}{a}, -a^3b\right)}{f(-a^2,-b^2)}\phi(ab)\,\psi(a^2b^2).$$

Hence

$$\frac{af\left(-\frac{b}{a}, -a^3b\right)}{f(-a^2,-b^2)}\phi(ab)\,\psi(a^2b^2) = \sum_{m=0}^{\infty}\left(\frac{a^{m+1}b^m}{1-a^{2m+2}b^{2m}} - \frac{a^m b^{m+1}}{1-a^{2m}b^{2m+2}}\right).$$

Finally, if we replace a and b by q and q^5, and use (12.10.5), we obtain

$$q\psi(q^2)\,\psi(q^6) = \sum_{m=0}^{\infty}\left(\frac{q^{6m+1}}{1-q^{12m+2}} - \frac{q^{6m+5}}{1-q^{12m+10}}\right),$$

which is (12.10.1).

Thus everything is reduced to the proof of (12.11.1) and (12.11.2).

12.12. The first of these formulae is equivalent to

(12.12.1) $$\tfrac{1}{4} + \sum_{1}^{\infty}\frac{q^n}{1+q^{2n}}\cos 2n\pi v = \frac{1}{4\pi}\frac{\vartheta_1'(0,\tau)\,\vartheta_4(v+\frac{1}{2},\tau)}{\vartheta_2(0,\tau)\,\vartheta_3(v+\frac{1}{2},\tau)}.$$

Tannery and Molk prove formulae of this type by means of Cauchy's theorem. We can deduce (12.11.1) from (12.12.1) by putting

$$q = e^{\pi i\tau} = (ab)^{\frac{1}{2}}, \quad z = e^{2\pi i v} = \left(\frac{a}{b}\right)^{\frac{1}{2}},$$

and then making a "trivial" transformation. Ramanujan's proof is naturally very different. He deduces (12.12.1) from a remarkable formula[1] with many parameters, viz.

(12.12.2)

$$\frac{\Pi(xy,x^2)\,\Pi\left(\frac{x}{y},x^2\right)\Pi(-x^2,x^2)\,\Pi(-\alpha\beta x^2,x^2)}{\Pi(\alpha xy,x^2)\,\Pi\left(\frac{\beta x}{y},x^2\right)\Pi(-\alpha x^2,x^2)\,\Pi(-\beta x^2,x^2)} = 1 + \left(xy\frac{1-\alpha}{1-\beta x^2} + \frac{x}{y}\frac{1-\beta}{1-\alpha x^2}\right)$$

$$+\left\{(xy)^2\frac{(1-\alpha)(x^2-\alpha)}{(1-\beta x^2)(1-\beta x^4)} + \left(\frac{x}{y}\right)^2\frac{(1-\beta)(x^2-\beta)}{(1-\alpha x^2)(1-\alpha x^4)}\right\}$$

$$+\left\{(xy)^3\frac{(1-\alpha)(x^2-\alpha)(x^4-\alpha)}{(1-\beta x^2)(1-\beta x^4)(1-\beta x^6)} + \dots\right\} + \dots.$$

[1] 16.17.

This formula seems to be new. It is however deduced from one which is familiar and probably goes back to Euler, viz.

$$\frac{\Pi(b,x)}{\Pi(-a,x)} = 1 + \frac{a+b}{1-x} + \frac{(a+b)(a+bx)}{(1-x)(1-x^2)} + \dots\text{[1]}$$

In the application of (12.12.2) to the proof of (12.11.1)

$$xy = a, \quad \frac{x}{y} = b, \quad \alpha = \beta = -1.$$

12.13. Finally (12.11.2) is equivalent to

$$\vartheta_3^2(v,\tau) - \vartheta_4^2(v,\tau) = 2\vartheta_2(0,2\tau)\vartheta_2(2v,2\tau),$$

which is derived by Tannery and Molk from the theory of Landen's quadratic transformation. Ramanujan deduces it from

$$f(a,b)f(c,d) - f(-a,-b)f(-c,-d) = 2af\left(\frac{b}{c}, \frac{c}{b}abcd\right)f\left(\frac{b}{d}, \frac{d}{b}abcd\right),\text{[2]}$$

where $$ab = cd.$$

The direct proof of this formula is quite simple, if we write the left-hand side as

$$2\sum_{m+n\text{ even}} a^{\frac{1}{2}m(m+1)}b^{\frac{1}{2}m(m-1)}c^{\frac{1}{2}n(n+1)}d^{\frac{1}{2}n(n-1)}$$

and put $$m+n = 2M, \quad m-n = 2N,$$

when the summation is over all integral M, N.

12.14. It is obvious that the argument of §§ 12.10–12.13, regarded simply as a proof of Legendre's equation, is very long and complicated, and will not bear comparison with the economical proofs of §§ 12.5–12.8. The comparison, however, is not at all a fair one. The proof of § 12.8, for example, is an *ad hoc* proof, a short cut to a particular formula; and it seems almost certain that Ramanujan must have possessed short cuts of the same kind. He certainly used similar methods sometimes (as is shown by the argument which I quoted in § 12.13), and the proof of § 12.8 is one which, whether he knew it or not, he could certainly have constructed. The proof which I have pieced together in §§ 12.10–12.13 is one of an entirely different kind; it wanders over large parts of the theory of elliptic functions, and the modular equation appears only as one of the climaxes of Ramanujan's reconstruction of the theory.

Singular moduli

12.15. I shall say less about Ramanujan's work in the closely connected theory of "singular moduli". A good deal of this work is already in print;[3] and Watson, in his series of papers on singular moduli, has proved all

[1] 16.2. There is a simple proof of this formula in the *Papers*, no. 11, 57–58.

[2] 16.29 (ii).

[3] *Papers*, no. 6, 23–39.

Ramanujan's results, and done a great deal to elucidate his probable method of procedure. I begin with a very short summary of the foundations of the orthodox theory.

It is familiar that

(12.15.1) $$\wp(\mu u, \omega_1, \omega_2) = R(\wp),$$

where $$\wp = \wp(u) = \wp(u, \omega_1, \omega_2),$$

and R is a rational function, when μ is an integer. Can this be true for other μ? If so, then $2\mu\omega_1$, $2\mu\omega_2$ are periods of $\wp(u)$, and

$$\mu\omega_1 = \alpha\omega_1 + \beta\omega_2, \quad \mu\omega_2 = \gamma\omega_1 + \delta\omega_2,$$

where α, β, γ, δ are integers with

(12.15.2) $$\beta \neq 0, \quad \gamma \neq 0, \quad \alpha\delta - \beta\gamma \neq 0.$$

We may suppose

(12.15.3) $$\beta > 0$$

(since otherwise we can change the sign of α, β, γ, δ and μ). Conversely, when these conditions are satisfied, $\wp(\mu u)$ is an elliptic function with periods ω_1, ω_2, and so a rational function of $\wp$ and $\wp'$. Finally, since it is even, it is a rational function of $\wp$ only.

The situation is not affected if we replace ω_1 and ω_2 by $\lambda\omega_1$ and $\lambda\omega_2$, so that only the ratio

$$\tau = \omega_1/\omega_2$$

is relevant. In these circumstances we shall say that "there is complex multiplication by μ for τ". For this, it is necessary and sufficient that

(12.15.4) $$\mu = \alpha + \beta\tau, \quad \mu\tau = \gamma + \delta\tau,$$

so that

(12.15.5) $$\tau = \frac{\gamma + \delta\tau}{\alpha + \beta\tau}$$

with integral α, β, γ, δ satisfying (12.15.2) and (12.15.3). Thus τ is the root, with positive imaginary part, of

(12.15.6) $$\beta\tau^2 + (\alpha - \delta)\tau - \gamma = 0.$$

It is easily verified that any number $A + B\mu$, where A and B are integers, is a multiplier for τ if μ is one.

12.16. We write

(12.16.1) $$\rho = (-\gamma, \alpha - \delta, \beta) > 0$$

and

(12.16.2) $$-\gamma = a\rho, \quad \alpha - \delta = b\rho, \quad \beta = c\rho.$$

Then

(12.16.3) $$a+b\tau+c\tau^2 = 0,$$

where

(12.16.4) $$a>0, \quad c>0, \quad b^2-4ac = -n<0,$$

and

(12.16.5) $$\tau = -\frac{b-i\sqrt{n}}{2c}.$$

Further, we write

(12.16.6) $$\alpha+\delta = \rho_1.$$

Then $$\rho_1+b\rho = 2\alpha \equiv 0 \pmod{2},$$

(12.16.7) $$\alpha = \tfrac{1}{2}(\rho_1+b\rho), \quad \beta = c\rho, \quad \gamma = -a\rho, \quad \delta = \tfrac{1}{2}(\rho_1-b\rho),$$

and

(12.16.8) $$\mu = \alpha+\beta\tau = \tfrac{1}{2}(\rho_1+i\rho\sqrt{n}).$$

Conversely, if there are integers a, b, c satisfying (12.16.4), and ρ, ρ_1 satisfying

(12.16.9) $$\rho>0, \quad \rho_1+b\rho \equiv 0 \pmod{2},$$

α, β, γ, δ are then defined by (12.16.7), τ by (12.16.5), and $\mu=\alpha+\beta\tau$, then there is complex multiplication by μ for τ.

Further, if

(12.16.10) $$\epsilon = 0 \ (b \text{ even}), \quad \epsilon = 1 \ (b \text{ odd}),$$

(12.16.11) $$M = \tfrac{1}{2}(-\epsilon+i\sqrt{n}),$$

then M is also a multiplier for τ. For it is easy to verify that

(12.16.12) $$\mu = \alpha-\tfrac{1}{2}(b-\epsilon)\rho+\rho M,$$

(12.16.13) $$M = \tfrac{1}{2}(b-\epsilon)+c\tau, \quad M\tau = -a-\tfrac{1}{2}(b+\epsilon)\tau.$$

Since the last equations are of the form (12.15.4), M is a multiplier (when μ is one).

On the other hand, if M is a multiplier for τ, it follows from (12.16.12), and the last remark of § 12.15, that μ is a multiplier.

12.17. If τ satisfies (12.15.5), A, B, C, D are integers with $AD-BC = 1$, and

(12.17.1) $$\tau' = \frac{C+D\tau}{A+B\tau},$$

so that

(12.17.2) $$J(\tau') = J(\tau),$$

then τ' satisfies an equation

(12.17.3) $$a'+b'\tau'+c'\tau'^2 = 0$$

with the same determinant as (12.16.3), so that b' has the same parity as b, and the ϵ' and M′ corresponding to τ' are the same as ϵ and M. It follows that there is multiplication for τ' by M, and therefore by any μ which is a multiplier for τ. The values of τ for which there is multiplication, and the corresponding multipliers, are determined uniquely by the values of $J(\tau)$.

12.18. If τ and any positive integer m are given, the values of

$$J\left(\frac{\gamma+\delta\tau}{\alpha+\beta\tau}\right),$$

for different α, β, γ, δ satisfying

$$\alpha\delta-\beta\gamma = m,$$

are the roots of an equation $\quad F(x, J) = 0,$

where $\quad J = J(\tau),$

of degree m. If τ satisfies (12.15.5), then

(12.18.1) $$F(J, J) = 0.$$

Hence each $J(\tau)$ corresponding to a τ for which there is complex multiplication satisfies an algebraic equation. We call these $J(\tau)$ singular invariants, and the corresponding $k^2(\tau)$ singular moduli; the singular moduli also satisfy algebraic equations. These equations are all soluble by radicals (though this depends on considerations of group theory of which Ramanujan knew nothing).

12.19. All this is connected with the theory of arithmetical forms. Suppose that

$$f = ax^2+bxy+cy^2,$$

where $\quad (a, b, c) = 1,$

is equivalent to $\quad f' = a'x'^2+b'x'y'+c'y'^2,$

with $\quad x' = \alpha x+\beta y, \quad y' = \gamma x+\delta y, \quad \alpha\delta-\beta\gamma = 1,$

so that

(12.19.1)

$$a = a'\alpha^2+b'\alpha\gamma+c'\gamma^2,\ b = 2a'\alpha\beta+b'(\beta\gamma+\alpha\delta)+2c'\gamma\delta,\ c = a'\beta^2+b'\beta\delta+c'\delta^2,$$

$$(a', b', c') = 1,$$

$$b^2-4ac = b'^2-4a'c' = -n.$$

Then b and b' have the same parity and, if τ and τ' are roots of

(12.19.2) $$a+b\tau+c\tau^2 = 0, \quad a'+b'\tau'+c'\tau'^2 = 0,$$

the M of § 12.16 has the same value for τ and τ'. Thus there is complex multiplication by M for both τ and τ'. Also τ and τ' satisfy

$$\tau' = \frac{\gamma+\delta\tau}{\alpha+\beta\tau} \qquad (\alpha\delta-\beta\gamma = 1) \tag{12.19.3}$$

and so

$$J(\tau') = J(\tau). \tag{12.19.4}$$

Conversely, suppose that (12.19.4) is true, that there is complex multiplication for τ and τ', and that τ and τ' satisfy equations (12.19.2), with $(a,b,c) = 1$ and $(a',b',c') = 1$. Since (12.19.4) is true, there are integers α, β, γ, δ satisfying (12.19.3). Hence the equation

$$a'(\alpha+\beta\tau)^2+b'(\alpha+\beta\tau)(\gamma+\delta\tau)+c'(\gamma+\delta\tau)^2 = 0$$

has the same roots as $\qquad a+b\tau+c\tau^2 = 0.$

From this it follows easily that a, b, c satisfy (12.19.1), and that the forms f and f' are equivalent.

Thus (12.19.4) is a necessary and sufficient condition for equivalence, and the degree of (12.18.1) is the class number $h(-n)$ for determinant $-n$.

It seems clear that Ramanujan knew nothing about the arithmetical side of the theory.

12.20. The theory appears in the clearest light when stated in terms of $J(\tau)$. For numerical work, other modular functions are more convenient, since the equations satisfied by $J(\tau)$ tend to have large coefficients.

It is usual to write

$$f(\tau) = q^{-\frac{1}{24}}\prod_1^\infty (1+q^{2n-1}) = \left(\frac{4}{kk'}\right)^{\frac{1}{12}},$$

$$f_1(\tau) = q^{-\frac{1}{24}}\prod_1^\infty (1-q^{2n-1}) = \left(\frac{4k'^2}{k}\right)^{\frac{1}{12}},$$

$$f_2(\tau) = 2^{\frac{1}{2}}q^{\frac{1}{12}}\prod_1^\infty (1+q^{2n}) = \left(\frac{4k^2}{k'}\right)^{\frac{1}{12}}.$$

Then $\qquad f^8 = f_1^8+f_2^8, \quad ff_1f_2 = \sqrt{2}.$

If $\qquad n = 4ac-b^2 = 8m+3 \not\equiv 0 \pmod 3,$

then the simplest invariant is

$$f\{\sqrt{(-n)}\} = f_n = f,$$

while if $\qquad n = 8m-1 \not\equiv 0 \pmod 3,$

it is $\qquad 2^{-\frac{1}{2}}f\{\sqrt{(-n)}\} = F_n = F.$

Ramanujan, however, uses the invariants

$$G_n = 2^{-\frac{1}{4}} f\{\sqrt{(-n)}\} = 2^{-\frac{1}{4}} f_n \quad (n \text{ odd}),$$
$$g_n = 2^{-\frac{1}{4}} f_1\{\sqrt{(-n)}\} \quad (n \text{ even}).$$

The equations connecting f, f_1 and f_2 with $J(\tau)$ are

$$j(\tau) = 1728\,J(\tau) = 256\,\frac{(1-k^2+k^4)^3}{k^4(1-k^2)^2}$$
$$= \frac{(f^{24}-16)^3}{f^{24}} = \frac{(f_1^{24}+16)^3}{f_1^{24}} = \frac{(f_2^{24}+16)^3}{f_2^{24}},$$

and f^{24}, $-f_1^{24}$, $-f_2^{24}$ are the three roots of

$$(x-16)^3 - xj(\tau) = 0.$$

12.21. Ramanujan obtained the values of the simplest of his singular moduli directly from modular equations. He gives the complete proof in one case only, viz. $n = 5$. Here

$$G_5 = 2^{-\frac{1}{4}} f\{\sqrt{(-5)}\} = \left(\frac{1}{2kk'}\right)^{\frac{1}{12}}$$

and

$$G_{\frac{1}{5}} = G_5$$

(from the transformation $\tau' = -1/\tau$).

If

$$u = 2^{-\frac{1}{4}} f(\tau), \quad v = 2^{-\frac{1}{4}} f(5\tau),$$

then

$$\left(\frac{u}{v}\right)^3 + \left(\frac{v}{u}\right)^3 = 2\left(u^2v^2 - \frac{1}{u^2v^2}\right).$$

This is Schläfli's form of the modular equation of degree 5, which occurs also in the notebook.[1] If $\tau = i/\sqrt{5}$, $q = e^{-\pi/\sqrt{5}}$, then

$$u = G_{\frac{1}{5}}, \quad v = G_5, \quad v = u,$$
$$u^4 - u^{-4} = 1,$$

and

$$G_5 = \left(\frac{\sqrt{5}+1}{2}\right)^{\frac{1}{4}}.$$

Similarly, if we use the form[2]

$$Q + \frac{1}{Q} + 7 = 2^{\frac{3}{2}}\left(P + \frac{1}{P}\right),$$

where

$$P = 2^{\frac{1}{2}}(kk'll')^{\frac{1}{4}}, \quad Q = \left(\frac{ll'}{kk'}\right)^{\frac{1}{4}},$$

of the modular equation of degree 7, we can show that

$$G_7 = 2^{\frac{1}{4}}.$$

Watson, in his paper **20**, works out the values of all of Ramanujan's new singular moduli which it seems likely that he could have found in this way.

[1] 19.13 (xiv). [2] 19.19 (ix).

12.22. There are also many cases in which Ramanujan gives the values of singular moduli which he can hardly have determined rigorously.

Let us suppose, to fix our ideas, that

$$n = 8m-1 \not\equiv 0 \pmod 3.$$

If the number of genera of classes of forms with determinant $-n$ is N, and the number of classes in each genus is r, so that the class number is $h = Nr$, then $j(\tau)$ and F_n satisfy equations of degree h, soluble by radicals. The solution of the equations depends on that of a number of subsidiary equations, the number and degree of these equations being specifiable in terms of N and r. Thus when $n = 7$, $N = 1$, $r = 1$, and $F_7 = 1$ is rational, and when $n = 23$, $N = 1$, $r = 3$, and F_{23} is given by a single cubic

$$F^3 - F - 1 = 0.$$

In Watson's paper **13** he deals with cases in which $N=1$, in **25** also with cases in which $N=2$ or $N=4$. When $N = 2$, two equations are required. Thus if $n = 55$, $N = 2$, $r = 2$, $h = 4$, F satisfies the quartic

$$F^4 - 2F^3 + F - 1 = 0$$

which reduces to $$(F-\tfrac{1}{2})^2 = \frac{3+2\sqrt{5}}{4}$$

by the adjunction of $\sqrt{5}$. If $n = 455$, $N = 4$, $r = 5$, $h = 20$; and F satisfies an equation of degree 20 which can be reduced to a quintic by the adjunction of $\sqrt{5}$ and $\sqrt{13}$.

Ramanujan did not know any of this arithmetical theory, and, when he was dealing with large values of n, where no modular equation was available, must have been guided a good deal by guesswork and intuition. Watson, in his papers **8**, **9**, and **19**, has made a very plausible reconstruction of the arguments which Ramanujan seems to have used. These all involve assumptions that certain numbers satisfy equations of appropriate degrees. In these assumptions, which always accord with the arithmetical theory, he seems to have been guided by a mixture of intuition and computation.

Thus in **8** Watson reconstructs Ramanujan's calculation of g_{210} (a value found also by Weber), and deduces one of the most striking of Ramanujan's results, viz. that

$$k = (\sqrt{2}-1)^2(2-\sqrt{3})(\sqrt{7}-\sqrt{6})^2(8-3\sqrt{7})(\sqrt{10}-3)(4-\sqrt{15})^2 \times(\sqrt{15}-\sqrt{14})(6-\sqrt{35})$$

when $$q = e^{-\pi\sqrt{210}}.$$

This result Ramanujan proved rigorously, granted the value of g_{210}, by a very curious algebraical lemma.

NOTES ON LECTURE XII

§ 12.1. The catalogue referred to on p. 20 (note on p. 10) contains Cayley, Greenhill, Tannery and Molk, A. C. Dixon's *The elementary properties of elliptic functions*, Appell and Lacour's *Principes de la théorie des fonctions elliptiques*, and the first edition of Whittaker's *Modern analysis*. It is however clear that Ramanujan had not read any account of elliptic functions based (like those in Tannery and Molk or Whittaker) on general function theory.

The quotation from Littlewood is from his review of the *Papers*.

§ 12.2. There are proofs of Jacobi's identity, and deductions of those of Euler and Gauss, in Hardy and Wright, §§ 19.8–9.

§ 12.3. See Watson's lecture **10**, Jacobi, *Fundamenta nova*, § 41, and Enneper, *Elliptische Funktionen*, § 13.

The last sentence is justified by the proof (from no. 18 of the *Papers*) reproduced in § 9.2.

§ 12.5. See Cayley, chs. VII and VIII (especially pp. 188–190), Greenhill, ch. X (especially p. 323). There is a better account of the general theory in Enneper, ch. IX (especially pp. 225–237), but Jacobi's treatment of the special case $n = 3$ is stated very shortly in an appendix (p. 518).

§ 12.9. See Tannery and Molk, ii, 163–166, and iv, 230–231, 242–244.

§ 12.12. For (12.12.1) see Tannery and Molk, iii, 120–134 and iv, 105 (Table CVIII).

(12.12.2) reduces to Jacobi's formula quoted in § 12.2 when $\alpha = \beta = 0$.

§ 12.13. See Tannery and Molk, ii, 114–119 and 268 (Table XLVII).

§§ 12.15–19. My sketch of the orthodox theory is based on Tannery and Molk, iv, 254–260.

§ 12.19. The class number here is the number of *primitive* classes of determinant $-n$; we have supposed throughout that $(a, b, c) = 1$. The whole argument is based on Kronecker's theory of forms $ax^2 + bxy + cy^2$, in which b is not necessarily even and the determinant is $b^2 - 4ac$: thus $x^2 + 7y^2$ has determinant -28. In the older theory of Gauss, the form is $ax^2 + 2bxy + cy^2$ and the determinant $b^2 - ac$, so that $x^2 + 7y^2$ has determinant -7.

I have in my possession a letter to Ramanujan from Professor W. E. H. Berwick which throws some light on the last sentence of this section. Professor Berwick, after acknowledging the receipt of Ramanujan's paper "Modular equations and approximations to π" (no. 6 of the *Papers*), and giving a number of references, explains that J satisfies an irreducible equation of degree h, and that k^2 can sometimes, but not always, be expressed rationally in terms of J. In particular he points out that $J(i\sqrt{257})$ and $k^2(i\sqrt{257})$ satisfy equations of degree 16 soluble by chains of four quadratics.

§ 12.20. See Watson (**13**).

§ 12.21. The proof for $n = 5$ is given in no. 6 of the *Papers*.

§ 12.22. All the examples are taken from Watson's papers.

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[In references to periodicals the following abbreviations are used: *JLMS*, Journal of the London Mathematical Society; *PLMS*, Proceedings of the London Mathematical Society; *PCPS*, Proceedings of the Cambridge Philosophical Society; *QJM*, Quarterly Journal of Mathematics; *OQJ*, Quarterly Journal of Mathematics (Oxford).]

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